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Gröbner–Shirshov basis for the braid group in the Artin–Garside generators

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Abstract

Using [Bokut, L., Fong, Y., Ke, W.-F., Shiao, L-S., 2003. Gröbner–Shirshov basis for the braid semigroup. In: Shum, K.-P. (Ed.), Advances in Algebra and Related Topics. Proceedings of the ICM2002 Satellite Conference on Algebra, Hong Kong. World Scientific, River Edge, pp. 14–25], we find a Gröbner–Shirshov basis S for the braid group B_{n+1} in the Artin–Garside generators. We prove that S-irreducible words of the B_{n+1} coincide with the Garside normal form words. It gives a new proof of the uniqueness of the Garside normal form of a word, as well as a new proof that the semigroup B^+_{n+1} of positive braids is a subsemigroup into B_{n+1}.

Keywords: Gröbner–Shirshov basis; Braid group; Garside normal form

1. Introduction and the theorem

Markov (1945) and Artin (1947) independently found a normal form for words in the braid group

$$B_{n+1} = gp\langle a_1, \dots a_n | a_{i+1}a_ia_{i+1} = a_ia_{i+1}a_i, 1 \le i \le n-1, a_ka_s = a_sa_k, k-s > 1\rangle.$$

It was proved in Bokut et al. (2007) that the Markov–Artin normal form leads to a Gröbner–Shirshov basis of B_{n+1} in the Artin–Burau generators with the so-called inverse tower order of words in the generators. Recall that the Artin–Burau generators are the elements

$$a_i, A_{ij} = a_{j-1} \dots a_{i+1}a_i^2a_{i+1}^{-1} \dots a_{j-1}^{-1},$$

where $1 \le i < j \le n+1$.

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In the paper Bokut et al. (2003), a Gröbner–Shirshov basis of the semigroup of positive braids B^+_{n+1} in the Artin generators a_i was found.

In this paper, we find a Gröbner–Shirshov basis of the braid group B_{n+1} in the Artin–Garside generators a_i, $1 \le i \le n$, Δ, Δ^{-1} (Garside, 1969). Here we have

$$\Delta = \Lambda_1\Lambda_2\ldots\Lambda_n, \quad \text{with } \Lambda_i = a_i\ldots a_1.$$

Let us order these generators

$$\Delta^{-1} < \Delta < a_1 < \ldots a_n.$$

We order words in this alphabet in the deg–lex way comparing two words first by theirs degrees (lengths) and then lexicographically when the degrees are equal.

By $V(j,i), W(j,i),\ldots,$ where $j \le i$, we understand positive words in the letters $a_j, a_{j+1},\ldots,a_i$. Also $V(i+1,i) = 1$, $W(i+1,i) = 1,\ldots$.

Given $V = V(1,i)$, let $V^{(k)}$, $1 \le k \le n-i$ be the result of shifting in V all indices of all letters by k, $a_1 \mapsto a_{k+1},\ldots a_i \mapsto a_{k+i}$, and we also use the notation $V^{(1)} = V'$. We write $a_{ij} = a_i a_{i-1}\ldots a_j$, $j \le i-1$, $a_{ii} = a_i$, $a_{ii+1} = 1$.

Theorem 1.1. *A Gröbner–Shirshov basis of B_{n+1} in the Artin–Garside generators consists of the following relations:*

$$a_{i+1}a_i V(1,i-1)W(j,i)a_{i+1j} = a_i a_{i+1} a_i V(1,i-1)a_{ij}W(j,i)', \tag{1.1}$$

$$a_s a_k = a_k a_s, \quad s-k \ge 2, \tag{1.2}$$

$$a_1V_1a_2a_1V_2\ldots V_{n-1}a_n\ldots a_1 = \Delta V_1^{(n-1)}V_2^{(n-2)}\ldots V'_{n-1}, \tag{1.3}$$

$$a_l\Delta = \Delta a_{n-l+1}, \quad 1 \le l \le n, \tag{1.4}$$

$$a_l\Delta^{-1} = \Delta^{-1}a_{n-l+1}, \quad 1 \le l \le n, \tag{1.5}$$

$$\Delta\Delta^{-1} = 1, \qquad \Delta^{-1}\Delta = 1, \tag{1.6}$$

where $1 \le i \le n-1$, $1 \le j \le i+1$, W begins with a_i if it is not empty, and $V_i = V_i(1,i)$.

Recall that a subset S of the free algebra $k\langle X\rangle$ over a field k on X is called a Gröbner–Shirshov set (basis) if every composition of elements of S is trivial. This definition goes back to Shirshov's 1962 paper (Shirshov, 1962). We recall the definition of triviality of a composition in the next section. In Section 3, we recall the definition of the Gröbner–Shirshov basis of a semigroup (group).

Let us define

$$\Lambda_i^{(-)} = a_i\ldots a_2, i \ge 2, \qquad \Lambda_1^{(-)} = 1, \qquad E_i = \Lambda_1\ldots\Lambda_{n-i}\Lambda_{n-i+1}^{(-)}\Lambda_{n-i+2}\ldots\Lambda_n,$$
$$1 \le i \le n.$$

Then

$$E_i a_i = \Delta, \quad \text{and so } a_i^{-1} = \Delta^{-1}E_i.$$

It follows that we do not need the letters a_i^{-1} and the relations $a_ia_i^{-1} = 1$, $a_i^{-1}a_i = 1$ in the above presentation of the group B_{n+1}.

We will write

$$\Lambda_i^{(--)} = a_i\ldots a_3, \quad i \ge 3, \ \Lambda_2^{(--)} = 1.$$

2. Proof of the theorem

Formulas (1.1)–(1.4) are valid in B^+_{n+1} for

$$W(j,i)a_{i+1j} = a_{i+1j}W', \qquad V_{i-1}\Lambda_i = \Lambda_i V'_{i-1}, \qquad a_i\Lambda_{i-1}\Lambda_i = \Lambda_{i-1}\Lambda_i a_1,$$
$$a_1\Lambda_{i+1}\ldots\Lambda_n = \Lambda_{i+1}\ldots\Lambda_n a_{n-i+1}.$$

Formula (1.5) follows from (1.4).

Here and henceforth the notations are the same as in Theorem 1.1.

We need to prove that all compositions of relations (1.1)–(1.6) are trivial. The triviality of compositions of (1.1) and (1.2) was proved in Bokut et al. (2003).

By "a word" we will mean a positive word in a_i, Δ; $u = v$ is either the equality in B^+_{n+1} or the graphical (letter-by-letter) equality (the meaning would be clear from the context).

We use the following notation for words u, v:

$$u \equiv v,$$

if u can be transformed to v by the eliminations of leading words of relations (1.1)–(1.4), i.e., by the eliminations of left parts of these relations. Actually, we will use an expansion of this notation meaning that $u \equiv v$ if

$$u \mapsto u_1 \mapsto u_2 \mapsto \cdots \mapsto u_k = v,$$

where $u_i < u$ for all i and each transformation is an application of (1.1)–(1.4) (so, in general, only the first transformation $u \mapsto u_1$ is the elimination of the leading word of (1.1)–(1.4)).

Another expansion of that formula is

$$u \equiv v (\text{mod } w)$$

meaning that u can be transformed to v as before and all $u_i < w$, $u \leq w$.

By abuse of notation, we take that in a word equivalence chain starting with a word u,

$$u \equiv v \equiv w \equiv t \ldots$$

each equivalence $v \equiv w$, $w \equiv t, \ldots$ is mod u.

This agrees with the definition of triviality of a composition (see Bokut (1972, 1976)). Namely, a composition $(f, g)_w$ is called trivial mod(S, w), if

$$(f, g)_w = \sum \alpha_i a_i s_i b_i,\ a_i \overline{s_i} b_i < w,$$

where $s_i \in S, a_i, b_i \in X^*, \alpha_i \in k$. Here $k\langle X\rangle$ is a free associative algebra over a field k on a set X, $S \subset k\langle X\rangle$, X^* is the set of all words in X, $\overline{s}$ is the leading monomial of a polynomial s. Recall that

$$(f, g)_w = fb - ag,\ w = \overline{f}b = a\overline{g},\ \deg(f) + \deg(g) > \deg(w),$$

or

$$(f, g)_w = f - agb,\ w = \overline{f} = a\overline{g}b.$$

Here w is called the ambiguity of the composition $(f, g)_w$, $a, b \in X^*$.

Let S be the set of polynomials corresponding to semigroup relations $u_i = v_i, u_i > v_i$, $(f, g)_w = u - v$ is a composition of two "semigroup" polynomials $f, g \in S$, $u, v \in X^*$, $u, v < w$. The triviality of $(f, g)_w$ mod(S, w) means that in the previous sense

$$u \equiv t(\mathrm{mod}\ w), \qquad v \equiv t(\mathrm{mod}\ w)$$

for some word t.

Take $V = V(j, i), 2 \le j \le i$. By $V^{(-k)}$, where $1 \le k \le j-1$, we will mean the result of shifting in V the indices of all letters by $-k$, $a_j \mapsto a_{j-k}, \dots, a_i \mapsto a_{i-k}$.

Take $\Delta_i = \Lambda_1 \dots \Lambda_i$, $V = V(1, i)$, $V^{\Delta_i^{\pm 1}} = \Delta_i^{\mp 1} V \Delta_i^{\pm 1}$. Then $V^{\Delta_i^{\pm 1}}$ is equal in B_{n+1} to the word that is the result of substitutions $a_j \mapsto a_{i-j+1}$, $1 \le j \le i$ in V. By abuse of notation, we will identify $V^{\Delta_i^{\pm 1}}$ with this word.

We need the formulas:

$$\Lambda_i W(2,i) \equiv W^{(-1)}\Lambda_i, \qquad \Lambda_i^{(-)} W(3,i) \equiv W^{(-1)}\Lambda_i^{(-)} \tag{2.1}$$

$$a_i \Lambda_{i-1}\Lambda_i^{(-)} \equiv \Lambda_{i-1}\Lambda_i, \qquad a_i\Lambda_{i-1}\Lambda_i \equiv \Lambda_{i-1}\Lambda_i a_1, \tag{2.2}$$

$$a_i \Lambda_{i-1}\Lambda_i^{(--)} \equiv \Lambda_{i-1}^{(-)}\Lambda_i, \qquad a_i \Lambda_{i-1}^{(-)}\Lambda_i^{(-)} \equiv \Lambda_{i-1}^{(-)}\Lambda_i^{(-)} a_2, \tag{2.3}$$

$$a_i \Lambda_{i-1} V_{i-1}\Lambda_i \equiv \Lambda_{i-1}\Lambda_i a_1 V'_{i-1}, \tag{2.4}$$

$$a_i \Lambda_1 V_1 \dots \Lambda_{i-1} V_{i-1} \Lambda_i \equiv \Delta_i a_1 V_1^{(i-1)} \dots V'_{i-1}, \tag{2.5}$$

$$a_i V(1, i-1)\Lambda_1 V_1 \dots \Lambda_{i-1} V_{i-1}\Lambda_i \equiv \Delta_i a_1 V^{\Delta_i} V_1^{(i-1)} \dots V'_{i-1}, \tag{2.6}$$

$$a_i V(1, i-2)\Lambda_{i-1} W(2, i-1)\Lambda_i^{(-)} \equiv \Lambda_{i-1}\Lambda_i V^{(2)} W(2, i-1)', \tag{2.7}$$

$$W(1, i-1)'\Lambda_2^{(-)} \dots \Lambda_i^{(-)} = \Lambda_2^{(-)} \dots \Lambda_i^{(-)} W(1, i-1)^{\Delta_i}, \tag{2.8}$$

$$W(1, i-2)^{\Delta_{i-1}}\Lambda_i = \Lambda_i W W(1, i-2)^{\Delta_i}, \tag{2.9}$$

$$a_n \cdot a_{n-1}a_n \cdots a_1 \dots a_n \equiv \Delta, \tag{2.10}$$

where $i \ge 2$, $V_j = V_j(1, j)$, $1 \le j \le i$.

Formula (2.1) follows from (1.1).

Formula (2.2) can be proved by induction on $i \ge 2$. For $i = 2$, it is clear. Let $i > 2$. Then

$$a_i \Lambda_{i-1}\Lambda_i^{(-)} = a_i a_{i-1}\Lambda_{i-2} a_i \Lambda_{i-1}^{(-)} \equiv a_{i-1}a_i a_{i-1}\Lambda_{i-2}\Lambda_{i-1}^{(-)} \equiv a_{i-1}a_i\Lambda_{i-2}\Lambda_{i-1} \equiv \Lambda_{i-1}\Lambda_i.$$

Formula (2.3) can also be proved by induction on $i \ge 2$. For $i = 2$, it is clear. Let $i > 2$. Then

$$\begin{aligned} a_i \Lambda_{i-1}\Lambda_i^{(--)} &= a_i a_{i-1}\Lambda_{i-2} a_i \Lambda_{i-1}^{(--)} \equiv a_{i-1} a_i a_{i-1}\Lambda_{i-2}\Lambda_{i-1}^{(--)} \\ &\equiv a_{i-1} a_i \Lambda_{i-2}^{(-)}\Lambda_{i-1} \equiv \Lambda_{i-1}^{(-)}\Lambda_i. \end{aligned}$$

Formula (2.4) follows from (1.1) and (2.2).

Formula (2.5) follows from (1.1) and (2.2):

$$\begin{aligned} a_i \Lambda_1 V_1 \dots \Lambda_{i-1} V_{i-1}\Lambda_i &\equiv \Lambda_1 V_1 \dots \Lambda_{i-2} V_{i-2} a_i a_{i-1}\Lambda_{i-2} V_{i-1}\Lambda_i \\ &\equiv \Lambda_1 V_1 \dots \Lambda_{i-2} V_{i-2} a_{i-1} a_i a_{i-1}\Lambda_{i-2}\Lambda_{i-1} V'_{i-1} \\ &\equiv \Lambda_1 V_1 \dots \Lambda_{i-2} V_{i-2} a_{i-1} a_i \Lambda_{i-2}\Lambda_{i-1} a_1 V'_{i-1} \\ &\equiv \Lambda_1 V_1 \dots \Lambda_{i-2} V_{i-2}\Lambda_{i-1}\Lambda_i a_1 V'_{i-1} \equiv \Delta_i a_1 V_1^{(i-1)} \dots V'_{i-1}. \end{aligned}$$

Remark, that the last word is less than the first word in the above chain of equivalence formulas, though it can be greater than the word just before the last.

Formula (2.6) can be proved by induction on $i \geq 2$. It is clear for $i = 2$. Take $i > 2$. Then formula (2.6) follows from (2.5) by induction on the number k of letters a_{i-1} in $V(1, i-1)$. If $k = 0$, the result is clear. Take $k \geq 1$ and

$$V(1, i-1) = W(1, i-1)a_{i-1}T(1, i-2).$$

Then

$$a_{i-1}T(1, i-2)\Lambda_1 V_1 \ldots \Lambda_{i-2}V_{i-2}\Lambda_{i-1} \equiv \Delta_{i-1}a_1 T^{\Delta_{i-1}} V_1^{(i-2)} \ldots V'_{i-2}.$$

It follows that

$$\begin{aligned} & a_i W(1, i-1)a_{i-1}T(1, i-2)\Lambda_1 V_1 \ldots \Lambda_{i-1}V_{i-1}\Lambda_i \\ & \quad \equiv a_i W(1, i-1)\Delta_{i-1}a_1 T^{\Delta_{i-1}} V_1^{(i-2)} \ldots V'_{i-2}V_{i-1}\Lambda_i \\ & \quad \equiv a_i W(1, i-1)\Delta_i a_2 T^{\Delta_i} V_1^{(i-1)} \ldots V'_{i-1} \equiv \Delta_i a_1 W^{\Delta_i} a_2 T^{\Delta_i} V_1^{(i-1)} \ldots V'_{i-1}. \end{aligned}$$

Remark again that here the second and third words above are less than the first one for $i - 1 \geq 2$.

Formula (2.7) follows from (1.1) and (2.2).

Formula (2.8) follows from (1.2), (2.3) and (1.1).

Formula (2.9) follows from (1.1).

Formula (2.10) can be proved by induction on n. Indeed:

$$\begin{aligned} & a_n \cdot a_{n-1}a_n \cdots a_2 \ldots a_n \cdot a_1 \ldots a_n \equiv a_2 \cdot a_3 a_2 \cdots a_n \ldots a_2 \cdot a_1 \ldots a_n \\ & \quad \equiv a_2 \cdot a_3 a_2 \cdots a_{n-1} \ldots a_2 \cdot a_1 \ldots a_{n-1} \cdot a_n a_{n-1} \ldots a_1 \equiv \ldots \\ & \quad \equiv a_2 \cdot a_3 a_2 \cdot a_1 a_2 a_3 \cdot a_4 \ldots a_1 \cdots a_n \ldots a_1 \equiv a_2 \cdot a_1 a_2 \cdot a_3 a_2 a_1 \cdots a_n \ldots a_1 \equiv \Delta. \end{aligned}$$

Now let us check the composition of (1.1), $i + 1 = n$, and (1.3). Without loss of generality we may assume that $j = 1$.

The ambiguity is

$$w = a_n a_{n-1} V(1, n-2) W(1, n-1) \Lambda_1 V_1 \ldots \Lambda_{n-1} V_{n-1} a_{n1}.$$

Applying (1.3) to w, we obtain

$$w_1 = a_n a_{n-1} V(1, n-2) W(1, n-1) \Delta V_1^{(n-1)} \ldots V'_{n-1}.$$

Applying (1.1) to w, we obtain

$$w_2 = a_{n-1} a_n a_{n-1} V(1, n-2) \Lambda_{n-1} W' \Lambda'_1 V'_1 \ldots \Lambda'_{n-1} V'_{n-1}.$$

Applying (1.1) and (2.1)–(2.10) to w_1, w_2, we obtain

$$\begin{aligned} w_1 & \equiv \Delta a_1 a_2 V^{\Delta} W^{\Delta} V_1^{(n-1)} \ldots V'_{n-1}, \\ w_2 & = a_{n-1} a_n a_{n-1} V(1, n-2) \Lambda_{n-1} W' \Lambda_2^{(-)} V'_1 \ldots \Lambda_{n-1}^{(-)} V'_{n-2} \Lambda_n^{(-)} V'_{n-1} \\ & \equiv a_{n-1} a_n a_{n-1} V(1, n-2) \Lambda_{n-1} W' \Lambda_2^{(-)} \ldots \Lambda_n^{(-)} V_1^{(n-1)} \ldots V'_{n-1} (\mathrm{mod}\ w) \\ & \equiv a_{n-1} a_n a_{n-1} V(1, n-2) \Lambda_{n-1} \Lambda_2^{(-)} \ldots \Lambda_{n-1}^{(-)} \Lambda_n^{(-)} W^{\Delta} V_1^{(n-1)} \ldots V'_{n-1} (\mathrm{mod}\ w) \\ & \equiv a_{n-1} a_n a_{n-1} V(1, n-2) \Lambda_1 \Lambda_2 \ldots \Lambda_{n-2} \Lambda_{n-1} \Lambda_n^{(-)} W^{\Delta} V_1^{(n-1)} \ldots V'_{n-1} (\mathrm{mod}\ w) \\ & \equiv a_{n-1} a_n a_{n-1} \Lambda_1 \Lambda_2 \ldots \Lambda_{n-2} \Lambda_{n-1} \Lambda_n^{(-)} V^{\Delta} W^{\Delta} V_1^{(n-1)} \ldots V'_{n-1} (\mathrm{mod}\ w) \end{aligned}$$

$$\begin{aligned} &\equiv a_{n-1}a_n\Lambda_1\ldots\Lambda_{n-2}\Lambda_{n-1}\Lambda_n^{(--)}a_1a_2V^{\Delta}W^{\Delta}V_1^{(n-1)}\ldots V'_{n-1}(\text{mod } w)\\ &\equiv a_{n-1}\Lambda_1\ldots\Lambda_{n-2}\Lambda_{n-1}^{(-)}\Lambda_n a_1a_2V^{\Delta}W^{\Delta}V_1^{(n-1)}\ldots V'_{n-1}(\text{mod } w)\\ &\equiv \Delta a_1a_2V^{\Delta}W^{\Delta}V_1^{(n-1)}\ldots V'_{n-1}(\text{mod } w). \end{aligned}$$

The composition is checked.

There is the composition of (1.1), $i+1<n$ and (1.3). Again, we may assume that $j=1$.
The ambiguity is

$$w = a_{i+1}a_iV(1,i-1)W(1,i)\Lambda_1V_1\ldots\Lambda_{i+1}V_{i+1}\ldots V_{n-1}\Lambda_n.$$

Applying (1.3) to w, we obtain

$$\begin{aligned} w_1 &= a_{i+1}a_iV(1,i-1)W(1,i)\Delta V_1^{(n-1)}\ldots V'_{n-1}\\ &\equiv \Delta a_{n-i}a_{n-i+1}V^{\Delta}W^{\Delta}V_1^{(n-1)}\ldots V'_{n-1}. \end{aligned}$$

Applying (1.1) to w, we have

$$\begin{aligned} w_2 &= a_ia_{i+1}a_iV(1,i-1)\Lambda_iW(1,i)'\Lambda'_1V'_1\ldots\Lambda'_iV'_iV_{i+1}\ldots V_{n-1}\Lambda_n\\ &\equiv \Lambda_1\ldots\Lambda_{i+1}a_1a_2V^{\Delta_{i+1}}W^{\Delta_{i+1}}V_1^{(i)}\ldots V'_iV_{i+1}\Lambda_{i+2}\ldots V_{n-1}\Lambda_n(\text{mod } w)\\ &\equiv \Delta a_{n-i}a_{n-i+1}V^{\Delta}W^{\Delta}V_1^{(n-1)}\ldots V_i^{(n-i)}\ldots V'_{n-1}. \end{aligned}$$

Here we use the calculation of w_2 from the previous composition, substituting $n\mapsto i+1$.
The second composition is checked.
There is the composition of (1.3) and (1.1).
The ambiguity is

$$w = \Lambda_1V_1\ldots V_{n-1}a_{ni+2}a_{i+1}a_i\ldots a_1V(1,i-1)W(j,i)a_{i+1j}.$$

Applying (1.3) to w, we obtain

$$w_1 = \Delta V_1^{(n-1)}\ldots V'_{n-1}VWa_{i+1j}.$$

Applying (1.1) to w, we have

$$\begin{aligned} w_2 &= \Lambda_1V_1\ldots V_{n-1}a_{ni+2}a_ia_{i+1}a_i\ldots a_1V(1,i-1)a_{ij}W'\\ &\equiv \Lambda_1V_1\ldots V_{n-1}a_ia_{ni+2}a_{i+1}a_i\ldots a_1Va_{ij}W' \equiv \Delta V_1^{(n-1)}\ldots V'_{n-1}a_{i+1}Va_{ij}W'\\ &\equiv \Delta V_1^{(n-1)}\ldots V'_{n-1}Va_{i+1j}W' \equiv \Delta V_1^{(n-1)}\ldots V'_{n-1}VWa_{i+1j}. \end{aligned}$$

There is the composition of (1.1) and (1.4).
The ambiguity is

$$w = a_{i+1}a_iV(1,i-1)W(j,i)a_{i+1j+1}a_j\Delta.$$

Applying (1.4) to w, we obtain

$$\begin{aligned} w_1 &= a_{i+1}a_iVWa_{i+1j+1}\Delta a_{n-j+1} \equiv \Delta a_{n-i}a_{n-i+1}V^{\Delta}W^{\Delta}a_{n-i}a_{n-i+1}\ldots a_{n-j+1}\\ &\equiv \Delta a_{n-i}a_{n-i+1}V^{\Delta}a_{n-i}a_{n-i+1}\ldots a_{n-j+1}W^{\Delta(-1)}\\ &\equiv \Delta a_{n-i}a_{n-i+1}a_{n-i}V^{\Delta}a_{n-i+1}\ldots a_{n-j+1}W^{\Delta(-1)}. \end{aligned}$$

Applying (1.1) to w, we obtain

$$\begin{aligned} w_2 &= a_i a_{i+1} a_i V a_{ij} W' \Delta \equiv \Delta a_{n-i+1} a_{n-i} a_{n-i+1} V^{\Delta} a_{n-i+1} \dots a_{n-j+1} W'^{\Delta} \\ &\equiv \Delta a_{n-i} a_{n-i+1} a_{n-i} V^{\Delta} a_{n-i+1} \dots a_{n-j+1} W'^{\Delta}. \end{aligned}$$

The composition is trivial since $W^{\Delta(-1)} = W'^{\Delta}$.
The triviality of the composition of (1.1) with (1.5) is proved similarly.
There is the composition of (1.3) and (1.4).
The ambiguity is

$$w = \Lambda_1 V_1 \dots V_{n-1} a_n \dots a_2 a_1 \Delta.$$

Applying (1.3) to w, we have

$$w_1 = \Delta V_1^{(n-1)} \dots V'_{n-1} \Delta \equiv \Delta^2 V_1^{(n-1)\Delta} \dots V'^{\Delta}_{n-1}.$$

Applying (1.4) to w, we obtain

$$\begin{aligned} w_2 &= \Lambda_1 V_1 \dots V_{n-1} a_n \dots a_2 \Delta a_n \equiv \Delta \Lambda_1^{\Delta} V_1^{\Delta} \dots V_{n-1}^{\Delta} a_1 \dots a_{n-1} a_n \\ &= \Delta a_n V_1^{\Delta} a_n a_{n-1} \dots V_{n-1}^{\Delta} a_1 \dots a_n \\ &\equiv \Delta a_n \cdot a_{n-1} a_n \cdots a_1 \dots a_n V_1^{\Delta(-(n-1))} \dots V_{n-1}^{\Delta(-1)} \\ &\equiv \Delta^2 V_1^{(n-1)\Delta} \dots V'^{\Delta}_{n-1} \end{aligned}$$

for $V_i^{(j)\Delta} = V_i^{\Delta(-j)}, 2 \leq i + j \leq n$.
The composition is checked.
The triviality of composition of (1.3) and (1.5) is proved similarly.

3. Corollaries

Let $S \subset k\langle X\rangle$. A word u is called S-irreducible if $u \neq a\bar{s}b$, where $s \in S, a, b \in X^*$. Let $Irr(S)$ be the set of all S-irreducible words.

Recall Shirshov's Composition lemma (Shirshov, 1962; Bokut, 1972, 1976):

Let S be a Gröbner–Shirshov basis in $k\langle X\rangle$. If $f \in ideal(S)$, then $\overline{f} = a\bar{s}b, s \in S$. The converse is also true: If for any $f \in ideal(S)$ we have $\overline{f} = a\bar{s}b, s \in S$, then S be a Gröbner–Shirshov basis in $k\langle X\rangle$.

The Main corollary to this lemma is the following statement:

A subset $S \subset k\langle X\rangle$ is a Gröbner–Shirshov basis iff the set $Irr(S)$ is a linear basis for the algebra $k\langle X\rangle/ideal(S) = k\langle X|S\rangle$ generated by X with defining relations S.

Let $G = sgp\langle X|S\rangle$ be the semigroup generated by X with defining relations S. Then S is called a Gröbner–Shirshov basis of G if S is a Gröbner–Shirshov basis of the semigroup algebra $k(G)$, i.e., S is a Gröbner–Shirshov set in $k\langle X\rangle$. It follows from the Main corollary to the Composition lemma that in this case any word u in X is equal in G to a unique S-irreducible word $C(u)$, called the normal (canonical) form of u. If $G = gp\langle X|S\rangle$ is a group, then $G = sgp\langle X \cup X^{-1}|S_0\rangle$, where $S_0 = S \cup \{xx^{-1} = 1, x^{-1}x = 1, x \in X\}$. Then S_0 is called a Gröbner–Shirshov basis for the group G if it is a Gröbner–Shirshov basis for G as the semigroup.

Now let S be the set of relations (1.1)–(1.6), and let $C(u)$ be a normal form of a word $u \in B_{n+1}$. Then $C(u)$ has a form

$$C(u) = \Delta^k A,$$

where $k \in \mathbb{Z}$, and A a positive S-irreducible word in a_i's. Let us prove that $A \neq \Delta A_1$ in B_{n+1} for every positive word A_1. It would give that $C(u)$ is the Garside normal form for u.

We can switch to the semigroup B_{n+1}^+ since this semigroup is a subsemigroup in B_{n+1} (see, for example, Corollary 3.2). Note that we have no generator Δ in B_{n+1}^+, it is just the word in the Artin generators. According to Bokut et al. (2003), the set S_1 of (1.1) and (1.2) is a Gröbner–Shirshov basis of B_{n+1}^+. Then A is an S_1-irreducible word.

Suppose to the contrary that in B_{n+1}^+ we have

$$A = \Delta A_1,$$

where A_1 is a positive word. We may assume that A_1 is also an S_1-irreducible word. Then

$$A = a_1 \cdot a_2 a_1 \cdots a_n \ldots a_1 A_1$$

in the semigroup B_{n+1}^+. Let us prove that

$$A = C(a_1 \cdot a_2 a_1 \cdots a_n \ldots a_1 A_1) = \Lambda_1 V_1 \ldots \Lambda_{n-1} V_{n-1} \Lambda_n V,$$

where $=$ is the graphical equality, $V_i = V(1,i)$, $1 \leq i \leq n-1$, and $C(D)$ is the irreducible normal form of D in B_{n+1}^+. It would contradict the S-irreducibility of A.

More generally, let us prove that

$$C(\Lambda_1 W_1 \ldots \Lambda_{n-1} W_{n-1} \Lambda_n W_n) = \Lambda_1 V_1 \ldots \Lambda_{n-1} V_{n-1} \Lambda_n V_n,$$

where $W_i = W_i(1,i)$, $V_i = V_i(1,i)$, $1 \leq i \leq n$. If

$$B = \Lambda_1 W_1 \ldots \Lambda_{n-1} W_{n-1} \Lambda_n W_n$$

is an irreducible word, than we are done. Suppose, B contains the left part of a relation (1.1) and (1.2). If this word is a subword of W_i, $1 \leq i \leq n$, then the situation is clear: we can apply the relation to get a smaller word of the same form, and then we can use induction. Let the word be a subword of $\Lambda_k W_k$, $1 \leq k \leq n$, but not W_k. It may only be the left part of (1.1), $k \geq i+1$. Then we have

$$\begin{aligned} \Lambda_k W_k &= a_{ki+2} a_{i+1} a_i a_{i-1} \ldots a_1 V(1, i-1) W(j,i) a_{i+1j} T_k \\ &\equiv a_{ki+2} a_i a_{i+1} a_i a_{i-1} \ldots a_1 V(1, i-1) a_{ij} W' T_k \equiv a_i \Lambda_k W_{k1}, \end{aligned}$$

where $T_k = T_k(1,k)$, $W_{k1} = W_{k1}(1,k)$. Substituting this expression of $\Lambda_k W_k$ in B, we obtain a positive word D which is smaller than B and has the same form. By induction, we are done.

As a result, we have the following

Corollary 3.1. *The S-irreducible normal form of each word of B_{n+1} coincides with the Garside normal form of the word.*

Proof. Recall (Garside, 1969) that Garside normal form $G(u)$ of $u \in B_{n+1}$ is

$$G(u) = \Delta^k A,$$

where $u = G(u)$ in B_{n+1}, $k \in \mathbb{Z}$, and A is a positive word in a_i, $A \neq \Delta A_1$ for every positive word A_1, and A is the minimal word with these properties. We have proved that the S-irreducible normal form $C(u)$ has these properties. □

Corollary 3.2 (*Garside (1969)*). *The semigroup of positive braids B_{n+1}^+ can be embedded into a group.*

Proof. From Bokut et al. (2003) and Theorem 1.1, it follows immediately that two positive braid words are equal in the group B_{n+1} iff they are equal in the semigroup B^+_{n+1}. It means that B^+_{n+1} is the subsemigroup of B_{n+1}. □

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