JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 134 (2001) 143–164

www.elsevier.com/locate/cam

Analysis of completely discrete finite element method for a free boundary diffusion problem with absorption

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Received 24 February 1999; received in revised form 28 March 2000

Abstract

Convergence of truncation methods is obtained for a free boundary problem in $\mathbb{R}^2$ with an absorption depending on space and time. Error estimates are derived for the discretization, in space by a P_1-finite element method and in time by a backward Euler method. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Free boundary; Parabolic; Finite elements; Backward Euler method

1. Introduction

In the vineyard, the grapes are often attacked by a butterfly, the Eudémis which favoured the development of a mushroom. This mushroom leads to alteration to the quality of the grapes; so an effective struggle against this insect is essential. In order to protect environment, researchers of INRA have settled a new method: The female emits an odorous substance (the pheromone) in a very small quantity. This pheromone has been synthetised and it has been proved that the presence of this substance causes interference to olfactory communications between the insects and then leads to a decline of the eggs and so of the population. So it is important to know the quantity of pheromone which is necessary to put in a field to lead to a significant drop of the population.

This substance is contained in small "diffusors" which are spread about the field, then it is diffused and is absorbed by the medium and it may be spread out by the wind.

So, the quantity c of pheromone in a field is the solution of a free boundary problem of the following form:

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PII: S0377-0427(00)00534-3

Let Ω be a bounded domain in $\mathbb{R}^2$ with a smooth boundary Γ; $\boldsymbol{V}$ is the velocity of the wind which satisfies $\operatorname{div} \boldsymbol{V} = 0$ ($\boldsymbol{V}$ is solution of Navier–Stokes equations); α is a diffusion coefficient ($\alpha > 0$), f is an absorption coefficient ($f \geqslant 0$);

c satisfies

$$c_t - \alpha\Delta c + \operatorname{div}(\boldsymbol{V}c) = -f, \quad x \in \omega(t),\ t > 0, \tag{1.1}$$

where $\omega(t) = \{x \in \Omega / c(x,t) > 0\}$ with the boundary conditions: $c = 0$ *on* $\Gamma \times (0,T)$; $c = \partial c/\partial n = 0$ *on* $\partial\omega(t)$ ($\boldsymbol{n}$ is the outer normal to $\partial\omega$) and the initial condition: $c(x,0) = c_0(x) \geqslant 0, x \in \Omega$ ($c_0(x) > 0$ *on* $\omega(0)$ and $\omega(0)$ is strictly included in Ω).

In this paper, we study a numerical method which is a generalization of the truncation method proposed in [1]. These authors have proved the convergence of the method for a constant absorption when the domain is $\mathbb{R}$ and with finite difference methods under the stability condition: $(\Delta t/h^2) \leqslant C^{te}$ (Δt is the time step and h is the space step).

Here, we use a truncation method for an absorption depending on space and time, a backward Euler method in time and a P_1-finite element method in Ω, a bounded domain in $\mathbb{R}^2$. By using the error estimates obtained in [8] in $L^\infty(\Omega)$ concerning parabolic problems, we obtain the error estimates for the numerical method. Numerical results are presented in [6].

An outline of this paper is as follows:

- In Section 2, we present two numerical methods using truncations which have been used to obtain a nonnegative solution. Since the proof of the convergence is obtained by comparing their respective solutions, we need to study these two methods all together.
- In Section 3, we proceed with the study of the error due to the truncation only.
- In Section 4, we analyse the semi-discretization in time for these two truncation methods.
- Section 5, finally, is devoted to the analysis of the complete discretization in space and time.

2. Definition of the numerical method

We denote by A the maximal positive operator of domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ defined by

$$\forall u \in D(A), \quad Au = -\alpha\Delta u + \operatorname{div}(\boldsymbol{V}u) \tag{2.1}$$

and by a the function defined on $\mathbb{R}$ by

$$a(s) = \begin{cases} 0 & \text{if } s \leqslant 0, \\ 1 & \text{if } s > 0. \end{cases} \tag{2.2}$$

By using these notations, problem (1.1) may be written

$$\begin{aligned} &c_t + Ac = -f a(c) \quad \text{in } \Omega, \\ &c(0) = c_0 \quad \text{in } \Omega. \end{aligned} \tag{2.3}$$

This problem has a unique continuous solution if $f \in L^\infty(0,T; W^{1,\infty}(\Omega))$, $c_0 \in W^{2,\infty}(\Omega)$ [9].

If we use a backward Euler method to solve (2.3), we obtain the following scheme:

Given an approximation c^n of $c(t_n)$, ($t_n = n\Delta t$ and Δt is the time step), c^{n+1} is solution of the problem:

$$(I + \Delta t A)c^{n+1} = c^n - \Delta t f(t_{n+1}) a(c_{n+1}). \tag{2.4}$$

In practice, this method cannot be used due to the term $a(c^{n+1})$; if we linearize this method, we get

$$(I + \Delta tA)c^{n+1} = c^n - \Delta t f(t_n)a(c_n), \tag{2.5}$$

but this scheme does not guarantee the positiveness of c^{n+1}. So we use a truncation method; several techniques may be used:

In the first case, we compute an intermediate value u^{n+1} solution of

$$(I + \Delta tA)u^{n+1} = c^n \tag{2.6}$$

and c^{n+1} is then defined by

$$c^{n+1} = \text{Max}(u^{n+1} - \Delta t f(t_{n+1}), 0). \tag{2.7}$$

In the second method, we define u^n by

$$u^n = \text{Max}(c^n - \Delta t f(t_{n+1}), 0) \tag{2.8}$$

and c^{n+1} is solution of

$$(I + \Delta tA)c^{n+1} = u^n. \tag{2.9}$$

In the two cases, c^{n+1} is a nonnegative function. The truncation may be used after k time steps where k must be chosen in an optimal manner. Eqs. (2.6), (2.9) are discretized by using a P_1-finite element method.

We shall prove the convergence of these methods all together, since the proof is obtained by comparing their respective solutions. This convergence will be proved in three steps: First, we estimate the error between the exact solution and the approximations obtained in replacing $(I+\Delta tA)^{-1}$ by e^{-tA} (semi-group generated by A) in (2.6), (2.9). Then we compare these approximations to the values of c obtained in (2.7), (2.9); this will give us an optimal value of k. By using the monotonicity of the operator $(I+\Delta tA)^{-1}$, we then prove that the truncation may be used at each time step. Finally, we analyze the error due to the space discretization of (2.6), (2.9).

3. Definition of the truncation methods

Let τ be a fixed time step; this time step will correspond to the time interval between two corrections of the solution by a truncation. If u is a function defined on Ω, we denote

$$Mu = \text{Max}(u, 0) \tag{3.1}$$

and $E(t) = e^{-tA}$ the semi-group operator associated to A.

We consider the following approximations of the solution c:

First approximation:

$$\begin{aligned} &c_1(0) = c_0, \\ &c_1((n+1)\tau) = M[E(\tau)c_1(n\tau) - \tau f((n+1)\tau)]. \end{aligned} \tag{3.2}$$

Second approximation:

$$\begin{aligned} &c_2(0) = c_0, \\ &c_2((n+1)\tau) = E(\tau)M(c_2(n\tau) - \tau f((n+1)\tau)). \end{aligned} \tag{3.3}$$

In the next lemmas, we compare these two approximations to c. The first approximation gives a lower bound to the exact solution and the second approximation gives an upper bound to this quantity.

The $L^\infty(\Omega)$-norm is denoted by $\|.\|$.

Lemma 3.1. *For* $n \geqslant 1$, *we have the inequality*

$$c_2(n\tau) \leqslant c_1(n\tau) + \tau f(n\tau) + \tau \sum_{j=1}^{n-1} \|f(j\tau) - f((j+1)\tau)\|. \tag{3.4}$$

Proof. Let us prove the result first in the case $n = 1$. We have the inequality

$$c_2(\tau) \leqslant M(c_2(\tau) - \tau f(\tau)) + \tau f(\tau) \tag{3.5}$$

and since $M(c_0 - \tau f(\tau)) \leqslant c_0$, by using (3.3), we get $c_2(\tau) \leqslant E(\tau)c_0$.

Then, from (3.5), we obtain

$$c_2(\tau) \leqslant M(E(\tau)c_0 - \tau f(\tau)) + \tau f(\tau)$$

that is $c_2(\tau) \leqslant c_1(\tau) + \tau f(\tau)$.

So, the result is proved for $n = 1$; we prove the general case recurrently.

Let us denote $\tilde{c}_0 = M(c_0 - \tau f(\tau))$ and $c_1(n\tau; \tilde{c}_0)$ the solution obtained by the first approximation with the initial data $\tilde{c}_0$. We prove recurrently the following estimate:

$$M(c_2(n\tau) - \tau f(n\tau)) \leqslant c_1(n\tau; \tilde{c}_0) + \tau \sum_{j=1}^{n-1} \|f(j\tau) - f((j+1)\tau)\|. \tag{3.6}$$

For $n = 1$, since $c_2(\tau) = E(\tau)\tilde{c}_0$, we get immediately: $M(c_2(\tau) - \tau f(\tau)) = \tilde{c}_1(\tau; \tilde{c}_0)$.

We suppose that (3.6) is satisfied for $j \leqslant n$; from the inequality

$$M(c_2(n\tau) - \tau f((n+1)\tau)) \leqslant M(c_2(n\tau) - \tau f(n\tau)) + \tau \|f(n\tau) - f((n+1)\tau)\|,$$

we get

$$c_2((n+1)\tau) \leqslant E(\tau)M(c_2(n\tau) - \tau f(n\tau)) + \tau \|f(n\tau) - f((n+1)\tau)\|$$

and by using the recurrence hypothesis, we obtain

$$c_2((n+1)\tau) \leqslant E(\tau)c_1(n\tau; \tilde{c}_0) + \tau \sum_{j=1}^{n} \|f(j\tau) - f(j+1)\tau\|.$$

It follows that

$$M(c_2((n+1)\tau) - \tau f((n+1)\tau)) \leqslant c_1((n+1)\tau; \tilde{c}_0) + \tau \sum_{j=1}^{n} \|f(j\tau) - f((j+1)\tau)\|$$

and (3.6) is obtained at the step $n + 1$.

From the following inequalities:

$c_2((n+1)\tau) \leqslant M(c_2((n+1)\tau) - \tau f((n+1)\tau)) + \tau f((n+1)\tau)$, and $c_1(n\tau; \tilde{c}_0) \leqslant c_1(n\tau)$, we obtain the result. □

Lemma 3.2. *For* $n \geqslant 1$, *if* $Af \in L^\infty((0,T)\times\Omega)$, *we have the inequality*

$$c_1(n\tau) \leqslant c(n\tau) + \sum_{j=0}^{n-1} \int_{j\tau}^{(j+1)\tau} \|f(s) - f((j+1)\tau)\| \,\mathrm{d}s + \tau^2 \sum_{j=1}^{n} \|Af(j\tau)\|.$$

Proof. For $n = 1$, we have

$$c_1(\tau) \leqslant M\left(E(\tau)c_0 - \int_0^\tau E(\tau - s)f(s)\,\mathrm{d}s\right) + M\left(\int_0^\tau E(\tau - s)f(s)\,\mathrm{d}s - \tau f(\tau)\right).$$

Using the definition of a in (2.2), we get

$$M\left(E(\tau)c_0 - \int_0^\tau E(\tau - s)f(s)\,\mathrm{d}s\right) \leqslant M\left(E(\tau)c_0 - \int_0^\tau E(\tau - s)a(c(s))f(s)\,\mathrm{d}s\right) = c(\tau),$$

hence we obtain

$$c_1(\tau) \leqslant c(\tau) + M\left(\int_0^\tau E(\tau - s)f(s)\,\mathrm{d}s - \tau f(\tau)\right).$$

We estimate now the second term of the right member of this inequality

$$M\left(\int_0^\tau E(\tau - s)f(s)\,\mathrm{d}s - \tau f(\tau)\right)$$
$$\leqslant \left\|\int_0^\tau E(\tau - s)(f(s) - f(\tau))\,\mathrm{d}s\right\| + \left\|\int_0^\tau (E(\tau - s) - I)f(\tau)\,\mathrm{d}s\right\|$$

and since $E(\tau - s) - I = -\int_0^{\tau - s} AE(\xi)\,\mathrm{d}\xi$, we deduce

$$\left\|\int_0^\tau (E(\tau - s) - I)f(\tau)\,\mathrm{d}s\right\| \leqslant \tau^2 \,\|Af(\tau)\|.$$

We then obtain

$$M\left(\int_0^\tau E(\tau - s)f(s)\,\mathrm{d}s - \tau f(\tau)\right) \leqslant \int_0^\tau \|f(s) - f(\tau)\|\,\mathrm{d}s + \tau^2\|Af(\tau)\|$$

and

$$c_1(\tau) \leqslant c(\tau) + \int_0^\tau \|f(s) - f(\tau)\|\,\mathrm{d}s + \tau^2\|Af(\tau)\|.$$

If $n > 1$, the proof is analogous. □

Lemma 3.3. *For* $n \geqslant 1$, *if* $Af \in L^\infty((0,T)\times\Omega)$, *we have the inequality*

$$c(n\tau) \leqslant c_2(n\tau) + \sum_{j=0}^{n-1} \int_{j\tau}^{(j+1)\tau} \|f(s) - f((j+1)\tau)\|\,\mathrm{d}s + \frac{\tau^2}{2} \sum_{j=1}^{n} \|Af(j\tau)\|. \tag{3.7}$$

Proof. For $n = 1$, $\varepsilon > 0$, we define a function $\tilde{c}$ on $[0, \tau]$ by

$$\tilde{c}(t) = E(t)M(c_0 - tf(\tau)) + \varepsilon + \int_0^t \|f(s) - f(\tau)\|\,\mathrm{d}s + \frac{t^2}{2}\|Af(\tau)\|. \tag{3.8}$$

This function satisfies

$$\tilde{c}_t(t) + A\tilde{c}(t) = E(t)\frac{\partial}{\partial t}M(c_0 - tf(\tau)) + \|f(t) - f(\tau)\| + t\|Af(\tau)\|,$$
$$\tilde{c}(0) = c_0 + \varepsilon.$$

Besides, $(\partial/\partial t)M(c_0 - tf(\tau)) \geqslant -f(\tau)$, hence we get

$$\tilde{c}_t(t) + A\tilde{c}(t) \geqslant -(E(t) - I)f(\tau) - f(t) + t\|Af(\tau)\|$$

and since, $\|(E(t) - I)f(\tau)\| \leqslant t\|Af(\tau)\|$ and $a(\tilde{c}) = 1$, it follows that

$$\tilde{c}_t(t) + A\tilde{c}(t) \geqslant -f(t)a(\tilde{c}).$$

Then, for any $\varepsilon > 0$, we have $\tilde{c}(t) \geqslant c(t)$, $t \in [0,\tau]$, in particular, we obtain: $\lim_{\varepsilon\to 0}\tilde{c}(\tau) \geqslant c(\tau)$, that is

$$c_2(\tau) + \int_0^\tau \|f(s) - f(\tau)\|\,\mathrm{d}s + \frac{\tau^2}{2}\|Af(\tau)\| \geqslant c(\tau).$$

So, we have proved the estimate for $n = 1$.

For $n > 1$, the proof is analogous; we define a function $\tilde{c}$ on $[n\tau, (n+1)\tau]$ by

$$\tilde{c}(t) = E(t - n\tau)M(c(n\tau) - (t - n\tau)f((n+1)\tau)) + \varepsilon$$
$$+ \int_{n\tau}^{t} \|f(s) - f((n+1)\tau)\|\,\mathrm{d}s + \frac{(t - n\tau)^2}{2}\|Af((n+1)\tau)\|.$$

Then $\tilde{c}$ satisfies

$$\tilde{c}_t + A\tilde{c} \geqslant -a(\tilde{c})f(t),$$
$$\tilde{c}(n\tau) = c(n\tau) + \varepsilon$$

hence, $\lim_{\varepsilon\to 0}\tilde{c}((n+1)\tau) \geqslant c((n+1)\tau)$, that is

$$E(\tau)M(c(n\tau) - \tau f((n+1)\tau)) + \int_{n\tau}^{(n+1)\tau} \|f(s) - f((n+1)\tau)\|\,\mathrm{d}s + \frac{\tau^2}{2}\|Af((n+1)\tau)\| \geqslant c((n+1)\tau)$$

and recurrently, we get

$$c((n+1)\tau) \leqslant c_2((n+1)\tau) + \sum_{j=0}^{n} \int_{j\tau}^{(j+1)\tau} \|f(s) - f((j+1)\tau)\|\,\mathrm{d}s + \frac{\tau^2}{2}\sum_{j=1}^{n+1} \|Af(j\tau)\|. \qquad \square$$

From these lemmas, we easily deduce the following theorem:

Theorem 3.4. *If f, Af, $\partial f/\partial t \in L^\infty((0,T)\times\Omega)$, there a positive constant C depending only on f and T such that*

$$\|c(n\tau) - c_i(n\tau)\| \leqslant C\tau \quad \textit{for } i = 1,2 \textit{ and } n\tau \leqslant T. \tag{3.9}$$

4. Semi-discretization in time

In this part, we study the semi-discretization in time of the two previous approximations of c obtained by using a backward Euler method.

Let us denote by Δt the time increment; the approximation of $E(t_n)$ with $t_n = n\Delta t$, using a backward Euler method will be $(I + \Delta tA)^{-n}$.

We note $r(z) = (1 + z)^{-1}$ and we suppose that $\tau = k\Delta t$, $k \in \mathbb{N}$. The approximations c_i^{nk} of $c_i(n\tau)$, $1 \leqslant i \leqslant 2$, at the time level $t_{nk} = nk\Delta t = n\tau$ are defined by

$$\begin{aligned} &c_1^0 = c_0,\\ &c_1^{(n+1)k} = M(r^k(\Delta tA)c_1^{nk} - \tau f((n+1)\tau)), \end{aligned} \tag{4.1}$$

$$\begin{aligned} &c_2^0 = c_0,\\ &c_2^{(n+1)k} = r^k(\Delta tA)M(c_2^{nk} - \tau f((n+1)\tau)). \end{aligned} \tag{4.2}$$

We estimate the errors between c_i^{nk} and $c_i(n\tau)$, $1 \leqslant i \leqslant 2$, in $L^\infty(\Omega)$, then by using theorem (3.4), we obtain the error estimates between c_i^{nk} and $c(nk\Delta t)$. In order to obtain error estimates, we shall use the following convergence result for holomorphic semi-groups [4]: There exists a positive constant C such that

$$\|E(n\Delta t) - r^n(\Delta tA)\|_{\mathscr{L}(L^\infty(\Omega),L^\infty(\Omega))} \leqslant \frac{C}{n} \tag{4.3}$$

and the estimate for the backward Euler method [10]:

$$\|r^n(\Delta tA)\|_{\mathscr{L}(L^\infty(\Omega),L^\infty(\Omega))} \leqslant 1. \tag{4.4}$$

Let us introduce the following notations: for $u \in L^\infty(\Omega)$, we define

$$\begin{aligned} &F_1^n(u) = M(E(\tau)u - \tau f(n\tau)),\\ &F_{1\Delta t}^n(u) = M(r^k(\Delta tA)u - \tau f(n\tau)), \end{aligned} \tag{4.5}$$

$$\begin{aligned} &F_2^n(u) = E(\tau)M(u - \tau f(n\tau)),\\ &F_{2\Delta t}^n(u) = r^k(\Delta tA)M(u - \tau f(n\tau)) \end{aligned} \tag{4.6}$$

and the expressions of $c_i(n\tau)$ and c_i^{nk} $(i = 1, 2)$ may be written:

$$c_i(n\tau) = F_i^n \ldots F_i^1(c_0), \tag{4.7}$$

$$c_i^{nk} = F_{i\Delta t}^n \ldots F_{i\Delta t}^1(c_0). \tag{4.8}$$

Lemma 4.1. *For $u, v \in L^\infty(\Omega)$, $i = 1, 2$, we have the estimate*

$$\|F_i^n(u) - F_i^n(v)\| \leqslant \|u - v\|. \tag{4.9}$$

Proof. We prove first the estimate for $i = 1$:

$$F_1^n(u) - F_1^n(v) = M(E(\tau)u - \tau f(n\tau)) - M(E(\tau)v - \tau f(n\tau))$$

hence, we get

$$\|F_1^n(u) - F_1^n(v)\| \leqslant \|E(\tau)(u - v)\| \leqslant \|u - v\|.$$

For $i = 2$, we have

$$F_2^n(u) - F_2^n(v) = E(\tau)M(u - \tau f(n\tau)) - E(\tau)M(v - \tau f(n\tau)),$$

hence we get

$$\|F_2^n(u) - F_2^n(v)\| \leqslant \|M(u - \tau f(n\tau)) - M(v - \tau f(n\tau))\| \leqslant \|u - v\|. \qquad \square$$

Lemma 4.2. *For $u \in L^\infty(\Omega)$, $n \geqslant 0$, $i = 1, 2$, we have*

$$\|F_{i\Delta t}^n(u)\| \leqslant \|u\|. \tag{4.10}$$

Proof. For $i = 1$, since f is a positive function, we obtain $\|F_{1\Delta t}^n(u)\| \leqslant \|r^k(\Delta t A)u\| \leqslant \|u\|$. For $i = 2$, we get $\|F_{2\Delta t}^n(u))\| \leqslant \|M(u - \tau f(n\tau))\| \leqslant \|u\|$. $\square$

Lemma 4.3. *For $u \in L^\infty(\Omega)$, $i = 1, 2$, we have*

$$\|F_{i\Delta t}^n(u) - F_i^n(u)\| \leqslant \frac{C}{k}\|u\|. \tag{4.11}$$

Proof. For $i = 1$, we have

$$F_{1\Delta t}^n(u) - F_1^n(u) = M(r^k(\Delta t A)u - \tau f(n\tau)) - M(E(\tau)u - \tau f(n\tau)),$$

hence by using (4.3), we get $\|F_{1\Delta t}^n(u) - F_1^n(u)\| \leqslant \|(r^k(\Delta t A) - E(\tau))u\| \leqslant (C/k)\|u\|$.
For $i = 2$, we obtain

$$\|F_{2\Delta t}^n(u) - F_2^n(u)\| \leqslant \frac{C}{k}\|M(u - \tau f(n\tau))\| \leqslant \frac{C}{k}\|u\|. \qquad \square$$

By using these three lemmas, we can estimate now the errors between c_i^{nk} and $c_i(n\tau)$ $(i = 1, 2)$.

Theorem 4.4. *For $n \geqslant 0$, we have the estimate*

$$\|c_i^{nk} - c_i(n\tau)\| \leqslant C\frac{n}{k}\ \|c_0\|, \quad i = 1, 2. \tag{4.12}$$

Proof. From (4.7) and (4.8), we immediately obtain the equality

$$\begin{aligned} c_i^{nk} - c_i(n\tau) = {} & \sum_{j=2}^{n-1} (F_i^n \dots F_i^{j+1} F_{i\Delta t}^j \dots F_{i\Delta t}^1(c_0) - F_i^n \dots F_i^j F_{i\Delta t}^{j-1} .. F_{i\Delta t}^1(c_0)) \\ & + (F_i^n \dots F_i^2 F_{i\Delta t}^1(c_0) - F_i^n \dots F_i^2 F_i^1(c_0)) + (F_{i\Delta t}^n \dots F_{i\Delta t}^1(c_0) - F_i^n F_{i\Delta t \dots}^{n-1} F_{i\Delta t}^1(c_0)). \end{aligned}$$

By using lemma (4.1), we get

$$\begin{aligned} \|c_i^{nk} - c_i(n\tau)\| \leqslant {} & \sum_{j=2}^{n} \|F_{i\Delta t}^j \dots F_{i\Delta t}^1(c_0) - F_i^j F_{i\Delta t}^{j-1} \dots F_{i\Delta t}^1(c_0)\| \\ & + \|F_{i\Delta t}^1(c_0) - F_i^1(c_0)\|. \quad \square \end{aligned}$$

Now, by using (4.10) and (4.11), we obtain

$$\|c_i^{nk} - c_i(n\tau)\| \leqslant \frac{C}{k} \sum_{j=2}^{n} \|F_{i\Delta t}^{j-1} \dots F_{i\Delta t}^1(c_0)\| + \frac{C}{k}\|c_0\| \leqslant C\frac{n}{k}\|c_0\|.$$

By using this result together with theorem (3.4), we easily obtain the estimates between c_i^{nk} and $c(n\tau)$ for $i = 1, 2$.

Theorem 4.5. *If $f, Af, (\partial f/\partial t) \in L^\infty((0,T) \times \Omega)$, there is a positive constant C depending on c_0, f, T such that*

$$\|c_i^{nk} - c(n\tau)\| \leqslant C\left(\tau + \frac{n}{k}\right) \quad \textit{for } n\tau \leqslant T. \tag{4.13}$$

It is now possible to choose τ (or k) in such a manner that this error is minimal. Since $n\tau = nk\Delta t \leqslant T$, estimate (4.13) may be written

$$\|c_i^{nk} - c(n\tau)\| \leqslant C_1\left(k\Delta t + \frac{1}{k^2\Delta t}\right).$$

Then this quantity is minimal for $k = \mathrm{O}(\Delta t^{-2/3})$ and we get an error of order $\mathrm{O}(\Delta t^{1/3})$:

$$\|c_i^{nk} - c(n\tau)\| \leqslant C\Delta t^{1/3}, \quad i = 1, 2, \; n\tau \leqslant T. \tag{4.14}$$

Remark 4.6. The operator A is a maximal sectorial operator: There is some constant θ_0 $(0 \leqslant \theta_0 \leqslant \pi/2)$ depending on $\|V\|$ such that $\forall u \in D(A), (Au, u) \in S_{\theta_0}, (S_{\theta_0} = \{z \in \mathbb{C}/|\arg(z)| < \theta_0\})$. So we may use a strongly $A(\theta) -$ stable method $(\theta > \theta_0)$ to discretize $E(t)$. In this case, if $r(z)$ is a rational approximation of e^{-z}, we have not estimate (4.4), but $\|r^n(\Delta tA)\|_{\mathscr{L}(L^2(\Omega),L^2(\Omega))} \leqslant 1$ and if the method is of order p, we have the estimate $\|E(n\Delta t) - r^n(\Delta tA)\|_{\mathscr{L}(L^2(\Omega),L^2(\Omega))} \leqslant C/n^p$ [5]. Then by choosing $k = \mathrm{O}(\Delta t^{-2/(p+2)})$, we obtain the error estimate in $L^2(\Omega)$: $\|c_i^{nk} - c(n\tau)\|_{\mathscr{L}^2(\Omega)} \leqslant C\Delta t^{p/(p+2)}$.

In practice, the truncation is done at each time step; we shall prove now that we keep the same error estimate in that case, if we use a backward Euler approximation of $E(t)$. This will be done by using the fact that the operator $r(\Delta tA) = (I + \Delta tA)^{-1}$ satisfies a positiveness property

$$u \geqslant 0 \;\Rightarrow\; r(\Delta tA)u \geqslant 0 \tag{4.15}$$

(this is not the case for the other classical methods) and by comparing the solutions obtained by the two truncation methods.

Let us introduce some notations: if $\tau = k\Delta t$ and $u \in L^\infty(\Omega)$, we define

$$F_{1\Delta t}(t;\tau)u = M(r^k(\Delta tA)u - \tau f(t)), \tag{4.16}$$

$$F_{2\Delta t}(t;\tau)u = r^k(\Delta tA)M(u - \tau f(t)). \tag{4.17}$$

We shall compare the approximations of c obtained by using the truncation at each time step or at each k step only.

Lemma 4.7. *If $u \geqslant 0$, $u \in L^\infty(\Omega)$, $\tau_1 = k_1\Delta t$, $\tau_2 = k_2\Delta t$, $Af \in L^\infty((0,T)\times\Omega)$, $(\partial f/\partial t) \in L^\infty((0,T)\times\Omega)$, we have the inequalities*

$$F_{1\Delta t}(\tau_1+\tau_2;\tau_1+\tau_2)u \leqslant F_{1\Delta t}(\tau_1+\tau_2;\tau_1)F_{1\Delta t}(\tau_2;\tau_2)u + C_1\tau_1\tau_2, \tag{4.18}$$

$$F_{2\Delta t}(\tau_1+\tau_2;\tau_1)F_{2\Delta t}(\tau_2;\tau_2)u \leqslant F_{2\Delta t}(\tau_1+\tau_2;\tau_1+\tau_2)u + C_1\tau_1\tau_2, \tag{4.19}$$

where C_1 is a positive constant depending only on f.

Proof. From (4.16), we have the inequality

$$F_{1\Delta t}(\tau_1+\tau_2;\tau_1+\tau_2)u \leqslant M(r^{k_1}(\Delta tA)(r^{k_2}(\Delta tA)u-\tau_2 f(\tau_2))-\tau_1 f(\tau_1+\tau_2)) + \tau_2 M(r^{k_1}(\Delta tA)f(\tau_2)-f(\tau_1+\tau_2))$$

and by using the positiveness property of the backward Euler method, we obtain

$$F_{1\Delta t}(\tau_1+\tau_2;\tau_1+\tau_2)u \leqslant F_{1\Delta t}(\tau_1+\tau_2;\tau_1)F_{1\Delta t}(\tau_2;\tau_2)u + \tau_2\|(r^{k_1}(\Delta tA)-I)f(\tau_2)\| + \tau_2\|f(\tau_2)-f(\tau_1+\tau_2)\|.$$

Besides, we have the equality

$$r^{k_1}(\Delta tA)-I=\sum_{l=0}^{k_1-1} r^l(\Delta tA)(r(\Delta tA)-I) = -\Delta t\sum_{l=0}^{k_1-1} r^{l+1}(\Delta tA)A.$$

Hence, we deduce

$$\|(r^{k_1}(\Delta tA)-I)f(\tau_2)\| \leqslant k_1\Delta t\|Af(\tau_2)\|, \tag{4.20}$$

it follows that

$$F_{1\Delta t}(\tau_1+\tau_2;\tau_1+\tau_2)u \leqslant F_{1\Delta t}(\tau_1+\tau_2;\tau_1)F_{1\Delta t}(\tau_2;\tau_2)u + \tau_1\tau_2\|Af(\tau_2)\| + \tau_1\tau_2\left\|\frac{\partial f}{\partial t}\right\|_{L^\infty((0,T)\times\Omega)}.$$

For the second method, we have the equality

$$F_{2\Delta t}(\tau_1+\tau_2;\tau_1)F_{2\Delta t}(\tau_2;\tau_2)u = r^{k_1}(\Delta tA)M(r^{k_2}(\Delta tA)M(u-\tau_2 f(\tau_2))-\tau_1 f(\tau_1+\tau_2)).$$

Since

$$M(u-\tau_2 f(\tau_2)) \leqslant M(u-(\tau_1+\tau_2)f(\tau_1+\tau_2)) + M((\tau_1+\tau_2)f(\tau_1+\tau_2)-\tau_2 f(\tau_2)),$$

we easily get by using (4.15)

$$\begin{aligned} &F_{2\Delta t}(\tau_1+\tau_2;\tau_1)F_{2\Delta t}(\tau_2,\tau_2)u \\ &\quad\leqslant r^{k_1+k_2}(\Delta tA)M(u-(\tau_1+\tau_2)f(\tau_1+\tau_2)) \\ &\qquad + r^{k_1}(\Delta tA)M(r^{k_2}(\Delta tA)M((\tau_1+\tau_2)f(\tau_1+\tau_2)-\tau_2 f(\tau_2))-\tau_1 f(\tau_1+\tau_2)).\end{aligned}$$

It follows that

$$F_{2\Delta t}(\tau_1+\tau_2;\tau_1)F_{2\Delta t}(\tau_2,\tau_2)u \leqslant F_{2\Delta t}(\tau_1+\tau_2;\tau_1+\tau_2)u + \|r^{k_2}(\Delta tA)M((\tau_1+\tau_2)f(\tau_1+\tau_2)-\tau_2 f(\tau_2))-\tau_1 f(\tau_1+\tau_2)\|.$$

Then by using (4.4) and (4.20), we obtain that the second part of this inequality is bounded by $\tau_2\|f(\tau_1+\tau_2)-f(\tau_2)\|+\tau_1\tau_2\|Af(\tau_1+\tau_2)\|$.

Estimate (4.19) follows immediately. □

Lemma 4.8. *Under the hypothesis of lemma* (4.7), *for* $\tau=k\Delta t$, *we have the inequality*

$$F_{1\Delta t}(t;\tau)u \leqslant F_{2\Delta t}(t;\tau)u+\tau^2\|Af(t)\|. \tag{4.21}$$

Proof. From (4.16), by using the positiveness of the operator $r(\Delta tA)$ and the inequality $u \leqslant M(u - \tau f(t)) + \tau f(t)$, we get

$$F_{1\Delta t}(t;\tau)u \leqslant M(r^k(\Delta tA)M(u-\tau f(t)) + \tau(r^k(\Delta tA) - I)f(t)),$$

then, by using (4.21), it follows that

$$F_{1\Delta t}(t;\tau)u \leqslant F_{2\Delta t}(t;\tau)u + \tau^2\|Af(t)\|. \qquad \square$$

From these lemmas, we deduce the following theorem:

Theorem 4.9. *If Af and $\partial f/\partial t \in L^\infty((0,T)\times\Omega)$, there a positive constant K depending only on f such that, if $\tau = k\Delta t$:*

$$F_{1\Delta t}(\tau;\tau)c_0 \leqslant F_{1\Delta t}(\tau;\Delta t)F_{1\Delta t}((k-1)\Delta t;\Delta t)\ldots F_{1\Delta t}(\Delta t;\Delta t)c_0 + K\tau^2. \tag{4.22}$$

$$\begin{aligned}&F_{1\Delta t}(\tau;\Delta t)F_{1\Delta t}((k-1)\Delta t;\Delta t)\ldots F_{1\Delta t}(\Delta t;\Delta t)c_0\\ &\quad\leqslant F_{2\Delta t}(\tau;\Delta t)\ldots F_{2\Delta t}(\Delta t;\Delta t)c_0 + K\Delta t\tau.\end{aligned} \tag{4.23}$$

$$F_{2\Delta t}(\tau;\Delta t)\ldots F_{2\Delta t}(\Delta t;\Delta t)c_0 \leqslant F_{2\Delta t}(\tau;\tau)c_0 + K\tau^2. \tag{4.24}$$

Proof. We prove first inequality (4.22) by using (4.18)

$$F_{1\Delta t}(\tau;\tau)c_0 \leqslant F_{1\Delta t}(\tau;\Delta t)F_{1\Delta t}((k-1)\Delta t;(k-1)\Delta t)c_0 + C_1(k-1)\Delta t^2$$

whence, by repeated application,

$$F_{1\Delta t}(\tau;\tau)c_0 \leqslant F_{1\Delta t}(\tau;\Delta t)F_{1\Delta t}((k-1)\Delta t;\Delta t)\ldots F_{1\Delta t}(\Delta t;\Delta t)c_0 + C_1\sum_{j=1}^{k-1} j\Delta t^2,$$

that is

$$F_{1\Delta t}(\tau;\tau)c_0 \leqslant F_{1\Delta t}(\tau;\Delta t)\ldots F_{1\Delta t}(\Delta t;\Delta t)c_0 + \frac{C_1}{2}(k-1)k\Delta t^2.$$

We deduce (4.22) with $K = C_1/2$.

The proof of (4.24) is analogous by using (4.19) and (4.23) is obtained from (4.21). $\square$

By using these inequalities, we may now compare the solutions obtained by a truncation at each time step or every k steps.

We denote $\tilde{c}_i^j$, (i = 1, 2) the solutions obtained at the time level $t_j = j\Delta t$ by the two different methods, using the truncation at each time step, that is

$$\tilde{c}_1^0 = c_0,$$
$$\tilde{c}_1^{j+1} = M(r(\Delta tA)\tilde{c}_1^j - \Delta t f((j+1)\Delta t)),$$

$$\tilde{c}_2^0 = c_0,$$
$$\tilde{c}_2^{j+1} = r(\Delta tA)M(\tilde{c}_2^j - \Delta t f((j+1)\Delta t)).$$

Theorem 4.10. *If f, Af and $(\partial f/\partial t) \in L^\infty((0,T)\times\Omega)$, if $Ac_0 \in L^\infty(\Omega)$, we have the estimate*:

$$\|\tilde{c}_i^j - c(j\Delta t)\| \leqslant C\Delta t^{1/3}, \quad i=1,2, \tag{4.25}$$

where C is a constant depending only on f, c_0, T.

Proof. From inequalities (4.22)–(4.24), we get $c_1^k \leqslant \tilde{c}_1^k + Kk^2\Delta t^2$, $\tilde{c}_1^k \leqslant \tilde{c}_2^k + Kk\Delta t^2, \tilde{c}_2^k \leqslant c_2^k + Kk^2\Delta t^2$, whence, by repeated application, we obtain at the time level $t = nk\Delta t$, $c_1^{nk} \leqslant \tilde{c}_1^{nk} + Ktk\Delta t, \tilde{c}_1^{nk} \leqslant \tilde{c}_2^{nk} + Kt\Delta t, \tilde{c}_2^{nk} \leqslant c_2^{nk} + Ktk\Delta t$, and we deduce

$$\begin{aligned} c_1^{nk} - c(nk\Delta t) - Ktk\Delta t &\leqslant \tilde{c}_1^{nk} - c(nk\Delta t) \\ &\leqslant \tilde{c}_2^{nk} - c(nk\Delta t) + Kt\Delta t \leqslant c_2^{nk} - c(nk\Delta t) + K(k+1)t\Delta t. \end{aligned}$$

Hence by using (4.13), we get

$$\|\tilde{c}_i^{nk} - c(nk\Delta t)\| \leqslant C\left(k\Delta t + \frac{t}{k\Delta t}\right) + K(k+1)t\Delta t, \quad i=1,2$$

and the optimal result is obtained for $k = \mathrm{O}(\Delta t^{-2/3})$, which gives the estimate

$$\|\tilde{c}_i^{nk} - c(nk\Delta t)\| = \mathrm{O}(\Delta t^{1/3}), \quad i=1,2.$$

Now, if $Ac_0 \in L^\infty(\Omega)$, the following estimates hold:

$$\|\tilde{c}_i^{nk} - \tilde{c}_i^{nk+j}\| \leqslant C\tau, \quad 0 \leqslant j \leqslant k; \quad \|c(nk\Delta t) - c((nk+j)\Delta t)\| \leqslant C\tau, \quad 0 \leqslant j \leqslant k;$$

and estimate (4.25) follows immediately. □

5. Complete discretization

For the discretization of the problem in space, we use a P_1-finite element method.

5.1. Notations

If Ω is a convex bounded domain, we consider a family of regular quasi-uniform triangulations $\mathscr{T}_h$, $(h>0)$ of subdomains Ω_h of Ω: $\Omega_h = \bigcup_{K\in\mathscr{T}_h} K$.

For any $K \in \mathscr{T}_h$, we set $h(K) =$ *diameter of* K, $h(K) \leqslant h$.

Let us denote by V_h the finite dimensional subspace of $H_0^1(\Omega)$ defined by

$$V_h = \{v_h \in C^0(\bar{\Omega}) / \forall K \in \mathscr{T}_h,\ v_{h|K} \in P_1,\ v_{h|\Gamma} = 0\} \tag{5.1}$$

(P_1 is the space of polynoms of degree $\leqslant 1$).

A_h is the operator of $\mathscr{L}(V_h, V_h)$ defined by

$$\forall u_h, v_h \in V_h, \quad (A_h u_h, v_h) = \int_\Omega (\alpha\, \overrightarrow{\mathrm{grad}\, u_h}.\, \overrightarrow{\mathrm{grad}\, v_h} + \boldsymbol{V}\, \overrightarrow{\mathrm{grad}\, u_h}\, v_h)\,\mathrm{d}x \tag{5.2}$$

If $v \in C^0(\bar{\Omega}) \cap H_0^1(\Omega)$, its Lagrange interpolate denoted by $\pi_h v$ is defined by

$$\pi_h v \in V_h \text{ and } \pi_h v(S) = v(S) \text{ for any interior vertex of } \mathcal{T}_h. \tag{5.3}$$

The standard L^2-projection onto V_h is denoted by P_h and defined by

$$\forall v \in L^2(\Omega), \quad P_h v \in V_h \quad \text{and} \quad (P_h v, v_h) = (v, v_h), \quad \forall v_h \in V_h, \tag{5.4}$$

where $(.,.)$ denotes the L^2-inner product.

The elliptic or Ritz projection $H_0^1(\Omega) \to V_h$ is denoted by r_h and defined by $\forall u \in H_0^1(\Omega)$, $r_h u \in V_h$ and

$$(A_h r_h u, v_h) = \int_\Omega (\alpha\, \overrightarrow{\text{grad}\, u_h}.\, \overrightarrow{\text{grad}\, v_h} + \boldsymbol{V}.\, \overrightarrow{\text{grad}\, u_h}\, v_h)\, \mathrm{d}x, \quad \forall v_h \in V_h. \tag{5.5}$$

Besides, we have the equality

$$A_h^{-1} P_h = r_h A^{-1}. \tag{5.6}$$

In order to define the approximations of c, we shall use the discrete operator: $r(\Delta t A_h) = (I + \Delta t A_h)^{-1}$ and we shall need the positiveness of this operator which impose some hypothesis on the triangulation.

Lemma 5.1. *We suppose that all the angles θ_K of the triangulation are acute and satisfy*

$$\theta_K \leqslant \theta_0 < \frac{\pi}{2} \tag{5.7}$$

and

$$h^2 \frac{\sin \theta_0}{12} + 2\Delta t h |\boldsymbol{V}| \leqslant \alpha \Delta t \operatorname{cotan} \theta_0. \tag{5.8}$$

Then, if $u_h \in V_h$ and $u_h \geqslant 0$, we have the inequality

$$r(\Delta t A_h) u_h(x) \geqslant 0, \quad \forall x \in \Omega \tag{5.9}$$

and for $u_h \in V_h$, we have

$$\| r(\Delta t A_h) u_h \| \leqslant \| u_h \|. \tag{5.10}$$

Proof. It is analogous to the proof made in [10]. Let $\{\varphi_i\}_{i=1}^N$ a basis of V_h (N is the number of interior vertices of $\mathcal{T}_h$). We note $v_h = r(\Delta t A_h) u_h$; v_h and u_h admit the representations: $u_h = \sum_{j=1}^N u_j \varphi_j, v_h = \sum_{j=1}^N v_j \varphi_j$ with $u_j = u(a_j), v_j = v(a_j)$ ($a_j (1 \leqslant j \leqslant N)$ are the vertices of $\mathcal{T}_h$).

By using matrix notations, the equality $v_h = r(\Delta t A_h) u_h$ may be written: $(M + \Delta t B)V = MU$ where V and U are the vectors of $\mathbb{R}^N$ of components $(v_j)_{j=1,N,} (u_j)_{j=1,N}$ and M and B the matrices: $M = (m_{ij})_{1 \leqslant i,j \leqslant N}$, $m_{ij} = (\varphi_j, \varphi_i)$, $B = (b_{ij})_{1 \leqslant i,j \leqslant N}$, $b_{ij} = (A_h \varphi_j, \varphi_i)$.

The matrix $M + \Delta t B$ is a positive-definite symmetric matrix; hence, we get $V = (M + \Delta t B)^{-1} MU$ and denoting $C = (M + \Delta t B)^{-1} M$, we may write: $V = CU$.

The coefficients m_{ij} $(1 \leqslant i,\ j \leqslant N)$ are nonnegative; the diagonal coefficients of $M + \Delta t B$ are positive; then if the nondiagonal coefficients of $M + \Delta t B$ satisfy $m_{ij} + \Delta t b_{ij} \leqslant 0$ $(i \neq j)$, the matrix C is positive $(c_{ij} \geqslant 0,\ 1 \leqslant i,\ j \leqslant N)$.

We have the equalities

$$m_{ij} = \sum_{K\in\mathcal{T}_h} \int_K \varphi_i\varphi_j \,\mathrm{d}x \quad \text{and} \quad \int_K \varphi_i\varphi_j \,\mathrm{d}x = \frac{1}{12}\mathrm{mes}(K) \quad \text{if } a_i, a_j \in K,$$

$$b_{ij} = \sum_{K\in\mathcal{T}_h} \int_K (\alpha\, \overrightarrow{\mathrm{grad}\,\varphi_i}\, \overrightarrow{\mathrm{grad}\,\varphi_j} + \boldsymbol{V}\, \overrightarrow{\mathrm{grad}\,\varphi_j}\, \varphi_i)\,\mathrm{d}x$$

and

$$\int_K \overrightarrow{\mathrm{grad}\,\varphi_j}\, \overrightarrow{\mathrm{grad}\,\varphi_i}\, \mathrm{d}x = -\cos\theta_{ijK}\frac{\delta_{iK}\delta_{jK}}{4\ \mathrm{mes}(K)},$$

where θ_{ijK} is the angle of K at the vertex opposite to a_ia_j; δ_{iK} is the length of the side of K opposite to a_i.

Besides, we have the estimate: $|\int_K \boldsymbol{V}\,\overrightarrow{\mathrm{grad}\,\varphi_j}\,\varphi_i\,\mathrm{d}x| \leqslant |\boldsymbol{V}|\delta_{jK}$.

Then, the contribution of the triangle K to the coefficient $m_{ij} + \Delta t b_{ij}$ is given by

$$\frac{1}{12}\mathrm{mes}(K) - \alpha\Delta t\cos\theta_{ijK}\frac{\delta_{iK}\delta_{jK}}{4\,\mathrm{mes}(K)} + \Delta t h|\boldsymbol{V}|$$

and since $\mathrm{mes}(K) = \frac{1}{2}\delta_{iK}\delta_{jK}\sin\theta_{ijK,}$ this quantity is bounded by

$$\frac{h^2}{24}\sin\theta_0 - \alpha\frac{\Delta t}{2}\cot\mathrm{an}\,\theta_0 + \Delta t h|\boldsymbol{V}|.$$

A sufficient condition to obtain $m_{ij} + \Delta t b_{ij} \leqslant 0$ is then

$$\frac{h^2}{12}\sin\theta_0 + 2h\Delta t|\boldsymbol{V}| \leqslant \alpha\Delta t\,\mathrm{cotan}\,\theta_0.$$

(This condition will be realized if $h^2 \leqslant C\Delta t$, C depending on θ_0 and $h \leqslant h_0$, h_0 depending on $|\boldsymbol{V}|$, α and θ_0.)

The second part of the proof of the theorem is analogous to the proof in [10]; it is easy to obtain $\sum_{j=1}^{N} c_{ij} \leqslant 1,\ i = 1,\ldots,N.$ □

In order to obtain the error estimates, we use the following convergence result for holomorphic semi-groups (cf. [8]): There exists $C > 0$ and $a > 0$ such that

$$\|E(t_n) - r^n(\Delta t A_h)P_h\|_{\mathcal{L}(L^\infty(\Omega),L^\infty(\Omega))} \leqslant C\frac{|\ln h|}{n} \tag{5.11}$$

provided that Δt satisfies

$$\Delta t \geqslant ah^2|\ln h|^3 \quad \text{and} \quad h \leqslant \mathrm{e}^{-1}. \tag{5.12}$$

We prove now an estimate which will be useful in the proof of several lemmas.

Lemma 5.13. *There exists a positive constant C such that if $Ag \in L^\infty(\Omega)$, the following estimate holds*:

$$\|r^k(\Delta t A_h)\pi_h g - g\| \leqslant C(h^2|\ln h| + k\Delta t)\|g\|_{2,\infty,\Omega}. \tag{5.13}$$

Proof. We have

$$\|r^k(\Delta t A_h)\pi_h g - g\| \leqslant \|r^k(\Delta t A_h)(\pi_h - r_h)g\| + \|(r^k(\Delta t A_h) - I)r_h g\| + \|(r_h - I)g\|.$$

It has been proved [3] that $\|(\pi_h - r_h)g\| \leqslant Ch^2|\ln h|\|g\|_{2,\infty,\Omega}$, $\forall g \in W^{2,\infty}(\Omega)$, hence from (5.10), we get $\|r^k(\Delta t A_h)(\pi_h - r_h)g\| \leqslant Ch^2|\ln h|\|g\|_{2,\infty,\Omega}$. In the same manner as for (4.20), we get $\|(r^k(\Delta t A_h) - I)r_h g\| \leqslant k\Delta t\|A_h r_h g\|$ and from (5.6), we obtain $\|(r^k(\Delta t A_h) - I)r_h g\| \leqslant k\Delta t\|P_h A g\| \leqslant Ck\Delta t\|g\|_{2,\infty,\Omega}$, since $\|P_h u\| \leqslant C\|u\|$.

Since, the following estimate holds [3,7] $\|(r_h - I)g\| \leqslant Ch^2|\ln h|\|g\|_{2,\infty,\Omega}$, we deduce

$$\|r^k(\Delta t A_h)\pi_h g - g\| \leqslant C(h^2|\ln h| + k\Delta t)\|g\|_{2,\infty,\Omega}. \qquad \square$$

5.2. Definition of the numerical approximation

We define two approximations of c in an analogous manner to (4.1), (4.2): the approximations $c_{ih}^{nk} \in V_h$, $(i = 1, 2)$ of c at the time level $nk\Delta t$ are defined by

$$\begin{aligned} c_{1h}^0 &= \pi_h c_0, \\ c_{1h}^{(n+1)k} &= \pi_h M(r^k(\Delta t A_h)c_{1h}^{nk} - \tau f((n+1)\tau)), \end{aligned} \tag{5.14}$$

$$\begin{aligned} c_{2h}^0 &= \pi_h c_0, \\ c_{2h}^{(n+1)k} &= r^k(\Delta t A_h)\pi_h M(c_{2h}^{nk} - \tau f((n+1)\tau)). \end{aligned} \tag{5.15}$$

It has been necessary to introduce the interpolate operator π_h since the positive part of a function of V_h is not in V_h.

In order to estimate the errors between c_{ih}^{nk} and $c(nk\Delta t)$, we use an analogous method as for the discretization in time.

Let us denote

$$\forall u_h \in V_h, \quad F_{1\Delta th}^n(u_h) = \pi_h M(r^k(\Delta t A_h)u_h - \tau f(n\tau)), \tag{5.16}$$

$$\forall u \in L^\infty(\Omega), \quad F_{2\Delta th}^n(u) = r^k(\Delta t A_h)\pi_h M(u - \tau f(n\tau)). \tag{5.17}$$

Lemma 5.3. *For $u_h \in V_h$, we have*

$$\|F_{1\Delta th}^n(u_h)\| \leqslant \|u_h\|. \tag{5.18}$$

For $u \in L^\infty(\Omega)$,

$$\|F_{2\Delta th}^n(u)\| \leqslant \|u\|. \tag{5.19}$$

Proof. From (5.16), we get immediately $\|F_{1\Delta th}^n(u_h)\| \leqslant \|M(r^k(\Delta t A_h)u_h - \tau f(n\tau))\|$; and since f is positive, we get $\|F_{1\Delta th}^n(u_h)\| \leqslant \|r^k(\Delta t A_h)u_h\| \leqslant \|u_h\|$.

In the same manner, we obtain $\|F_{2\Delta th}^n(u)\| \leqslant \|\pi_h M(u - \tau f(n\tau))\| \leqslant \|u\|$. $\square$

Lemma 5.4. *There exists $C > 0$ and $a > 0$ such that, for $u_h \in V_h$, $f \in L^\infty(0,T;W^{1,\infty}(\Omega))$, we have the estimate*

$$\|F^n_{1\Delta th}(u_h) - F^n_1(u_h)\| \leqslant C\left(\frac{|\ln h|}{k} + \frac{h}{\sqrt{k\Delta t}}\right)\|u_h\| + Chk\Delta t\|f(n\tau)\|_{1,\infty,\Omega} \tag{5.20}$$

if Δt satisfies (5.12).

Proof. From (4.5) and (5.18), we get immediately

$$\begin{aligned}\|F^n_{1\Delta th}(u_h) - F^n_1(u_h)\| \leqslant\ & \|\pi_h M(r^k(\Delta t A_h)u_h - \tau f(n\tau)) - \pi_h M(E(\tau)u_h - \tau f(n\tau))\| \\ & + \|(\pi_h - I)M(E(\tau)u_h - \tau f(n\tau))\|.\end{aligned}$$

Besides,

$$\|\pi_h(M(r^k(\Delta t A_h)u_h - \tau f(n\tau)) - M(E(\tau)u_h - \tau f(n\tau)))\| \leqslant \|(r^k(\Delta t A_h) - E(\tau))u_h\|$$

and from (5.11), there exists $C > 0$ and $a > 0$ such that

$$\|(r^k(\Delta t A_h) - E(\tau))u_h\| \leqslant C\left(\frac{|\ln h|}{k}\right)\|u_h\|.$$

Further, we have [2]

$$\begin{aligned}\|(\pi_h - I)M(E(\tau)u_h - \tau f(n\tau))\| &\leqslant Ch\|E(\tau)u_h - \tau f(n\tau)\|_{1,\infty,\Omega} \\ &\leqslant Ch\left(\frac{\|u_h\|}{\sqrt{\tau}} + \tau\|f(n\tau)\|_{1,\infty,\Omega}\right).\end{aligned}$$

We deduce

$$\|F^n_{1\Delta th}(u_h) - F^n_1(u_h)\| \leqslant C\left(\frac{|\ln h|}{k} + \frac{h}{\sqrt{k\Delta t}}\right)\|u_h\| + Ch\tau\|f(n\tau)\|_{1,\infty,\Omega}. \qquad \square$$

Lemma 5.5. *There exists $C > 0$ and $a > 0$ such that for $u \in W^{1,\infty}(\Omega)$, $f \in L^\infty(0,T;W^{1,\infty}(\Omega))$, we have the estimate*

$$\|F^n_{2\Delta th}(u) - F^n_2(u)\| \leqslant C\left(\frac{|\ln h|}{k}\right)\|u\| + Ch\|u\|_{1,\infty,\Omega} + Chk\Delta t\|f(n\tau)\|_{1,\infty,\Omega} \tag{5.21}$$

if Δt satisfies (5.12).

Proof. From (5.17) and (4.6), we get

$$\|F^n_{2\Delta th}(u) - F^n_2(u)\| \leqslant \|(r^k(\Delta t A_h) - E(\tau))\pi_h M(u - \tau f(n\tau))\| + \|E(\tau)(\pi_h - I)M(u - \tau f(n\tau))\|.$$

If Δt satisfies (5.12), we have from (5.11):

$$\|(r^k(\Delta t A_h) - E(\tau))\pi_h M(u - \tau f(n\tau))\| \leqslant C\left(\frac{|\ln h|}{k}\right)\|u\|$$

and

$$\|E(\tau)(\pi_h - I)M(u - \tau f(n\tau))\| \leqslant \|(\pi_h - I)M(u - \tau f(n\tau))\| \leqslant Ch\|u - \tau f(n\tau)\|_{1,\infty,\Omega},$$

hence we get

$$\|F^n_{2\Delta th}(u) - F^n_2(u)\| \leqslant C\left(\frac{|\ln h|}{k}\right)\|u\| + Ch\|u\|_{1,\infty,\Omega} + Chk\Delta t\|f(n\tau)\|_{1,\infty,\Omega}. \qquad \square$$

From these two lemmas, we obtain the following theorem.

Theorem 5.6. *There exists $C>0$ and $a>0$ such that, for $c_0 \in W^{1,\infty}(\Omega)$, $f \in L^\infty(0,T;W^{1,\infty}(\Omega))$, the estimate*

$$\|c^{nk}_{ih} - c_i(n\tau)\| \leqslant C\left(\frac{h}{(k\Delta t)^{3/2}} + \frac{|\ln h|}{k^2\Delta t}\right)\|c_0\| + Ch\|c_0\|_{1,\infty,\Omega} + Ch\|f\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}, \quad i=1,2 \tag{5.22}$$

holds, provided that the time step satisfies (5.12) *and $nk\Delta t \leqslant T$.*

Proof. We have the equality

$$\begin{aligned} c^{nk}_{1h} - c_1(n\tau) = \sum_{j=2}^{n-1} & (F^n_1 \dots F^{j+1}_1 F^j_{1\Delta th} \dots F^1_{1\Delta th}(c^0_{1h}) - F^n_1 \dots F^j_1 F^{j-1}_{1\Delta th} \dots F^1_{1\Delta th}(c^0_{1h})) \\ & + F^n_{1\Delta th} \dots F^1_{1\Delta th}(c^0_{1h}) - F^n_1 F^{n-1}_{2\Delta th} \dots F^1_{1\Delta th}(c^0_{1h}) \\ & + F^n_1 \dots F^2_1 F^1_{1\Delta th}(c^0_{1h}) - F^n_1 \dots F^1_1(c^0_{1h}) \\ & + F^n_1 \dots F^1_1(c^0_{1h}) - F^n_1 \dots F^1_1(c_0). \end{aligned}$$

By using (4.9), we obtain

$$\begin{aligned} \|c^{nk}_{1h} - c_1(n\tau)\| \leqslant \sum_{j=2}^{n} & \|F^j_{1\Delta th} \dots F^1_{1\Delta th}(c^0_{1h}) - F^j_1 F^{j-1}_{1\Delta th} \dots F_{1\Delta th}(c^0_{1h})\| \\ & + \|F^1_{1\Delta th}(c^0_{1h}) - F^1_1(c^0_{1h})\| + \|c_0 - \pi_h c_0\| \end{aligned}$$

and from (5.20) and (5.18), we get

$$\|c^{nk}_{1h} - c(n\tau)\| \leqslant C\left(\frac{n|\ln h|}{k} + \frac{nh}{(k\Delta t)^{1/2}}\right)\|c_0\| + Cnk\Delta th\|f\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + Ch\|c_0\|_{1,\infty,\Omega}.$$

Since, $nk\Delta t \leqslant T$, we deduce

$$\|c^{nk}_{1h} - c_1(n\tau)\| \leqslant C\left(\frac{|\ln h|}{k^2\Delta t} + \frac{h}{(k\Delta t)^{3/2}}\right)\|c_0\| + Ch\|f\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + Ch\|c_0\|_{1,\infty,\Omega}.$$

For the case $i=2$, we use the equality

$$\begin{aligned} c^{nk}_{2h} - c_2(n\tau) = \sum_{j=2}^{n-1} & (F^n_{2\Delta th} \dots F^{j+1}_{2\Delta th} F^j_2 \dots F^1_2(c_0) - F^n_{2\Delta th} \dots F^j_{2\Delta th} F^{j-1}_2 \dots F^1_2(c_0)) \\ & + F^n_2 \dots F^1_2(c_0) - F^n_{2\Delta th} F^{n-1}_2 \dots F^1_2(c_0) \\ & + F^n_{2\Delta th} \dots F^2_{2\Delta th} F^1_2(c_0) - F^n_{2\Delta th} \dots F^1_{2\Delta th}(c_0) \\ & + F^n_{2\Delta th} \dots F^1_{2\Delta th}(c^0_{2h}) - F^n_{2\Delta th} \dots F^1_{2\Delta th}(c_0). \end{aligned}$$

Besides for $u, v \in L^\infty(\Omega)$, we have

$$\|F^n_{2\Delta th}(u) - F^n_{2\Delta th}(v)\| \leqslant \|\pi_h M(u - \tau f(n\tau)) - \pi_h M(v - \tau f(n\tau))\| \leqslant \|u - v\|,$$

hence we get

$$\|c^{nk}_{2h} - c_2(n\tau)\| \leqslant \sum_{j=2}^{n} \|F^j_2 \dots F^1_2(c_0) - F^j_{2\Delta th} F^{j-1}_2 \dots F^1_2(c_0)\| + \|F^n_2 \dots F^1_2(c_0) - F^n_{2\Delta th} F^{n-1}_2 \dots F^1_2(c_0)\| + \|c_0 - c^0_{2h}\|.$$

Then by using (5.21), we obtain

$$\|c^{nk}_{2h} - c_2(n\tau)\| \leqslant C \sum_{j=2}^{n} \left(\frac{|\ln h|}{k}\right) \|F^{j-1}_2 \dots F^1_2(c_0)\| + Ch \sum_{j=2}^{n} \|F^{j-1}_2 \dots F^1_2(c_0)\|_{1,\infty,\Omega} + Cnhk\Delta t \|f\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + C\frac{|\ln h|}{k}\|c_0\| + Ch\|c_0\|_{1,\infty,\Omega}.$$

Besides, if $u \in L^\infty(\Omega), \|F^j_2(u)\|_{W^{1,\infty}(\Omega)} \leqslant (C/\sqrt{\tau})\|M(u - \tau f(j\tau)\| \leqslant (C/\sqrt{k\Delta t})\|u\|$.

We deduce immediately

$$\|c^{nk}_{2h} - c_2(n\tau)\| \leqslant C\left(\frac{n|\ln h|}{k} + \frac{nh}{\sqrt{k\Delta t}}\right)\|c_0\| + Cnk\Delta th\|f\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + Ch\|c_0\|_{1,\infty,\Omega}.$$

We obtain immediately the error estimates between c^{nk}_{ih} and $c(nk\Delta t)$. □

Theorem 5.7. *For* $i = 1, 2$, *if* $c_0 \in W^{1,\infty}(\Omega)$ *and* $f, Af, (\partial f/\partial t) \in L^\infty(0, T \times \Omega)$, *the following estimate holds*:

$$\|c^{nk}_{ih} - c(n\tau)\| \leqslant C\left(k\Delta t + \frac{|\ln h|}{k^2\Delta t} + \frac{h}{(k\Delta t)^{3/2}}\right) \tag{5.23}$$

if Δt *satisfies* (5.12), $nk\Delta t \leqslant T$.

This estimate proceeds immediately from (5.22), (3.9).

We can choose k in order that this error is minimal. We obtain easily: if there exists $b > 0$ such that $bh^{6/5}|\ln h|^{-1} \leqslant \Delta t$, we can choose $k = \mathrm{O}(\Delta t^{-2/3}|\ln h|^{1/3})$ and we get

$$\|c^{nk}_{ih} - c_i(n\tau)\| \leqslant C\Delta t^{1/3}|\ln h|^{1/3}.$$

If Δt is chosen such that $\Delta t \leqslant bh^{6/5}|\ln h|^{-1}$, we can choose $k = \mathrm{O}(h^{2/5}\Delta t^{-1})$ and we get

$$\|c^{nk}_{ih} - c_i(n\tau)\| \leqslant Ch^{2/5}. \tag{5.24}$$

For the complete discretization, we prove also that we obtain the same error estimate if the truncation is done at each time step.

We introduce the following notations:

$$F_{1\Delta th}(t;\tau)u_h = \pi_h M(r^k(\Delta t A_h)u_h - \tau f(t)), \tag{5.25}$$

$$F_{2\Delta th}(t;\tau)u = r^k(\Delta t A_h)\pi_h M(u - \tau f(t)) \tag{5.26}$$

and we prove lemmas analogous to lemmas (4.7), (4.8).

Lemma 5.8. *If* $\tau_1 = k_1\Delta t,\ \tau_2 = k_2\Delta t$, *if* $Af(\tau_2) \in L^\infty(0, T\times\Omega), (\partial f/\partial t) \in L^\infty(0, T\times\Omega)$, *we have*

$$F_{1\Delta th}(\tau_1+\tau_2;\tau_1+\tau_2)u_h \leqslant F_{1\Delta th}(\tau_1+\tau_2;\tau_2)F_{1\Delta th}(\tau_2;\tau_2)u_h + C_1(\tau_1\tau_2+\tau_2h^2|\ln h|), \tag{5.27}$$

$$F_{2\Delta th}(\tau_1+\tau_2;\tau_1)F_{2\Delta th}(\tau_2;\tau_2)u \leqslant F_{2\Delta th}(\tau_1+\tau_2;\tau_1+\tau_2)u + C_1\tau_1(\tau_2+h^2|\ln h|), \tag{5.28}$$

where C_1 *is a positive constant depending on* f.

Proof. For $i = 1$, we have the equality

$$F_{1\Delta th}(\tau_1+\tau_2;\tau_1+\tau_2)u_h = \pi_h M(r^{k_1+k_2}(\Delta t A_h)u_h - (\tau_1+\tau_2)f(\tau_1+\tau_2)).$$

From the inequality,

$$r^{k_2}(\Delta t A_h)u_h \leqslant \pi_h M(r^{k_2}(\Delta t A_h)u_h - \tau_2 f(\tau_2)) + \tau_2\pi_h f(\tau_2),$$

we get

$$\begin{aligned}F_{1\Delta th}(\tau_1+\tau_2;\tau_1+\tau_2)u_h &\leqslant F_{1\Delta th}(\tau_1+\tau_2;\tau_1)F_{1\Delta th}(\tau_2;\tau_2)u_h\\ &\quad + \tau_2\pi_h M(r^{k_1}(\Delta t A_h)\pi_h f(\tau_2) - f(\tau_1+\tau_2)).\end{aligned}$$

Let us bound the last term of this inequality

$$\begin{aligned}\|\tau_2\pi_h M(r^{k_1}(\Delta t A_h)\pi_h f(\tau_2) - f(\tau_1+\tau_2))\| &\leqslant \tau_2\|r^{k_1}(\Delta t A_h)\pi_h f(\tau_2) - f(\tau_2)\|\\ &\quad + \tau_2\|f(\tau_2) - f(\tau_1+\tau_2)\|\end{aligned}$$

and by using (5.13), we get

$$\begin{aligned}&\tau_2\|\pi_h M(r^{k_1}(\Delta t A_h)\pi_h f(\tau_2) - f(\tau_1+\tau_2))\|\\ &\quad \leqslant C\tau_2(h^2|\ln h| + \tau_1)\|f(\tau_2)\|_{2,\infty,\Omega} + \tau_1\tau_2\left\|\frac{\partial f}{\partial t}\right\|_{L^\infty(0,T\times\Omega)}.\end{aligned}$$

In the second case, from (5.28), we get

$$\begin{aligned}F_{2\Delta th}(\tau_1+\tau_2;\tau_1)F_{2\Delta th}(\tau_2;\tau_2)u &= r^{k_1}(\Delta t A_h)\pi_h M(F_{2\Delta th}(\tau_2;\tau_2)u - \tau_1 f(\tau_1+\tau_2))\\ &= r^{k_1}(\Delta t A_h)\pi_h M[r^{k_2}(\Delta t A_h)\pi_h M(u - \tau_2 f(\tau_2)) - \tau_1 f(\tau_1+\tau_2)].\end{aligned}$$

Moreover, we have the inequality

$$\pi_h M(u - \tau_2 f(\tau_2)) \leqslant \pi_h M(u - (\tau_1+\tau_2)f(\tau_1+\tau_2)) + \pi_h M((\tau_1+\tau_2)f(\tau_1+\tau_2) - \tau_2 f(\tau_2)).$$

We deduce

$$\begin{aligned}&F_{2\Delta th}(\tau_1+\tau_2;\tau_1)F_{2\Delta th}(\tau_2;\tau_2)u\\ &\quad \leqslant r^{k_1+k_2}(\Delta t A_h)\pi_h M(u - (\tau_1+\tau_2)f(\tau_1+\tau_2))\\ &\qquad + \|r^{k_2}(\Delta t A_h)\pi_h M((\tau_1+\tau_2)f(\tau_1+\tau_2) - \tau_2 f(\tau_2) - \tau_1\pi_h f(\tau_1+\tau_2)\|.\end{aligned}$$

It remains to bound the second term of the right member

$$\begin{aligned}&\|r^{k_2}(\Delta t A_h)\pi_h M((\tau_1+\tau_2)f(\tau_1+\tau_2) - \tau_2 f(\tau_2)) - \tau_1\pi_h f(\tau_1+\tau_2)\|\\ &\quad \leqslant \|r^{k_2}(\Delta t A_h)[\pi_h M((\tau_1+\tau_2)f(\tau_1+\tau_2) - \tau_2 f(\tau_2)) - \tau_1\pi_h f(\tau_1+\tau_2)]\|\\ &\qquad + \tau_1\|(r^{k_2}(\Delta t A_h) - I)\pi_h f(\tau_1+\tau_2)\|.\end{aligned}$$

The first term of this inequality is bounded by

$$\tau_2\|f(\tau_1+\tau_2)-f(\tau_2)\|.$$

In the same manner as in the first case, we obtain that the second term is bounded by

$$C(h^2|\ln h|+k\Delta t)\|f\|_{2,\infty,\Omega}.$$

Hence we get

$$F_{2\Delta th}(\tau_1+\tau_2;\tau_1)F_{2\Delta th}(\tau_2;\tau_2)u\leqslant F_{2\Delta th}(\tau_1+\tau_2;\tau_1+\tau_2)u+C(\tau_1\tau_2+\tau_1h^2|\ln h|). \quad \square$$

Lemma 5.9. *If $f\in L^\infty(0,T;W^{2,\infty}(\Omega)),(\partial f/\partial t)\in L^\infty(0,T\times\Omega)$, we have the inequality*

$$F_{1\Delta th}(t;\tau)u_h\leqslant F_{2\Delta th}(t;\tau)u_h+C\tau(\tau+h^2|\ln h|)\|f(t)\|_{2,\infty,\Omega}. \tag{5.29}$$

Proof. From (5.25), we get: $F_{1\Delta th}(t;\tau)u_h=\pi_hM(r^k(\Delta tA_h)u_h-\tau f(t))$ and by using the inequality: $u_h\leqslant\pi_hM(u_h-\tau f(t))+\tau\pi_hf(t)$, we obtain $F_{1\Delta th}(t;\tau)u_h\leqslant F_{2\Delta th}(t;\tau)u_h+\tau\|\pi_hM(r^k(\Delta tA_h)\pi_hf(t)-f(t))\|$. Inequality (5.29) follows immediately from lemma (5.2). □

By using these inequalities, we may compare the solutions obtained by a truncation at each time step or every k steps in the same manner as for the semi-discretization in time.

We denote $\tilde{c}_{ih}^j$ $(i=1,2)$ the solutions obtained at the time level $t_j=j\Delta t$ by the three different methods using the truncation at each time step, that is

$$\tilde{c}_{ih}^0=\pi_hc_0,\quad i=1,2,$$

$$\tilde{c}_{1h}^{j+1}=\pi_hM(r(\Delta tA_h)\tilde{c}_{1h}^j-\Delta tf((j+1)\Delta t)), \tag{5.30}$$

$$\tilde{c}_{2h}^{j+1}=r(\Delta tA_h)\pi_hM(\tilde{c}_{2h}^j-\Delta tf((j+1)\Delta t)) \tag{5.31}$$

and we easily obtained the following theorem analogous to theorem (4.12):

Theorem 5.10. *For $i=1,2$, if $Ac_0\in L^\infty(\Omega)$ and $f,Af,(\partial f/\partial t)\in L^\infty(0,T\times\Omega)$, we obtain the estimate*:

$$\|\tilde{c}_{ih}^j-c_i(j\Delta t)\|\leqslant C(\Delta t^{1/3}|\ln h|^{1/3}+h^{2/5})\quad i=1,2, \tag{5.32}$$

provided the time step satisfies: $ah^2|\ln h|^3\leqslant\Delta t$, *where C is a constant depending on f,c_0,T.*

This estimate proceeds of the optimal choice of k in (5.23). In this inequality, the second term $|\ln h|/k^2\Delta t$ is the error due to the approximation of the operator $E(k\Delta t)$ by the operator $r^k(\Delta tA_h)$ which is repeated n times. This term will give an error of $O(\Delta t^{1/3}|\ln h|)$ with an optimal choice of k. The third term of (5.23) $h/(k\Delta t)^{3/2}$ is due to the interpolation error: since the positive part of a function of V_h is not in that space, it is necessary to interpolate the solution obtained after a truncature; with an optimal choice of k, this term will give an error of $O(h^{2/5})$.

In the two following figures, we represent the concentrations of pheromone obtained with a small diffusor situated in the centre of a rectangular field with a W–E wind and a constant absorption at the time $t=1$ and 10 (see Figs. 1 and 2).

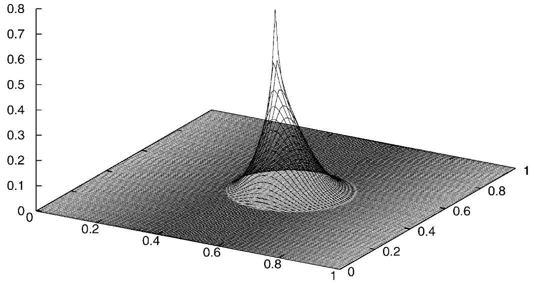

Fig. 1. Concentration at $t = 1$.

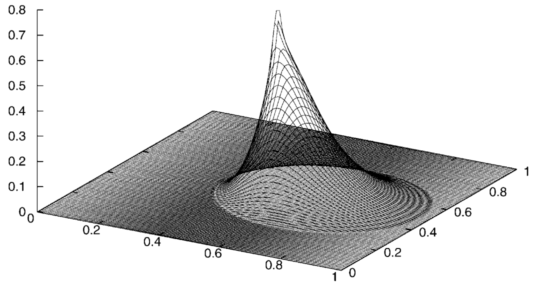

Fig. 2. Concentration at $t = 10$.

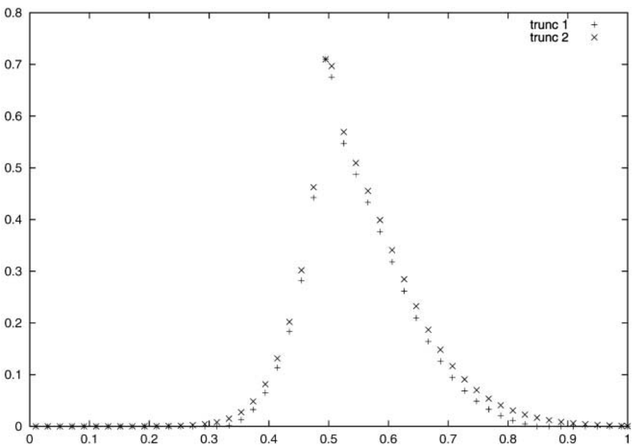

Fig. 3. Comparison between the two truncature methods.

The Fig. 3 represents the results obtained along the W–E median line of the rectangular field by the two truncature methods at the same time level: there is very little difference between these two methods and we may verify that the first approximation is always below the second.

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