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Continua and their σ-ideals

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Abstract

It is proved that a coanalytic invariant σ-ideal of continua of the plane is either G_δ or it is complete coanalytic. The structure of the complete lattice of invariant σ-ideals of continua is investigated.

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Introduction and notation

A *continuum* is a compact connected (non-empty) metric space. It is *non-degenerate* if it contains more than one point. For X a Polish space, let $C(X)$ be the set of all *subcontinua* of X, that is all subspaces of X that are continua. This is a closed subset of $K(X)$ (the Polish space of all compact subsets of X endowed with the Vietoris topology), so it is a Polish space in its own right. Since every compact metric space can be embedded in $\mathbb{R}^{\mathbb{N}}$—actually in $[0,1]^{\mathbb{N}}$—$C(\mathbb{R}^{\mathbb{N}})$ (or $C([0,1]^{\mathbb{N}})$) can be regarded as the Polish space of all continua (up to homeomorphism) and provides the setting in which continuum theory is usually performed.

Given a class $\mathcal{F} \subseteq C(X)$ of continua, one can investigate the descriptive set theoretic complexity of $\mathcal{F}$ (see, for example, [3,6,2]). Knowing that $\mathcal{F}$ is Γ-complete for some class Γ of sets in Polish spaces entails that the complexity of any definition of $\mathcal{F}$ is bounded be-

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low by the complexity of Γ. As a reference for these and other basic concepts of descriptive set theory used in this paper, see [4].

In various mathematical contexts, one can formalize notions of *smallness* using the concept of σ-ideals. For continua this can be stated as follows.

Definition. Let $\emptyset \neq \mathcal{I} \subseteq C(X)$. Then $\mathcal{I}$ is a *σ-ideal of continua* if:

(1) $\mathcal{I}$ is *hereditary*, that is $\forall C \in \mathcal{I}\ \forall C' \in C(X)\ (C' \subseteq C \Rightarrow C' \in \mathcal{I})$;
(2) if $\forall n \in \mathbb{N}\ C_n \in \mathcal{I}$ and if $\bigcup_{n\in\mathbb{N}} C_n$ is a continuum, then $\bigcup_{n\in\mathbb{N}} C_n \in \mathcal{I}$.

In other words, a non-empty class of continua is a σ-ideal if it is closed under subcontinua and under countable unions that are continua. Examples of σ-ideals of continua often come from dimension theory (continua of dimension not greater than some fixed n, countable dimensional continua, continua whose non-degenerate subcontinua are infinite dimensional, ...); other examples include Suslinian continua, continua contained in countable unions of homeomorphic copies of a given continuum, ... More examples will be discussed later.

This paper investigates structural properties for σ-ideals of continua in affine spaces $\mathbb{R}^n$, $2 \leqslant n \leqslant \omega$, where for $m \leqslant n$ the space $\mathbb{R}^m$ will always be thought of as naturally embedded in $\mathbb{R}^n$.

In Section 1 a dichotomy result for σ-ideals of continua is proved, along the lines of Kechris, Louveau and Woodin dichotomy for σ-ideals of compact sets [5] according to which a σ-ideal of compact sets in a Polish space is either G_δ or complete coanalytic. To state this, call $\mathcal{F} \subseteq C(X)$ *invariant* if it is closed under homeomorphism.

Theorem. *Let $\mathcal{I}$ be a coanalytic invariant σ-ideal of continua of the plane $\mathbb{R}^2$. Then either $\mathcal{I}$ is G_δ or it is $\mathbf{\Pi}^1_1$-complete.*

Corollary. *Let $\mathcal{I} \subseteq C(\mathbb{R}^n)$, $2 \leqslant n \leqslant \omega$, be a coanalytic invariant class of continua. If $\mathcal{I} \cap C(\mathbb{R}^2)$ is a σ-ideal of continua and it is not G_δ, then $\mathcal{I}$ is $\mathbf{\Pi}^1_1$-complete.*

The hypotheses about $\mathcal{I}$ being invariant can be relaxed quite a bit, but they cannot just be dropped.

Section 2 remarks that it is immaterial whether the study of invariant σ-ideals of continua is undertaken in affine spaces $\mathbb{R}^n$ or the corresponding cubes $[0,1]^n$. The choice for $\mathbb{R}^n$ is made to have the affine structure of such spaces available, but same results hold in $[0,1]^n$.

The family of all invariant σ-ideals of continua of $\mathbb{R}^{\mathbb{N}}$ is a complete lattice under inclusion, which will be denoted by $\mathcal{L}$. Indeed, $\mathcal{L}$ has a least element ($F_1(\mathbb{R}^{\mathbb{N}})$, the set of all singletons), a biggest element ($C(\mathbb{R}^{\mathbb{N}})$) and, given any non-empty family $\mathcal{G} \subseteq \mathcal{L}$, $\bigcap \mathcal{G}$ is an invariant σ-ideal of continua, so $\inf \mathcal{G} = \bigcap \mathcal{G}$; consequently also $\sup \mathcal{G}$ exists and it is the intersection of all invariant σ-ideals of continua containing $\bigcup \mathcal{G}$.

Starting with Section 3 the structure of $\mathcal{L}$ is studied. It follows from the above that if $\emptyset \neq \mathcal{F} \subseteq C(\mathbb{R}^{\mathbb{N}})$ then there is a smallest invariant σ-ideal of continua $\mathcal{F}^\sigma$ containing $\mathcal{F}$; $\mathcal{F}^\sigma$ is called the *invariant σ-ideal of continua generated by $\mathcal{F}$*. Since $\mathcal{F}^\sigma$ is the closure of $\mathcal{F}$ under homeomorphism, subcontinua and the $\aleph_0$-ary operation of countable unions that are continua, one might expect that the construction of $\mathcal{F}^\sigma$ from $\mathcal{F}$ would require transfinite induction up to ω_1. However this is not the case.

Theorem. *Let $\emptyset \neq \mathcal{F} \subseteq C(\mathbb{R}^\mathbb{N})$ and let $X \in C(\mathbb{R}^\mathbb{N})$. Then $X \in \mathcal{F}^\sigma$ if and only if X is included in a countable union of continua each of which is homeomorphic to some member of $\mathcal{F}$.*

Concerning the descriptive complexity of $\mathcal{F}^\sigma$, the following is proved.

Theorem. *Let $\emptyset \neq \mathcal{F} \subseteq C(\mathbb{R}^\mathbb{N})$. Suppose that $\mathcal{K} = \{K \in K(\mathbb{R}^\mathbb{N}) \mid \exists C \in \mathcal{F}$ (K is embeddable in C)$\}$ is a coanalytic subset of $K(\mathbb{R}^\mathbb{N})$. Then $\mathcal{F}^\sigma \in \mathbf{\Pi}^1_1(C(\mathbb{R}^\mathbb{N}))$.*

As a consequence, it will follow that the invariant σ-ideal of continua generated by the arcs is $\mathbf{\Pi}^1_1$-complete.

In Section 4, as a consequence of the observation that $\mathcal{L} \setminus \{C(\mathbb{R}^\mathbb{N})\}$ has a biggest element $\mathcal{I}_\infty$ which is complete coanalytic, a purely descriptive set theoretic argument shows the non-existence of a universal continuum for non-universal continua (that is for continua not containing Hilbert cubes). A countable chain of invariant σ-ideals of continua (which are *big* in some sense) is given whose supremum is less than $\mathcal{I}_\infty$.

It is worth remarking here that several arguments in this paper are based on Baire category theorem or the following simple consequence of it: if X is a non-degenerate continuum and if $X_0, X_1, \ldots$ are closed subsets of X that cover X, then there is $n \in \mathbb{N}$ such that X_n contains a non-degenerate continuum.

1. A dichotomy theorem

This section is devoted to prove the following theorem.

Theorem 1. *Let C be a fixed non-degenerate continuum of the plane $\mathbb{R}^2$. Suppose $\mathcal{I}$ is a coanalytic σ-ideal of continua of the plane containing all continua which are obtainable from C by an affine non-degenerate orientation preserving transformation. Then either $\mathcal{I}$ is G_δ or it is $\mathbf{\Pi}^1_1$-complete.*

The case of main interest is when $\mathcal{I}$ is closed under homeomorphism. So it is worth stating explicitly this particular case: a coanalytic invariant σ-ideal of continua in the plane is either G_δ or $\mathbf{\Pi}^1_1$-complete.

Proof of Theorem 1. The proof is achieved establishing a sequence of propositions.

Proposition 2. *Let $K \in K(\mathbb{R}^2)$ have cardinality at least 2 and let A, A' be fixed distinct points of K. Then there is a continuous function $f_{KAA'} : (\mathbb{R}^2)^2 \to K(\mathbb{R}^2) \setminus \{\emptyset\}$ such that, when P, P' are distinct points of $\mathbb{R}^2$, then K is homeomorphic with $f_{KAA'}(P, P')$ via an affine orientation preserving transformation of the plane which sends A to P and A' to P'.*

Proof. Let $f_{KAA'}(P, P') = \varphi\psi\tau(K)$ where $\varphi, \psi, \tau : \mathbb{R}^2 \to \mathbb{R}^2$ are defined as follows:

- τ is the rotation with centre the origin O such that $\tau(A') - \tau(A)$ is concordly parallel to $P' - P$ (the identity if $P = P'$);
- ψ is defined by $\forall z \in \mathbb{R}^2\ \psi(z) = \frac{\|P'-P\|}{\|A'-A\|} z$;
- φ is the translation sending $\psi\tau(A)$ to P.

To see continuity of $f_{KAA'}$, assume (P_n, P'_n) is a sequence of pairs of points converging to (P, P'); let τ, ψ, φ be as above for constructing $f_{KAA'}(P, P')$ and let $\tau_n, \psi_n, \varphi_n$ be the corresponding functions used in producing $f_{KAA'}(P_n, P'_n)$. If $P \neq P'$ then $\varphi_n\psi_n\tau_n(K)$ converges to $\varphi\psi\tau(K)$ since the distance $d(\varphi_n\psi_n\tau_n(x), \varphi\psi\tau(x))$ converges uniformly to 0 for $x \in K$. If $P = P'$, then $\lim_{n\to\infty} \operatorname{diam}(\varphi_n\psi_n\tau_n(K)) = 0$ and $\lim_{n\to\infty} \varphi_n\psi_n\tau_n(K) = \varphi\psi\tau(K) = \{P\}$.

Finally, if $P \neq P'$ then φ, ψ, τ are non-degenerate affine orientation preserving transformations of the plane and $\varphi\psi\tau(A) = P$, $\varphi\psi\tau(A') = P'$. □

Proposition 3. *The following functions are continuous*:

(1) $G : K(\mathbb{R}^2) \setminus \{\emptyset\} \to \mathbb{R}^2$ *assigning to each compact subspace* $K \subseteq \mathbb{R}^2$ *the middle point of the bottom basis of the smallest, possibly degenerate rectangle* $R(K)$*, with edges parallel to the coordinate axes, containing* K;

(2) $G' : K(\mathbb{R}^2) \setminus \{\emptyset\} \to \mathbb{R}^2$ *assigning to* $K \in K(\mathbb{R}^2) \setminus \{\emptyset\}$ *the middle point of the top basis of* $R(K)$;

(3) $w : K(\mathbb{R}^2) \setminus \{\emptyset\} \to \mathbb{R}_0^+$ *assigning to* K *the width of* $R(K)$.

Proof. Let $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the projections onto the first and second coordinates, respectively.

(1) Observe that

$$\forall K \in K(\mathbb{R}^2) \setminus \{\emptyset\}, \quad G(K) = \left(\frac{\min \pi_1(K) + \max \pi_1(K)}{2}, \min \pi_2(K)\right);$$

moreover the minimum and maximum functions $K(\mathbb{R}) \setminus \{\emptyset\} \to \mathbb{R}$ are continuous since, for $H, H' \in K(\mathbb{R}) \setminus \{\emptyset\}$, $|\min H - \min H'| \leqslant d_H(H, H')$ and similarly for max. So G is a composition of continuous functions.

(2) Similar to (1), as

$$\forall K \in K(\mathbb{R}^2) \setminus \{\emptyset\}, \quad G'(K) = \left(\frac{\min \pi_1(K) + \max \pi_1(K)}{2}, \max \pi_2(K)\right).$$

(3) $\forall K \in K(\mathbb{R}^2) \setminus \{\emptyset\}\ w(K) = \max \pi_1(K) - \min \pi_1(K)$. □

Fix a non-degenerate continuum $C \subseteq \mathbb{R}^2$ and $A, A' \in C$ be such that $d(A, A') = \operatorname{diam}(C)$. Define a function $F : K(\mathbb{R}^2) \setminus \{\emptyset\} \to K(\mathbb{R}^2) \setminus \{\emptyset\}$ according to the following cases.

Case 1: C *is not a segment*. To define $F(K)$ consider two subcases.

Subcase 1a: $G(K) \neq G'(K)$. Then let $F(K) = \sigma(f_{CAA'}(G(K), G'(K)))$, where $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ has equations

$$\begin{cases} X = \frac{w(K)}{3wf_{CAA'}(G(K),G'(K))}\big(x - \pi_1 G(K)\big) + \pi_1 G(K), \\ Y = y. \end{cases}$$

So $F(K)$ is obtained by shrinking, or enlarging, $f_{CAA'}(G(K), G'(K))$ along the horizontal direction, leaving the vertical line through $G(K)$ and $G'(K)$ fixed, in such a way that $wF(K) = \frac{w(K)}{3}$.

Subcase 1b: $G(K) = G'(K)$. Let $\vec{v}$ be the unitary vector orthogonal to $A' - A$ such that $(\vec{v}, A' - A)$ is positively oriented and r the line through A of direction $\vec{v}$; let $A - \alpha\vec{v}$, $A + \beta\vec{v}$ be the end points of the orthogonal projection of C on r. Then let $F(K) = [\pi_1 G(K) - a, \pi_1 G(K) + b] \times \{\pi_2 G(K)\}$, where $a, b \in \mathbb{R}_0^+$ are such that $b + a = \frac{w(K)}{3}$ and, if $w(K) \neq 0$, (a, b) is proportional to (α, β).

Case 2: *C is a segment*. Then let $F(K)$ be the possibly degenerate segment with extrema $G(K), G'(K)$.

Proposition 4. *The function F is continuous.*

Proof. Suppose first C is not a segment. If K is such that $G(K) \neq G'(K)$ (note that these K form an open set) the continuity of F in K follows using continuity of the coefficients of σ as functions of K. If $G(K) = G'(K)$, that is the convex hull of K is a, possibly degenerate, horizontal segment centered in $G(K)$ with length $w(K)$, then $F(K)$ is a, possibly degenerate, horizontal segment containing $G(K)$ with length $\frac{w(K)}{3}$. If $F(K)$ is not a point the pieces of $F(K)$ to the left and to the right of $G(K)$ are proportional to the extension of the corresponding parts of C around the line through A, A'. Let K_n be a sequence in $K(\mathbb{R}^2) \setminus \{\emptyset\}$ converging to K. Then $\lim_{n\to\infty} G(K_n) = \lim_{n\to\infty} G'(K_n) = G(K), \lim_{n\to\infty} w(K_n) = w(K)$; since also the left-right proportions are preserved, $\lim_{n\to\infty} F(K_n) = F(K)$.

If on the other hand C is a segment, $F(K) = f_{CAA'}(G(K), G'(K))$. Again F is a continuous function. □

So $F(K)$ is a continuum, homeomorphic to C if $R(K)$ is neither a segment nor a point, such that $G(K) \in F(K), G'(K) \in F(K), R(F(K)) \subseteq R(K)$, the horizontal bases of $R(F(K))$ lie on the corresponding bases of $R(K)$ and $R(F(K))$ is at least $\delta = \frac{w(K)}{6}$ far from the vertical edges of $R(K)$.

Proposition 5. *For every continuum $C' \subseteq \mathbb{R}^2$, $C' \cap F(C') \neq \emptyset$.*

Proof. If $R(C')$ is a segment or a point then $G(C') \in C' \cap F(C')$.

Suppose otherwise and suppose $C' \cap F(C') = \emptyset$. Then there is $\varepsilon \in \left]0, \frac{\delta}{2}\right[$ such that the open ε-neighborhood O_ε of $F(C')$ is disjoint from C'. O_ε is connected, so it is arcwise connected (see, for example, [9, Theorem 8.26]). Since $G(C'), G'(C') \in O_\varepsilon$, this means that there is an arc $L \subseteq O_\varepsilon$ with an end point on each horizontal basis of $R(C')$ such that L is disjoint from the vertical edges of $R(C')$ and $R(C') \setminus L$ has two connected components.

Since $L \cap C' = \emptyset$, and C' has points both on the left vertical edge and on the right vertical edge of $R(C')$, a contradiction is reached. □

Proposition 6. *The following functions are continuous*:

(1) $t : K(\mathbb{R}^2) \setminus \{\emptyset\} \to K(\mathbb{R}^2) \setminus \{\emptyset\}$ *defined by* $t(K) = K \cup F(K)$;
(2) $L : (K(\mathbb{R}^2) \setminus \{\emptyset\})^2 \to K(\mathbb{R}^2) \setminus \{\emptyset\}$ *defined by* $L(K, K') = f_{CAA'}(G(K), G(K'))$.

Moreover t *maps continua to continua.*

Proof. (1) By continuity of F and the union. (2) By continuity of G and $f_{CAA'}$. The last assertion follows from Proposition 5. □

Proposition 7. *For* $\mathcal{K} \in K(K(\mathbb{R}^2) \setminus \{\emptyset\})$ *let*

$$T(\mathcal{K}) = \{t(K)\}_{K \in \mathcal{K}},$$
$$\Lambda(\mathcal{K}) = \{L(K, K') \mid (K, K') \in \mathcal{K}^2\},$$
$$\Theta(\mathcal{K}) = \bigcup T(\mathcal{K}) \cup \bigcup \Lambda(\mathcal{K}).$$

Then $\Theta : K(K(\mathbb{R}^2) \setminus \{\emptyset\}) \to K(\mathbb{R}^2)$ *is continuous. If* $\mathcal{K} \subseteq C(\mathbb{R}^2)$ *and* $\mathcal{K}$ *is not empty then* $\Theta(\mathcal{K}) \in C(\mathbb{R}^2)$.

Proof. Indeed $T(\mathcal{K})$ and $\Lambda(\mathcal{K})$ are compact, by the continuity of t and L, and compact union of compact sets is compact. Moreover, union is continuous.

If $\mathcal{K} \subseteq C(\mathbb{R}^2), \mathcal{K} \neq \emptyset$, note that each element of $T(\mathcal{K})$ is connected, as well as each element of $\Lambda(\mathcal{K})$. Moreover, for $K, K' \in \mathcal{K}$, the continuum $L(K, K')$ links $t(K), t(K')$ so $\Theta(\mathcal{K})$ is connected too. □

Now the proof is completed using above propositions and the basic idea of the proof of the dichotomy of Kechris, Louveau and Woodin as in [4, Theorem 33.3].

Assume $\mathcal{I}$ is not G_δ. Then by [4, Theorem 21.18] there is a Cantor set $E \subseteq C(\mathbb{R}^2)$ such that $Q = E \cap \mathcal{I}$ is countable dense in E. Moreover $\mathcal{I}$ must contain some non-degenerate continua. Fix such non-degenerate continuum $C \in \mathcal{I}$ and let $\Theta|_{K(E)\setminus\{\emptyset\}} : K(E) \setminus \{\emptyset\} \to C(\mathbb{R}^2)$ be as above, for this C. It is enough to establish

$$\forall \mathcal{K} \in K(E) \setminus \{\emptyset\} \quad \big(\Theta(\mathcal{K}) \in \mathcal{I} \iff \mathcal{K} \subseteq Q\big),$$

since $K(Q)$ is $\mathbf{\Pi}^1_1$-complete. If $\mathcal{K} \nsubseteq Q$, then $\Theta(\mathcal{K}) \notin \mathcal{I}$ as $\mathcal{I}$ is a σ-ideal of continua. So suppose $\mathcal{K} \subseteq Q$. If $\mathcal{I}$ contains some segments (and so contains all of them), then $\Theta(\mathcal{K}) \in \mathcal{I}$ since $\Theta(\mathcal{K})$ is a continuum which is countable union of continua in $\mathcal{I}$. If $\mathcal{I}$ contains no segments then $F(K)$ is not a segment for any $K \in \mathcal{K}$ and again $\Theta(\mathcal{K})$ is a countable union of elements of $\mathcal{K}$ and continua obtained from C by an affine orientation preserving transformation. So again $\Theta(\mathcal{K}) \in \mathcal{I}$. □

Now think of $\mathbb{R}^2$ as a subspace of $\mathbb{R}^n$, for $2 \leqslant n \leqslant \omega$.

Corollary 8. *Let C be a fixed non-degenerate continuum of the plane $\mathbb{R}^2$ and $\mathcal{I} \subseteq C(\mathbb{R}^n)$, $2 \leqslant n \leqslant \omega$, be a coanalytic class of continua containing all continua of the plane obtainable from C by affine non-degenerate orientation preserving transformations. If $\mathcal{I} \cap C(\mathbb{R}^2)$ is a σ-ideal of continua and it is not G_δ, then $\mathcal{I}$ is $\mathbf{\Pi}^1_1$-complete.*

Proof. By applying Theorem 1 and its proof to $\mathcal{I} \cap C(\mathbb{R}^2)$. □

A way to prove that a given subset of a Polish space is not G_δ is to show that it is meagre and dense. Since the class Ψ of pseudo-arcs is dense G_δ, Corollary 8 gives the following.

Corollary 9. *Let $\mathcal{I} \subseteq C(\mathbb{R}^n)$, $2 \leqslant n \leqslant \omega$, be a coanalytic class of continua closed under planar affine non-degenerate orientation preserving transformations of some fixed planar continuum C. If $\mathcal{I} \cap C(\mathbb{R}^2)$ is a σ-ideal of continua dense in $C(\mathbb{R}^2)$ and does not contain pseudo-arcs, then $\mathcal{I}$ is $\mathbf{\Pi}^1_1$-complete.*

Note that, since the class Ψ of pseudo-arcs is the only comeagre homeomorphism class in $C(\mathbb{R}^n)$, $2 \leqslant n \leqslant \omega$, not containing pseudo-arcs is a necessary and sufficient condition for an invariant class of continua for being meagre.

The main tool in the proof of Theorem 1 was the function Θ—actually its restriction $\Theta : K(C(\mathbb{R}^2)) \setminus \{\emptyset\} \to C(\mathbb{R}^2)$—depending on a fixed non-degenerate planar continuum $C \in \mathcal{I}$. To summarize its features, let me recall here the steps of its definition (restricted to $K(C(\mathbb{R}^2)) \setminus \{\emptyset\}$).

- $G : C(\mathbb{R}^2) \to \mathbb{R}^2$ assigns to C' the midpoint of the bottom basis of the least (possibly degenerate) rectangle $R(C')$, with edges parallel to the coordinate axes, containing C';
- $F : C(\mathbb{R}^2) \to C(\mathbb{R}^2)$ is a continuous function such that, if $R(C')$ is not a degenerate rectangle, $F(C')$ is the image of C by a non-degenerate affine orientation preserving transformation, $C' \cap F(C') \neq \emptyset$, $G(C') \in F(C')$ (if $R(C')$ is a degenerate rectangle, then $F(C')$ is either a segment or a singleton, containing $G(C')$);
- $t : C(\mathbb{R}^2) \to C(\mathbb{R}^2)$ is defined by $t(C') = C' \cup F(C')$;
- $L : (C(\mathbb{R}^2))^2 \to C(\mathbb{R}^2)$ is a continuous function such that, if $G(C') \neq G(C'')$, then $L(C', C'')$ is image of C by an affine non-degenerate orientation preserving transformation and $G(C'), G(C'') \in L(C', C'')$ (if $G(C') = G(C'')$, then $L(C', C'') = \{G(C')\}$);
- finally, $\Theta(\mathcal{K}) = \bigcup_{C' \in \mathcal{K}} t(C') \cup \bigcup_{(C', C'') \in \mathcal{K}^2} L(C', C'')$.

As in [5] for the original dichotomy, using Θ it is possible to generalize the argument employed and get the following.

Theorem 10. *Let $\mathcal{I}$ be a coanalytic σ-ideal of continua of $\mathbb{R}^2$ satisfying the hypothesis of Theorem* 1. *Let $B \subseteq \mathcal{I}$ and B_Θ the class of continua obtained by applying the function Θ to countable elements of $K(B)$ (thus $B_\Theta \subseteq \mathcal{I}$). If there exists an analytic set A with $B_\Theta \subseteq A \subseteq \mathcal{I}$, then there is a G_δ set H with $B \subseteq H \subseteq \mathcal{I}$.*

Proof. Deny. Then we can apply [4, Theorem 21.22] to $C(\mathbb{R}^2) \setminus \mathcal{I}$ and B. This gives a Cantor set $F \subseteq B \cup (C(\mathbb{R}^2) \setminus \mathcal{I})$ such that $F \cap B$ is homeomorphic with $\mathbb{Q}$. Consider the continuous function $\varphi = \Theta|_{K(F)\setminus\{\emptyset\}} : K(F) \setminus \{\emptyset\} \to C(\mathbb{R}^2)$. Then, for $L \in K(F) \setminus \{\emptyset\}$,

$$L \subseteq F \cap B \iff \Theta(L) \in B_\Theta \iff \Theta(L) \in \mathcal{I},$$

so $\varphi^{-1}(B_\Theta) = \varphi^{-1}(\mathcal{I}) = K(F \cap B) \setminus \{\emptyset\}$, which is $\mathbf{\Pi}^1_1$-complete. Hence there cannot be any analytic A satisfying $B_\Theta \subseteq A \subseteq \mathcal{I}$. $\square$

Using the construction carried out in Propositions 2 through 7 the following is also achieved.

Theorem 11. *Let $\mathcal{I}$ be an analytic σ-ideal of continua of $\mathbb{R}^2$ satisfying the closure hypothesis of Theorem* 1. *Then $\mathcal{I}$ is actually G_δ.*

Proof. This is similar to the proof of [5, Section 1, Theorem 11]. Instead of $K(E)$ use $C(\mathbb{R}^2)$ and the function Θ instead of union. Note that Saint Raymond's Lemma 10 used and stated there for a compact metric space E, works for any Polish space (embed it in its compactification). $\square$

Remark. Unlike the dichotomy of Kechris, Louveau and Woodin for σ-ideals of compact sets, there cannot be a similar general dichotomy for σ-ideals of continua, so it is necessary to require some closure hypotheses, like the ones stated in Theorems 1 and 11. Indeed the set of all continua included in $\mathbb{Q} \times \{0\}$ is a $\mathbf{\Sigma}^0_2$-complete σ-ideal of continua of the plane. It actually consists of degenerate continua; for a less trivial counterexample take the σ-ideal of continua $\mathcal{I} \subseteq C(\mathbb{R}^2)$ formed by all vertical segments of the plane having rational abscissae and their points. So $\mathcal{I} \in \mathbf{\Sigma}^0_2(C(\mathbb{R}^2))$ since, for $X \in C(\mathbb{R}^2)$,

$$X \in \mathcal{I} \iff \pi_1(X) \in F_1(\mathbb{R}) \wedge \min \pi_1(X) \in \mathbb{Q}$$

(where $F_1(\mathbb{R})$ is the family of singletons of the real line) while $\mathbf{\Sigma}^0_2$-hardness of $\mathcal{I}$ is witnessed by the function $\mathbb{R} \to C(\mathbb{R}^2)$, $x \mapsto \{(x, 0)\}$.

2. Invariant σ-ideals in cubes

This paper studies σ-ideals of continua in Euclidean spaces $\mathbb{R}^n$ or $\mathbb{R}^\mathbb{N}$. This allows to use the affine structure of such spaces, which is not available in cubes $[0, 1]^n$ or the Hilbert cube $[0, 1]^\mathbb{N}$, where the study of continua is usually performed. Note however that, dealing with invariant σ-ideals of continua, this is not restrictive at all. Indeed the following holds.

Proposition 12. *Let $2 \leqslant n \leqslant \omega$. For $\mathcal{F} \subseteq C(\mathbb{R}^n)$, denote by $[\mathcal{F}]_n \subseteq C(\mathbb{R}^n)$ the saturation of $\mathcal{F}$ with respect to the relation of homeomorphism. Then*:

(1) *If $\mathcal{I}$ is an invariant class of continua in $\mathbb{R}^n$ then $\mathcal{I} \cap C([0, 1]^n)$ is an invariant class of continua in $[0, 1]^n$ and $\mathcal{I} = [\mathcal{I} \cap C([0, 1]^n)]_n$. Moreover $\mathcal{I}$ and $\mathcal{I} \cap C([0, 1]^n)$ are Wadge bireducible as subsets of $C(\mathbb{R}^n)$, $C([0, 1]^n)$, respectively. If in addition $\mathcal{I}$ is a σ-ideal of continua, then $\mathcal{I} \cap C([0, 1]^n)$ is a σ-ideal of continua.*

(2) *If $\mathcal{J}$ is an invariant class of continua in* $[0,1]^n$ *then* $[\mathcal{J}]_n$ *is an invariant class of continua in* $\mathbb{R}^n$ *and* $\mathcal{J} = [\mathcal{J}]_n \cap C([0,1]^n)$. *Moreover* $\mathcal{J}$ *and* $[\mathcal{J}]_n$ *are Wadge bireducible as subsets of* $C([0,1]^n)$, $C(\mathbb{R}^n)$, *respectively. If in addition* $\mathcal{J}$ *is a* σ*-ideal of continua, then* $[\mathcal{J}]_n$ *is a* σ*-ideal of continua.*

Proof. (1) $\mathcal{I} \cap C([0,1]^n)$ is closed under homeomorphism in $C([0,1]^n)$ since both $\mathcal{I}$ and $C([0,1]^n)$ are.

Let $g : \mathbb{R} \to]0,1[$ be a homeomorphism and let $\gamma : \mathbb{R}^n \to]0,1[^n$ be the n-fold Cartesian product of g. If $X \in \mathcal{I}$, then $\gamma(X) \subseteq [0,1]^n$ and $X, \gamma(X)$ are homeomorphic. So $\gamma(X) \in \mathcal{I} \cap C([0,1]^n)$ and $X \in [\mathcal{I} \cap C([0,1]^n)]_n$. Conversely, if $X \in [\mathcal{I} \cap C([0,1]^n)]_n$, then X is homeomorphic to some $X' \in \mathcal{I} \cap C([0,1]^n)$, thus $X \in \mathcal{I}$.

The function $X \in C(\mathbb{R}^n) \mapsto \gamma(X) \in C([0,1]^n)$ and the natural inclusion map witness that $\mathcal{I}$ and $\mathcal{I} \cap C([0,1]^n)$ are Wadge bireducible.

If $\mathcal{I}$ is a σ-ideal of continua, then such is $\mathcal{I} \cap C([0,1]^n)$, being the intersection of σ-ideals of continua.

(2) Closure under homeomorphism of $[\mathcal{J}]_n$ is granted by definition.

Now note that $\mathcal{J}$ is both included in $[\mathcal{J}]_n$ and $C([0,1]^n)$. Conversely, let $X \in [\mathcal{J}]_n \cap C([0,1]^n)$. Then there exists $X' \in \mathcal{J}$ such that X, X' are homeomorphic. Consequently $X \in \mathcal{J}$.

The natural inclusion map and the function $X \in C(\mathbb{R}^n) \mapsto \gamma(X) \in C([0,1]^n)$ grant Wadge bireducibility of $\mathcal{J}$ and $[\mathcal{J}]_n$.

Suppose now that $\mathcal{J}$ is a σ-ideal of continua. Closure under subcontinua of $[\mathcal{J}]_n$ is granted by this same property of $\mathcal{J}$. Suppose $\forall m \in \mathbb{N}\ X_m \in [\mathcal{J}]_n$ and that $\bigcup_{k\in\mathbb{N}} X_k$ is a continuum. Then $\gamma(\bigcup_{k\in\mathbb{N}} X_k) = \bigcup_{k\in\mathbb{N}} \gamma(X_k) \in \mathcal{J}$ and thus $\bigcup_{k\in\mathbb{N}} X_k \in [\mathcal{J}]_n$. □

As an illustration one can state the dichotomy theorem for an invariant σ-ideal $\mathcal{J}$ of continua in some cube. Note that if such a $\mathcal{J}$ contains a planar non-degenerate continuum, then (thinking of $[0,1]^2$ as naturally embedded in $[0,1]^n$ or $[0,1]^{\mathbb{N}}$) $\mathcal{J} \cap C([0,1]^2)$ is dense in $C([0,1]^2)$ since each homeomorphism class of (planar) non-degenerate continua is dense (among planar continua).

Corollary 13. *Let* $\mathcal{J} \subseteq C([0,1]^n)$, $2 \leqslant n \leqslant \omega$, *be a coanalytic invariant* σ*-ideal of continua. If* $\mathcal{J} \cap C([0,1]^2)$ *is not* G_δ (*for example, if* $\mathcal{J}$ *contains some non-degenerate planar continua but the pseudoarc is not in* $\mathcal{J}$), *then* $\mathcal{J}$ *is* $\mathbf{\Pi}^1_1$*-complete.*

Proof. Since $\mathcal{J}$ is coanalytic, by Proposition 12 one has that $[\mathcal{J}]_n$ is coanalytic too. On the other hand $[\mathcal{J}]_n \cap C(\mathbb{R}^2)$ is not G_δ, as $\mathcal{J} \cap C([0,1]^2)$ is not G_δ by the hypothesis. It follows from Corollary 8 that $[\mathcal{J}]_n$ is $\mathbf{\Pi}^1_1$-complete and thus $\mathcal{J}$ is $\mathbf{\Pi}^1_1$-complete by Proposition 12 again. □

3. Generating families

For $\emptyset \neq \mathcal{F} \subseteq C(\mathbb{R}^{\mathbb{N}})$ define $\mathcal{F}^\sigma$ as the smallest σ-ideal of continua containing $\mathcal{F}$ as a subset and invariant under homeomorphism. It will be called the *invariant* σ*-ideal of con-*

tinua generated by $\mathcal{F}$; it is the intersection of all invariant σ-ideals of continua containing $\mathcal{F}$. If $\mathcal{F}=\{X\}$, write X^σ for $\{X\}^\sigma$ and call it the *the invariantly principal* σ*-ideal of continua generated by* X. Also, let $\mathcal{F}^\perp$ be the family of all continua $Z\in C(\mathbb{R}^\mathbb{N})$ such that $Z^\sigma\cap\mathcal{F}^\sigma$ is the set of singletons; again, this is denoted $X^\perp$ when $\mathcal{F}=\{X\}$.

Lemma 14. *Let* $X, X', Y\in K(\mathbb{R}^\mathbb{N})$. *Assume* $X\subseteq Y$ *and suppose* X, X' *are homeomorphic. Then there is* $Y'\in K(\mathbb{R}^\mathbb{N})$ *such that* $X'\subseteq Y'$ *and* Y, Y' *are homeomorphic.*

Proof. Let $\prod_{n\in\mathbb{N}}[a_n,b_n], \prod_{n\in\mathbb{N}}[c_n,d_n]$ be Hilbert cubes in $\mathbb{R}^\mathbb{N}$ such that Y is contained in the pseudo-interior $\prod_{n\in\mathbb{N}}]a_n,b_n[$ of $\prod_{n\in\mathbb{N}}[a_n,b_n]$ and X' is included in the pseudo-interior $\prod_{n\in\mathbb{N}}]c_n,d_n[$ of $\prod_{n\in\mathbb{N}}[c_n,d_n]$. Let $f: X\to X'$ be a homeomorphism. Using [11, Theorem 6.3.4], f can be extended to a homeomorphism $\bar{f}:\prod_{n\in\mathbb{N}}[a_n,b_n]\to\prod_{n\in\mathbb{N}}[c_n,d_n]$. Let $Y'=\bar{f}(Y)$. □

If $\emptyset\neq\mathcal{F}\subseteq C(\mathbb{R}^\mathbb{N})$, then $\mathcal{F}^\sigma$ is the smallest subclass of $\mathbb{R}^\mathbb{N}$ closed under homeomorphism, subcontinua and countable unions that are continua. A typical manner to generate such a class is to start with $\mathcal{F}$ and use a transfinite process to close it successively under the three required conditions. For example,

$$\mathcal{F}^\sigma=\bigcup_{\alpha\in\omega_1}\mathcal{F}_\alpha,$$

where $\mathcal{F}_0$ is the class of all continua homeomorphic to some subcontinuum of members of $\mathcal{F}$ and, for $\alpha>0$, $\mathcal{F}_\alpha$ is obtained by taking all those countable unions of members of $\bigcup_{\beta\in\alpha}\mathcal{F}_\beta$ that happen to be continua and all subcontinua of such unions. This way, closure under subcontinua and countable unions that are continua is granted by the inductive construction, so it is enough to see that $\bigcup_{\alpha\in\omega_1}\mathcal{F}_\alpha$ is invariant and, in turn, it is enough to establish by induction that each $\mathcal{F}_\alpha$ is closed under homeomorphism. Invariance of $\mathcal{F}_0$ is required by definition. So let $\alpha>0$, let $X\in\mathcal{F}_\alpha$ and $Y\in C(\mathbb{R}^\mathbb{N})$ be homeomorphic to X. There are $X_0, X_1, \ldots\in\bigcup_{\beta\in\alpha}\mathcal{F}_\beta$ such that $\bigcup_{n\in\mathbb{N}}X_n$ is a continuum and $X\subseteq\bigcup_{n\in\mathbb{N}}X_n$. By Lemma 14, let $Z\in C(\mathbb{R}^\mathbb{N})$ be homeomorphic with $\bigcup_{n\in\mathbb{N}}X_n$ and such that $Y\subseteq Z$. Then there are subcontinua $Z_0, Z_1, \ldots$ of Z, with Z_n, X_n homeomorphic for each $n\in\mathbb{N}$, such that $Z=\bigcup_{n\in\mathbb{N}}Z_n$. By inductive hypothesis, $\forall n\in\mathbb{N}\ Z_n\in\bigcup_{\beta\in\alpha}\mathcal{F}_\beta$ and thus $Y\in\mathcal{F}_\alpha$.

However it turns out that this inductive process stops very early.

Theorem 15. *Let* $\emptyset\neq\mathcal{F}\subseteq C(\mathbb{R}^\mathbb{N})$. *Then* $\mathcal{F}^\sigma=\mathcal{F}_1$.

Proof. It will be shown $\mathcal{F}_2\subseteq\mathcal{F}_1$. Fix $X\in\mathcal{F}_2$. There are $X_0, X_1, \ldots\in\mathcal{F}_1$ such that $X\subseteq\bigcup_{n\in\mathbb{N}}X_n$ (and $\bigcup_{n\in\mathbb{N}}X_n$ is a continuum, but this will not be used). So, for each $n\in\mathbb{N}$ there are $X_{n0}, X_{n1}, \ldots\in\mathcal{F}_0$ such that $X_n\subseteq\bigcup_{m\in\mathbb{N}}X_{nm}$. It is then enough to show that, whenever $X\subseteq\bigcup_{n\in\mathbb{N}}Y_n$ with $Y_0, Y_1, \ldots\in\mathcal{F}_0$ (and it can be assumed $\forall n\in\mathbb{N}\ X\cap Y_n\neq\emptyset$), then $X\subseteq\bigcup_{n\in\mathbb{N}}Z_n$ where $Z_0, Z_1, \ldots\in\mathcal{F}_0$ and moreover $\bigcup_{n\in\mathbb{N}}Z_n$ a continuum. By the invariance under homeomorphism it can be further assumed that $X\subseteq[\frac{1}{3},\frac{2}{3}]^\mathbb{N}$. Fix $h\in\mathbb{N}$ and set $K_h=X\cap Y_h$. For each $x\in K_h$ let $]a_0,b_0[\times]a_1,b_1[\times\cdots\times]a_{r_x},b_{r_x}[\times[0,1]^\mathbb{N}\ni x$ be an open subset of $[0,1]^\mathbb{N}$ with diameter less than $\frac{1}{2(h+1)}$ where $a_i,b_i\in[\frac{1}{4},\frac{3}{4}]$. By compactness extract a finite subcovering $V_1,\ldots,V_{k_h}$ of K_h, where each V_j is of the form

$]u_1^j, v_1^j[\times\cdots\times]u_{r_j}^j, v_{r_j}^j[\times[0,1]^{\mathbb{N}}$; let $W_j = [u_1^j, v_1^j]\times\cdots\times[u_{r_j}^j, v_{r_j}^j]\times[0,1]^{\mathbb{N}}$, $R_j = K_h \cap W_j$. So the R_j are compact and cover K_h. Finally, let $Q_j = [u_1^j - \varepsilon, v_1^j + \varepsilon] \times \cdots \times [u_{r_j}^j - \varepsilon, v_{r_j}^j + \varepsilon] \times [0,1]^{\mathbb{N}}$, where $\varepsilon \in \mathbb{R}^+$ is such that each Q_j is still included in $[0,1]^{\mathbb{N}}$ and has diameter less than $\frac{1}{h+1}$. Now note that each Q_j is a product of closed intervals, it is homeomorphic to the Hilbert cube and it contains R_j in its pseudo-interior; using [11, Theorem 6.3.4], there is an embedding of Y_h into Q_j (let Z_{hj} be its range) which is identity on R_j. Note moreover that each point of Z_{hj} is less that $\frac{1}{h+1}$ apart from some point of $K_h \subseteq X$ (since actually $\operatorname{diam}(Z_{hj}) < \frac{1}{h+1}$). Do this for each $h \in \mathbb{N}$. So $X \subseteq \bigcup_{h,j} Z_{hj}$ and each Z_{hj} is in $\mathcal{F}_0$. It remains to show that $\bigcup_{h,j} Z_{hj}$ is compact. Let x_p be a sequence in $\bigcup_{h,j} Z_{hj}$; by thinning it can be assumed that x_p converges in $[0,1]^{\mathbb{N}}$ to some x. If $x_p \in Z_{hj}$ for infinitely many p, then $x \in Z_{hj}$. Otherwise it can be assumed that each x_p belongs to a different Z_{hj}; in this case $\lim_{p\to\infty} d(x_p, X) = 0$ and so $x \in X \subseteq \bigcup_{h,j} Z_{hj}$. □

This allows to state another synthetic description of $\mathcal{F}^\sigma$.

Corollary 16. *Let $\emptyset \neq \mathcal{F} \subseteq C(\mathbb{R}^{\mathbb{N}})$ and let $X \in C(\mathbb{R}^{\mathbb{N}})$. Then $X \in \mathcal{F}^\sigma$ if and only if X is included in a countable union of continua* (*so without requiring this union to be compact*) *each of which is homeomorphic to some member of $\mathcal{F}$.*

Proof. The proof of Theorem 15 established that $X \in \mathcal{F}^\sigma$ if and only if $X \subseteq \bigcup_{n\in\mathbb{N}} Y_n$ where each Y_n is embeddable in some $Z_n \in \mathcal{F}$. Using Lemma 14, replace each Y_n in the union with a continuum Z_n' homeomorphic with Z_n. □

Corollary 17. *Let X be a continuum such that each non-empty open subset of X contains a continuum homeomorphic with X. Let $\mathcal{F} \subseteq C(\mathbb{R}^{\mathbb{N}})$ and suppose $X \in \mathcal{F}^\sigma$. Then $X \in \mathcal{F}_0$.*

Proof. By Corollary 16, $X \subseteq \bigcup_{n\in\mathbb{N}} Y_n$, with each Y_n homeomorphic to an element from $\mathcal{F}$. Let $n \in \mathbb{N}$ be such that $X \cap Y_n$ contains an open non-empty subset U of X. Since U contains a homeomorphic copy of X, it turns out that X is embeddable in Y_n, so $X \in \mathcal{F}_0$. □

The following is also achieved.

Theorem 18. *Let $\emptyset \neq \mathcal{F} \subseteq C(\mathbb{R}^{\mathbb{N}})$. Suppose that $\mathcal{K} = \{K \in K(\mathbb{R}^{\mathbb{N}}) \mid \exists C \in \mathcal{F}$ (K is embeddable in C)$\}$ is a coanalytic subset of $K(\mathbb{R}^{\mathbb{N}})$. Then $\mathcal{F}^\sigma \in \mathbf{\Pi}_1^1(C(\mathbb{R}^{\mathbb{N}}))$.*

Proof. By the proof of Theorem 15, $\mathcal{F}^\sigma$ is the set of continua contained in some countable union of continua embeddable in elements of $\mathcal{F}$. Thus $\mathcal{F}^\sigma = C(\mathbb{R}^{\mathbb{N}}) \cap \mathcal{H}$, where $\mathcal{H} \subseteq K(\mathbb{R}^{\mathbb{N}})$ is the set of compact sets K such that there are $K_0, K_1, \ldots \in \mathcal{K}$ with $K \subseteq \bigcup_{n\in\mathbb{N}} K_n$. To conclude, let $\{U_n\}_{n\in\mathbb{N}}$ be an open basis for $\mathbb{R}^{\mathbb{N}}$. Using an argument from [1] note that, for each $K \in K(\mathbb{R}^{\mathbb{N}})$, $K \in \mathcal{H}$ if and only if for every compact non-empty $K' \subseteq K$ there is $n \in \mathbb{N}$ such that $U_n \cap K' \neq \emptyset$ and $\overline{U_n} \cap K' \in \mathcal{K}$. This gives a coanalytic definition for $\mathcal{H}$ and thus for $\mathcal{F}^\sigma$. □

To see an application of Theorem 18, consider the invariantly principal σ-ideal generated by the arc.

Theorem 19. *If $A \subseteq \mathbb{R}^{\mathbb{N}}$ is an arc, then the invariantly principal σ-ideal A^{σ} is a complete coanalytic subset of $C(\mathbb{R}^{\mathbb{N}})$.*

Note that once coanalyticity is established, completeness follows from Corollary 9. By Theorem 18, it is then enough to find a coanalytic characterization of compact subsets of arcs (note that this will entail that compact subsets of arcs actually form a Borel set, since such a class is analytic). The following characterization will do.

Theorem 20. *Let K be a compact metric space. Then K is homeomorphic to a subset of $[0,1]$ if and only if the following hold*:

(1) *each connected component of K is either a point or an arc*;
(2) *if L is a connected component of K and L is an arc with end points x, y, then $L \setminus \{x, y\}$ is open in K.*

Note that these conditions define indeed a coanalytic subset of $K(\mathbb{R}^{\mathbb{N}})$, since, after fixing an open basis $\{U_n\}_{n\in\mathbb{N}}$ for $\mathbb{R}^{\mathbb{N}}$, they can equivalently (though less transparently) be expressed by the conjunction of the following:

(a) if $C \in C(\mathbb{R}^{\mathbb{N}})$, then $C \subseteq K$ implies that C is either a point or an arc;
(b) for all $P, Q \in C(\mathbb{R}^{\mathbb{N}})$, $x \in \mathbb{R}^{\mathbb{N}}$, if P, Q are non-degenerate subcontinua of K and $P \cap Q = \{x\}$, then there is $n \in \mathbb{N}$ such that $x \in U_n$, $\overline{U_n} \cap K \subseteq P \cup Q$.

Recall here that classes of homeomorphism are Borel subsets of $C(\mathbb{R}^{\mathbb{N}})$.

It remains only to establish Theorem 20. This may well be a folklore result, however I was not able to find a reference in the literature.

Proof of Theorem 20. Any compact subset of $[0,1]$ satisfies the conditions. To prove the converse, let K be a (non-empty) compact metric space fulfilling (1) and (2) in the statement of the theorem and notice first that the family of connected components of K that are arcs is a null collection: for all $\varepsilon \in \mathbb{R}^+$ there are only finitely many of them of diameter greater than ε. Let $\widetilde{K} = \{x \in K \mid \operatorname{ord}(x, K) \leqslant 1\}$, $E = \{x \in K \mid \operatorname{ord}(x, K) = 1\}$; so $\widetilde{K}$ is obtained by removing all points of K that are interior of connected components of K that are arcs. To ease notation, let $p : E \to E$ assign to each end point of a connected component of K that is an arc the other end point. Note that $\widetilde{K}$ is a zero-dimensional compact subset of K and, by enlarging K, it may be assumed that $\widetilde{K}$ is perfect. So $\widetilde{K}$ is homeomorphic to the middle third Cantor set $C_{1/3} \subseteq [0,1]$; the idea is to build carefully a homeomorphism $\varphi : \widetilde{K} \to C_{1/3}$ in such a way that, if $x, y \in E$, $p(x) = y$, then the open interval $]\min(\varphi(x), \varphi(y)), \max(\varphi(x), \varphi(y))[$ is disjoint from $C_{1/3}$, so that finally the segment between $\varphi(x), \varphi(y)$ can be restored. To this aim a suitable Cantor scheme $\{C_t\}_{t\in 2^{<\omega}}$ on $\widetilde{K}$ will be constructed.

Let $\{Y_1, \dots, Y_h\}$ be a partition of $\widetilde{K}$ in finitely many clopen non-empty subsets of diameter less than $\frac{1}{2}$.

Claim. *For any clopen $Y \subseteq \widetilde{K}$, the set $\{x \in Y \mid x \in E \wedge p(x) \notin Y\}$ is finite.*

Proof. Deny and pick a sequence of distinct points $x_m \in Y \cap E$ such that $\forall m \in \mathbb{N}\ p(x_m) \notin Y$. By thinning, assume $\lim_{m\to\infty} x_m = x$. So $x \in Y$. Moreover, since the family of connected components of K that are arcs is null, $\lim_{m\to\infty} p(x_m) = x$, so eventually $p(x_m) \in Y$. □

Using the claim it is possible to refine the partition in such a way that each member contains at most one point $x \in E$ without containing the corresponding $p(x)$ too. Suppose indeed that $x_1, \dots, x_m$ are the distinct elements of $\{x \in Y_j \cap E \mid p(x) \notin Y_j\}$. Let $\widetilde{Y}_j$ be the decomposition space obtained from Y_j by identifying each $x, p(x)$ whenever both are in $Y_j \cap E$ and let $\pi : Y_j \to \widetilde{Y}_j$ be the natural map. Then $\widetilde{Y}_j$ is compact zero-dimensional: it is homeomorphic to the upper semicontinuous decomposition determined by the connected components of the space obtained from Y_j by restoring the original arc between every pair of points $x, p(x) \in Y_j \cap E$. So $\widetilde{Y}_j$ admits a partition into clopen sets such that $\pi(x_1), \dots, \pi(x_m)$ are in different elements of the partition. Consider the preimages under π of these sets.

So now $\widetilde{K}$ has been partitioned into sets $X_1, \dots, X_n$ such that:

(1) each X_i is clopen non-empty;
(2) each X_i has diameter less than $\frac{1}{2}$;
(3) for every i there is at most one $x_i \in E \cap X_i$ such that $p(x_i) \notin X_i$.

A clopen subset Z of $\widetilde{K}$ such that there is exactly one element $z \in E \cap Z$ with $p(z) \notin Z$ will be called *special*; the point z will be called the *characterizing* point of Z. Modulo reindexing, it can be assumed that in $X_1, \dots, X_n$ the special sets are listed first (say $X_1, \dots, X_{2k}$) and in such a way that if $x_1, \dots, x_{2k}$ are the corresponding characterizing points, then $p(x_{2j-1}) = x_{2j}$ for $1 \leqslant j \leqslant k$. Let $C_{0^i 1} = X_{i+1}$ for $0 \leqslant i < n-1$ and $C_{0^{n-1}} = X_n$. Set also $C_{0^i} = X_{i+1} \cup \dots \cup X_n$ for $0 \leqslant i < n-1$. Moreover, the points $x_1, x_3, \dots, x_{2k-1}$ get a label *left* and the points $x_2, x_4, \dots, x_{2k}$ get a label *right*. This will be retained throughout the construction and will indicate that, at subsequent partitionings, if a clopen set of the splitting contains x_i, for $i \leqslant 2k$ odd, as its (unique) already labeled element, then it will be listed last; if it contains x_i for $i \leqslant 2k$ even, it will be listed first (so it will appear, in the Cantor scheme, in the leftmost or rightmost position, respectively).

The next step will consist in repeating the process within each X_i, splitting it in clopen non-empty subsets of diameter less than $\frac{1}{3}$, taking care that at each stage of the construction the points x_{2j-1}, x_{2j} are kept adjacent, as well as for any other pair of end points of the connected components of K that are arcs that at some stage gets separated. So suppose that k stages of the construction have already been performed. The indices of the part of the Cantor scheme built so far form a finite binary tree; the terminal nodes are associated to finitely many clopen non-empty subsets of $\widetilde{K}$ of diameter less than $\frac{1}{k+1}$ forming a partition of $\widetilde{K}$. For each such Z le $t \in 2^{<\omega}$ be such that $C_t = Z$. Moreover, each such Z contains at most one element $z \in E$ with $p(z) \notin Z$ and, in this case, z has got a label *right* or *left*. Let $\{Z_1, \ldots, Z_l\}$ be a partition of Z in clopen subsets of diameter less that $\frac{1}{k+2}$ and such that each Z_r contains at most one element $z_r \in E$ with $p(z_r) \notin Z_r$. The only thing still needed is to decide the order in which the above list $Z_1, \ldots, Z_l$ is given, for making the next step in the construction of the Cantor scheme. There are three cases to consider.

Case 1: *Z is not special.* In this case proceed as at the first stage. List first, as $Z_1, \ldots, Z_{2q}$, the special sets of the partition in such a way that, if $z_1, \ldots, z_{2q}$ are the corresponding characterizing points, $p(z_1) = z_2$, $p(z_3) = z_4, \ldots, p(z_{2q-1}) = z_{2q}$. Assign label *left* to $z_1, z_3, \ldots, z_{2q-1}$ and label *right* to $z_2, z_4, \ldots, z_{2q}$.

Case 2: *Z is special and its characterizing point z has got label left.* Note that now there is an odd number of special sets in the partition of Z. It may be thus assumed that these are $Z_1, Z_2, \ldots, Z_{2q}, Z_l$ and, if $z_1, \ldots, z_q, z_l$ are their characterizing points, then $p(z_1) = z_2$, $p(z_3) = z_4, \ldots, p(z_{2q-1}) = z_{2q}$, $z_l = z$. Give $z_1, z_3, \ldots, z_{2q-1}$ label *left*, $z_2, z_4, \ldots, z_{2q}$ label *right* ($z_l = z$ retains its label *left*).

Case 3: *Z is special and its characterizing point z has got label right.* Again there is an odd number of special sets in the partition of Z. The listing is now arranged in such a way that $Z_1, Z_2, Z_3, \ldots, Z_{2q+1}$ are these special sets and, if $z_1, z_2, \ldots, z_{2q+1}$ are the corresponding characterizing points, then $z_1 = z$, $p(z_2) = z_3$, $p(z_4) = z_5, \ldots, p(z_{2q}) = z_{2q+1}$. The points $z_2, z_4, \ldots, z_{2q}$ get label *left*, while $z_3, z_5, \ldots, z_{2q+1}$ are marked *right* (and $z_1 = z$ keeps its label *right*).

In any of the three cases let $C_{t^\frown 0^r 1} = Z_{r+1}$ for $0 \leqslant r < l-1$ and let $C_{t^\frown 0^{l-1}} = Z_l$. Set also $C_{t^\frown 0^r} = Z_{r+1} \cup \cdots \cup Z_l$ for $0 \leqslant r < l-1$.

This way, each point of E gets labeled in the course of the construction.

Notice that $\forall \alpha \in 2^{\mathbb{N}}$ $\bigcap_{n \in \mathbb{N}} C_{\alpha|_n}$ is a singleton $\{x_\alpha\}$ and this defines a homeomorphism $\widetilde{K} \to 2^{\mathbb{N}}$, $x_\alpha \mapsto \alpha$. Composing this with the usual homeomorphism $2^{\mathbb{N}} \to C_{1/3}$ via ternary expansion, one gets the homeomorphism $\varphi : \widetilde{K} \to C_{1/3} \subseteq [0,1]$.

Now let $x \in E$. Suppose x got label *left* during the construction (same argument if it got label *right*). Then note that $p(x)$ got label *right* at the same stage. Let $u, u' \in 2^{<\omega}$ be such that $C_u, C_{u'}$ were the special sets appearing in the construction at the stage when x, $p(x)$ got their labels and having x, $p(x)$ as their respective characterizing points. Then either $u = t^\frown 1$, $u' = t^\frown 01$ for some $t \in 2^{<\omega}$ or $u = t^\frown 1$, $u' = t^\frown 0$ for some $t \in 2^{<\omega}$ (this case happens when the two special sets are the last ones listed in the corresponding partition). In both cases, $x = x_\alpha$, $p(x) = x_{\alpha'}$ where $\alpha = t^\frown 10^\infty$, $\alpha' = t^\frown 01^\infty$. Thus $\varphi p(x) < \varphi(x)$ and no points of $\operatorname{im}\varphi = C_{1/3}$ are between them, so $\varphi p(x), \varphi(x)$ are the extrema of one of the intervals that get deleted in the construction of the Cantor set $C_{1/3}$. For each such pair x, $p(x)$ restore the interval between them, thus getting

$$T = C_{\frac{1}{3}} \cup \bigcup_{x \in E} \bigl[\min\bigl(\varphi(x), \varphi p(x)\bigr), \max\bigl(\varphi(x), \varphi p(x)\bigr)\bigr].$$

Matching each connected component of K that is an arc with end points x, $p(x)$ with the corresponding $[\min(\varphi(x), \varphi p(x)), \max(\varphi(x), \varphi p(x))]$, the homeomorphism φ can be extended to an embedding $\psi : K \to [0,1]$ (whose range is T). □

A somewhat similar construction, involving the building of a specific embedding of a zero-dimensional separable metric space into the Cantor space, is in [7].

The following easy fact will be employed in the sequel.

Lemma 21. *If X is a connected Polish space with* $\mathrm{card}(X) > 1$ *and Y is a closed, zero-dimensional subspace of X, then Y is nowhere dense in X.*

Proof. If not, let U be a non-empty open subset of X with $U \subseteq Y$. There is then V non-empty clopen in Y such that $V \subseteq U$ (note that $V \neq X$ since $Y \neq X$). As V is closed in Y, it is closed in X too; moreover, being V open in Y, it is open in U and so open in X, contradicting connectedness of X. □

Corollary 22. *Let H be a non-degenerate continuum and, for each $m \in \mathbb{N}$, let K_m be closed in H. Suppose $H = \bigcup_{m\in\mathbb{N}} K_m$. Then, for some $\bar{m} \in \mathbb{N}$, $H \cap K_{\bar{m}}$ contains a non-degenerate continuum.*

Proof. Otherwise each $H \cap K_m$ would be zero-dimensional and thus, by Lemma 21, nowhere dense in H, contradicting Baire category theorem. □

Lemma 23. *Let $\emptyset \neq \mathcal{F} \subseteq C(\mathbb{R}^{\mathbb{N}})$ and let $Y \in \mathcal{F}^{\sigma}$ be a non-degenerate continuum. Then every open non-empty subset of Y contains a non-degenerate continuum homeomorphic to a subcontinuum of some element of $\mathcal{F}$.*

Proof. By Corollary 16, $Y \subseteq \bigcup_{n\in\mathbb{N}} Y_n$ where each Y_n is homeomorphic to some element $X_n \in \mathcal{F}$. Let $U \subseteq Y$ be open non-empty and let Z be a non-degenerate subcontinuum of U. By Corollary 22, there is $n \in \mathbb{N}$ such that $Z \cap Y_n$ contains a non-degenerate continuum W, which is thus homeomorphic to a subcontinuum of X_n. □

Lemma 24. *Let $Z \in C(\mathbb{R}^{\mathbb{N}})$, $\emptyset \neq \mathcal{F} \subseteq C(\mathbb{R}^{\mathbb{N}})$. Then $Z \in \mathcal{F}^{\perp}$ if and only if, for all $X \in \mathcal{F}$, the continua X, Z do not have common* (*up to homeomorphism*) *non-degenerate subcontinua.*

Proof. If Y is a non-degenerate continuum embeddable both in Z and in some continuum $X \in \mathcal{F}$, then $Y \in Z^{\sigma} \cap X^{\sigma} \subseteq Z^{\sigma} \cap \mathcal{F}^{\sigma}$.

Conversely, let $Y \in Z^{\sigma} \cap \mathcal{F}^{\sigma}$ be a non-degenerate continuum. By Lemma 23, Y contains a non-degenerate subcontinuum W embeddable in Z. Since $W \in \mathcal{F}^{\sigma}$, by Lemma 23 again W contains a non-degenerate subcontinuum T embeddable in some $X \in \mathcal{F}$ (and in Z too). □

Theorem 25. *Let $\emptyset \neq \mathcal{F} \subseteq C(\mathbb{R}^{\mathbb{N}})$. Then*:

(1) *$\mathcal{F}^{\perp}$ is an invariant σ-ideal of continua*;
(2) *$\mathcal{F} \subseteq \mathcal{F}^{\perp\perp}$* (*consequently $\mathcal{F}^{\sigma} \subseteq \mathcal{F}^{\perp\perp}$*) *and $\mathcal{F}^{\perp\perp\perp} = \mathcal{F}^{\perp}$*;
(3) *if $\mathcal{F} \in \mathbf{\Sigma}^1_1(C(\mathbb{R}^{\mathbb{N}}))$ then $\mathcal{F}^{\perp}$ is coanalytic.*

Proof. (1) If $Z \in \mathcal{F}^{\perp}$ and Y is a subcontinuum of Z, then $Y^{\sigma} \cap \mathcal{F}^{\sigma}$ does not contain non-degenerate continua, so $Y \in \mathcal{F}^{\perp}$.

Assume $\forall n \in \mathbb{N}\ Z_n \in \mathcal{F}^{\perp}$ be such that $Z = \bigcup_{n \in \mathbb{N}} Z_n$ is a continuum and suppose there is a non-degenerate subcontinuum $Y \subseteq Z$ that is embeddable in some $X \in \mathcal{F}$. By Corollary 22, for some $n \in \mathbb{N}$, $Y \cap Z_n$ must contain a non-degenerate subcontinuum T, which is thus embeddable in X, contradicting $Z_n \in \mathcal{F}^{\perp}$. By Lemma 24, $Z \in \mathcal{F}^{\perp}$.

Invariance under homeomorphism follows from Lemma 24 (or directly from the definition).

(2) By Lemma 24.

(3) By Lemma 24 and the fact that homeomorphism is an analytic relation on $C(\mathbb{R}^{\mathbb{N}})$. □

Corollary 26. *If $\mathcal{F} \subseteq C(\mathbb{R}^{\mathbb{N}})$ is an analytic class of continua having a member containing a pseudo-arc but there is a non-degenerate planar continuum Z such that no element of $\mathcal{F}$ contains a subcontinuum homeomorphic to a non-degenerate subcontinuum of Z, then $\mathcal{F}^{\perp}$ is a complete coanalytic subset of $C(\mathbb{R}^{\mathbb{N}})$.*

Proof. By Theorem 25, Corollary 9 and Lemma 24. □

The result in the particular case when $\mathcal{F}$ has the pseudo-arc as its unique element has been already obtained in [10].

Corollary 27. *Let $NHI \subseteq C(\mathbb{R}^{\mathbb{N}})$ be the set of continua containing no hereditarily indecomposable subcontinuum. Then NHI is $\mathbf{\Pi}^1_1$-complete.*

Proof. By Corollary 26 applied to the family $\mathcal{F}$ of hereditarily indecomposable continua, which is a G_δ subset of $C(\mathbb{R}^{\mathbb{N}})$ (see [8]). □

4. The maximal invariant σ-ideal of continua

For $n \geqslant 1$ let

$$\mathcal{I}_n = \{X \in C(\mathbb{R}^{\mathbb{N}}) \mid [0,1]^n \text{ does not embed into } X\}$$

and let

$$\mathcal{I}_\infty = \{X \in C(\mathbb{R}^{\mathbb{N}}) \mid [0,1]^{\mathbb{N}} \text{ does not embed into } X\}.$$

Then $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_3 \subseteq \cdots \subseteq \mathcal{I}_\infty$. Using the fact that, for $1 \leqslant n \leqslant \infty$, each open non-empty subset of $[0,1]^n$ contains a subcontinuum homeomorphic to $[0,1]^n$ and applying Baire category theorem, it turns out that each $\mathcal{I}_n$ is an invariant σ-ideal of continua and it is coanalytic by direct computation on the definition. Observe moreover that for $2 \leqslant n \leqslant \infty$, $\mathcal{I}_n$ is *big*, in the sense that its orthogonal $\mathcal{I}_n^{\perp}$ is just the class of singletons $F_1(\mathbb{R}^{\mathbb{N}})$ (this follows from the fact that $[0,1]^n$ contains subcontinua in which $[0,1]^n$ does not embed).

If $\mathcal{I}$ is an invariant σ-ideal of continua that does not coincide with the entire $C(\mathbb{R}^{\mathbb{N}})$, then $\mathcal{I} \subseteq \mathcal{I}_\infty$. So $\mathcal{I}_\infty$ is the maximal invariant σ-ideal of continua. In [6] it is proved that $\mathcal{I}_\infty$ is complete coanalytic.

Let $\mathcal{F}$ be an invariant hereditary class of continua and X be a continuum. Then X is *universal* for $\mathcal{F}$ if, for any continuum Y,

$$Y \in \mathcal{F} \iff Y \text{ can be continuously embedded in } X$$

(in particular, $X \in \mathcal{F}$). For example, the Hilbert cube, as well as any continuum containing a Hilbert cube, is universal for the class of all continua.

Proposition 28. *There is no universal continuum for the class of non-universal continua.*

Proof. For X a continuum, $\{Y \in C(\mathbb{R}^{\mathbb{N}}) \mid X \text{ contains a homeomorphic copy of } Y\}$ is analytic. Since $\mathcal{I}_\infty$ is the family of non-universal continua, the result follows from $\mathbf{\Pi}^1_1$-completeness of $\mathcal{I}_\infty$. □

Proposition 28 can be restated by saying that the preorder of embeddability among continua, which has a biggest element, namely the (biembeddability equivalence class of the) Hilbert cube, does not have a second biggest element. This implies also that if the continuum X does not contain a Hilbert cube, then there is a continuum Y strictly between X and $[0,1]^{\mathbb{N}}$ in the preorder of embeddability. Indeed, by Proposition 28 there is a non-universal Z not embeddable into X. Let Y ba a non-disjoint union of X and Z.

In $\mathcal{L}$ the chain $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \mathcal{I}_3 \subset \cdots$ is not enough to approach the maximal invariant σ-ideal $\mathcal{I}_\infty$.

Theorem 29. $\sup\{\mathcal{I}_n\}_{n \geqslant 1} \neq \mathcal{I}_\infty$.

Proof. For each $n \geqslant 1, h \in \mathbb{N}$, let $p_{nh} : [0,1] \to [0,1]^n$ be a $\frac{1}{n+h}$-map. Moreover let $\mathbb{Q} \cap]0,1[= \{q_{nh} \mid n \geqslant 1,\ h \in \mathbb{N}\}$ with each $Q_n = \{q_{nh}\}_{h \in \mathbb{N}}$ dense in $\mathbb{Q} \cap]0,1[$. Consider the upper semi-continuous decomposition M of $[0,1]^2$ whose elements are:

- the sets $\{q_{nh}\} \times p_{nh}^{-1}(\{a\})$, for $n \geqslant 1, h \in \mathbb{N}, a \in [0,1]^n$;
- the singletons $\{(t,x)\}$ for $t \notin \mathbb{Q} \cap]0,1[, x \in [0,1]$.

Let $\pi : [0,1]^2 \to M$ be the quotient map and define $\forall t \in [0,1]\ M_t = \pi(\{t\} \times [0,1])$. So

- for $n \geqslant 1, h \in \mathbb{N}$, $M_{q_{nh}}$ is an n-cell;
- for $t \notin \mathbb{Q} \cap]0,1[$, M_t is an arc.

Now it remains to show that $M \in \mathcal{I}_\infty \setminus \sup\{\mathcal{I}_n\}_{n \geqslant 1}$.

Claim. *If a continuum X is such that every non-empty open subset contains m-cells for unbounded values of m, then $X \notin \sup\{\mathcal{I}_n\}_{n \geqslant 1}$.*

Proof. Let $\mathcal{F} = \bigcup_{n=1}^\infty \mathcal{I}_n$, so that $\sup\{\mathcal{I}_n\}_{n \geqslant 1} = \mathcal{F}^\sigma$. If $X \in \mathcal{F}^\sigma$, by Corollary 16, $X \subseteq \bigcup_{n \in \mathbb{N}} X_n$, where $\forall n \in \mathbb{N}\ X_n \in \mathcal{F}$. Baire category theorem implies the existence of $n \in \mathbb{N}$ such that $X \cap X_n$ contains an open non-empty subset of X and so m-cells for unbounded values of m, contrary to $X_n \in \mathcal{F}$. □

Claim. *Every non-empty open subset U of M contains cells of unbounded dimension.*

Proof. Let $V = \pi^{-1}(U)$. Then V is a non-empty open subset of $[0,1]^2$, invariant with respect to the equivalence relation associated with the decomposition. Let $u, v, y \in [0,1]$ be such that $u < v$, $[u,v] \times \{y\} \subseteq V$. For any $n \geqslant 1$ there is $h \in \mathbb{N}$ such that $q_{nh} \in\,]u,v[$. Thus $\pi((\{q_{nh}\} \times [0,1]) \cap V)$ (which is contained in U) is an open non-empty subset of the n-cell $\pi(\{q_{nh}\} \times [0,1])$ and therefore contains an n-cell. □

Having established that $M \notin \sup\{\mathcal{I}_n\}_{n \geqslant 1}$, the proof will be completed by the following claim.

Claim. *No subcontinuum of M is homeomorphic to the Hilbert cube.*

Proof. Let $Q \subseteq M$ be homeomorphic to the Hilbert cube. Since Q cannot be contained in M_t for any $t \in [0,1]$, the set of $t \in [0,1]$ such that $Q \cap M_t \neq \emptyset$ is an interval $[a,b]$. If $t \in\,]a,b[\setminus \mathbb{Q}$ then M_t is an arc and $Q \cap M_t$ separates Q, which is impossible. □

Concerning Theorem 29 and the remark preceding it, it would be interesting to know the cofinality of $\mathcal{I}_\infty$ in $\mathcal{L}$.

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