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An Existence Result for First-Order Impulsive Functional Differential Equations in Banach Spaces

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Abstract—In this paper, the Leray-Schauder nonlinear alternative is used to investigate the existence of solutions to first-order impulsive initial value problems for functional differential equations in Banach spaces.

Keywords—Impulsive functional differential equations, Fixed point, Banach space.

1. INTRODUCTION

This paper is concerned with the existence of solutions for the initial value problem for functional differential equations with impulsive effects as

$$y' = f(t, y_t), \qquad t \in J = [0,T], \quad t \neq t_k, \quad k = 1, \dots, m, \tag{1}$$

$$\Delta y|_{t=t_k} = I_k\left(y\left(t_k^-\right)\right), \qquad k = 1, \dots, m, \tag{2}$$

$$y(t) = \phi(t), \qquad t \in [-r, 0], \tag{3}$$

where $f : J \times C([-r,0],E) \longrightarrow E$ is a given function, $\phi \in C([-r,0],E)$, $(0 < r < \infty)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(E,E)$ $(k = 1,2,\dots,m)$, are bounded, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-)$, and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively, and E a real Banach space with norm $|\cdot|$.

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For any continuous function y defined on the interval $[-r, T]$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], E)$ defined by

$$y_t(\theta) = y(t + \theta), \qquad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

Impulsive differential equations have become more important in recent years in some mathematical models of real world phenomena, especially in the biological or medical domain (see the monographs of Bainov and Simeonov [1], Lakshmikantham, Bainov and Simeonov [2], and Samoilenko and Perestyuk [3], and the papers of Agur, Cojocaru, Mazur, Anderson and Danon [4], Goldbeter, Li and Dupont [5]).

Very recently, an extension to functional differential equations with impulsive effects has been done by Yujun [6] by using the coincidence degree theory. For other results on functional differential equations, we refer the interested reader to the monograph of Erbe, Kong and Zhang [7], Hale [8], Henderson [9], and the survey paper of Ntouyas [10].

The fundamental tools used in the existence proofs of all above-mentioned works are essentially fixed-point arguments, nonlinear alternative, topological transversality [11], topological degree theory [12], or the monotone method combined with upper and lower solutions [13].

In this paper, we shall generalize the results of Frigon and O'Regan [14], which they considered for scalar impulsive differential equations. Our approach is based on the Leray-Schauder alternative [11].

This paper will be divided into three sections. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout Section 3. In Section 3, we shall establish an existence theorem for (1)–(3).

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

$C([-r, 0], E)$ is the Banach space of all continuous functions from $[-r, 0]$ into E with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

By $C(J, E)$, we denote the Banach space of all continuous functions from J into E with the norm

$$\|y\|_J := \sup\{|y(t)| : t \in J\}.$$

A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see for instance, [15].)

$L^1(J, E)$ denotes the Banach space of functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^T |y(t)|\, dt, \qquad \text{for all } y \in L^1(J, E).$$

In order to define the solution of (1)–(3), we shall consider the following spaces:

$$\Omega = \big\{y : [-r, T] \longrightarrow E : y_k \in C(J_k, E),\ k = 0, \dots, m, \text{ and there exist} \\ y\left(t_k^-\right) \text{ and } y\left(t_k^+\right),\ k = 1, \dots, m \text{ with } y\left(t_k^-\right) = y\left(t_k\right),\ y(t) = \phi(t),\ \forall\, t \in [-r, 0]\big\},$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_J,\ k = 0, \dots, m\},$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}]$, $k = 0, \dots, m$.

We shall also consider the set

$$\Omega^1 = \{y : J \longrightarrow E : y_k \in W^{1,1}(J_k, E), k = 0, \dots, m, \text{ and there exist } y\left(t_k^-\right) \text{ and } y\left(t_k^+\right),\ k = 1, \dots, m \text{ with } y\left(t_k^-\right) = y(t_k)\right\},$$

where $W^{1,1}(J_k, E)$ is the Sobolev class of absolutely continuous functions $y : J_k \longrightarrow E$.

The set Ω^1 is a Banach space with the norm

$$\|y\|_{\Omega^1} = \max\{\|y_k\|_{W^{1,1}(J_k,E)},\ k = 0, \dots, m\}.$$

DEFINITION 2.1. *A map $f : J \times C([-r, 0], E) \longrightarrow E$ is said to be an L^1-Carathéodory if*

(i) *$t \longmapsto f(t, u)$ is measurable for each $u \in C([-r, 0], E)$;*
(ii) *$u \longmapsto f(t, u)$ is continuous for almost all $t \in J$;*
(iii) *for each $k > 0$, there exists $g_k \in L^1(J, \mathbb{R}_+)$ such that*

$$|f(t, u)| \leq g_k(t), \qquad \text{for all } \|u\| \leq k \text{ and for almost all } t \in J.$$

So let us start by defining what we mean by a solution of problem (1)-(3).

DEFINITION 2.2. *A function $y \in \Omega \cap \Omega^1$ is said to be a solution of (1)–(3) if y satisfies the equation $y'(t) = f(t, y_t)$ a.e. on $J - \{t_1, \dots, t_m\}$ and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, \dots, m$.*

Our main result is based on the following.

LEMMA 2.3. NONLINEAR ALTERNATIVE. *(See [11].) Let X be a Banach space with $C \subset X$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $G : \overline{U} \longrightarrow C$ is a compact map. Then either,*

(i) *G has a fixed point in $\overline{U}$; or*
(ii) *there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda G(u)$.*

REMARK 2.4. By $\overline{U}$ and ∂U, we denote the closure of U and the boundary of U, respectively.

Let us introduce the following hypotheses which are assumed hereafter:

(H1) $f : J \times C([-r, 0], E) \longrightarrow E$ is an L^1-Carathéodory map;
(H2) there exists a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that

$$|f(t, u)| \leq p(t)\psi(\|u\|), \qquad \text{for a.e. } t \in J \text{ and each } u \in C([-r, 0], E)$$

with

$$\int_{t_{k-1}}^{t_k} p(s)\, ds < \int_{N_{k-1}}^{\infty} \frac{du}{\psi(u)}, \qquad k = 1, \dots, m + 1.$$

Here $N_0 = \|\phi\|$ and for $k = 2, \dots, m + 1$, we have

$$N_{k-1} = \sup_{y \in [-M_{k-2}, M_{k-2}]} |I_{k-1}(y)| + M_{k-2}, \qquad M_{k-2} = \Gamma_{k-1}^{-1}\left(\int_{t_{k-2}}^{t_{k-1}} p(s)\, ds\right)$$

with

$$\Gamma_l(z) = \int_{N_{l-1}}^{z} \frac{du}{\psi(u)}, \qquad z \geq N_{l-1}, \quad l \in \{1, \dots, m + 1\};$$

(H3) for each bounded $B \subseteq C([-r, T], E)$ and $t \in J$ the set

$$\left\{\phi(0) + \int_0^t f(s, y_s)\, ds + \sum_{0<t_k<t} I_k(y(t_k)) : y \in B\right\}$$

is relatively compact in E.

We have the following auxiliary result. In what follows, we will use the notation $\sum_{0<t_k<t}[y(t_k^+) - y(t_k)]$ to mean 0, when $k = 0$ and $0 < t < t_1$, and to mean $\sum_{i=1}^{k}[y(t_k^+) - y(t_k)]$, when $k \geq 1$ and $t_k < t \leq t_{k+1}$.

LEMMA 2.5. *If $y \in \Omega^1$, then*

$$y(t) = y(0) + \int_0^t y'(s)\,ds + \sum_{0<t_k<t} \left[y\left(t_k^+\right) - y(t_k) \right], \qquad \text{for } t \in J. \tag{4}$$

PROOF. Assume that $t_k < t \le t_{k+1}$ (here $t_0 = 0$, $t_{m+1} = T$). Then

$$\begin{aligned} y(t_1) - y(0) &= \int_0^{t_1} y'(s)\,ds, \\ y(t_2) - y\left(t_1^+\right) &= \int_{t_1}^{t_2} y'(s)\,ds, \\ &\vdots \\ y(t_k) - y\left(t_{k-1}^+\right) &= \int_{t_{k-1}}^{t_k} y'(s)\,ds, \\ y(t) - y\left(t_k^+\right) &= \int_{t_k}^{t} y'(s)\,ds. \end{aligned}$$

Adding these together, we get

$$y(t) - y(0) - \sum_{i=1}^{k} \left[y\left(t_i^+\right) - y(t_i) \right] = \int_0^t y'(s)\,ds,$$

i.e., equation (4) holds.

3. MAIN RESULT

THEOREM 3.1. *A priori* BOUNDS ON SOLUTIONS. *There exist constants $M_0, \dots, M_m$ such that, if $y \in \Omega^1$ is a solution of (1)–(3), then*

$$\sup\{|y(t)| : t \in [t_{k-1}, t_k]\} \le M_{k-1}, \qquad k = 1, \dots, m+1.$$

PROOF. Let y be a (possible) solution to (1)–(3). Then $y|_{[-r,t_1]}$ is a solution to

$$\begin{aligned} y'(t) &= f(t, y_t), &&\text{for a.e. } t \in (0, t_1), \\ y(t) &= \phi(t), && t \in [-r, 0]. \end{aligned}$$

Then for each $t \in [0, t_1]$

$$y(t) - \phi(0) = \int_0^t f(s, y_s)\,ds.$$

From (H2), we get

$$|y(t)| \le \|\phi\| + \int_0^t p(s)\psi(\|y_s\|)\,ds, \qquad t \in [0, t_1].$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \qquad 0 \le t \le t_1.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, t_1]$, by combining $\|y_t\| \le \mu(t)$, for $t \in [0, t_1]$ with the previous inequality, we have for $t \in [0, t_1]$

$$\mu(t) \le \|\phi\| + \int_0^t p(s)\psi(\mu(s))\,ds.$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$c = v(0) = \|\phi\|, \qquad \mu(t) \leq v(t), \qquad t \in [0, t_1]$$

and

$$v'(t) = p(t)\psi(\mu(t)), \qquad t \in [0, t_1].$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq p(t)\psi(v(t)), \qquad t \in [0, t_1].$$

This implies for each $t \in [0, t_1]$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \int_0^{t_1} p(s)\, ds.$$

In view of (H2), we obtain

$$|v(t^*)| \leq \Gamma_1^{-1}\left(\int_0^{t_1} p(s)\, ds\right) := M_0.$$

Since for every $t \in [0, t_1]$, $|y(t)| \leq \mu(t) \leq v(t)$, we have

$$\sup_{t \in [0, t_1]} |y(t)| \leq M_0.$$

Now, $y|_{[t_1, t_2]}$ is a solution to

$$\begin{aligned} y'(t) &= f(t, y_t), \qquad \text{for a.e. } t \in (t_1, t_2), \\ y\left(t_1^+\right) &= I_1(y(t_1)) + y(t_1). \end{aligned} \tag{6}$$

Note that

$$\begin{aligned} \left|y\left(t_1^+\right)\right| &\leq \sup_{y \in [-M_0, +M_0]} |I_1(y)| + \sup_{t \in [0, t_1]} |y(t)| \\ &\leq \sup_{y \in [-M_0, +M_0]} |I_1(y)| + M_0 := N_1. \end{aligned}$$

Then, for each $t \in [t_1, t_2]$

$$y(t) - y\left(t_1^+\right) = \int_{t_1}^{t} f(s, y_s)\, ds.$$

From (H2), we get

$$|y(t)| \leq N_1 + \int_0^t p(s)\psi(\|y_s\|)\, ds, \qquad t \in [t_1, t_2].$$

We consider the function μ_1 defined by

$$\mu_1(t) = \sup\{|y(s)| : t_1 \leq s \leq t\}, \qquad t_1 \leq t \leq t_2.$$

Let $t^* \in [t_1, t_2]$ be such that $\mu_1(t) = |y(t^*)|$. By combining $\|y_t\| \leq \mu_1(t)$, for $t \in [t_1, t_2]$ with the previous inequality, we have for $t \in [t_1, t_2]$

$$\mu_1(t) \leq N_1 + \int_{t_1}^{t} p(s)\psi(\mu_1(s))\, ds.$$

Let us take the right-hand side of the above inequality as $v_1(t)$, then we have

$$c = v_1(0) = N_1, \qquad \mu_1(t) \le v_1(t), \qquad t \in [t_1, t_2],$$

and

$$v_1'(t) = p(t)\psi(\mu_1(t)), \qquad t \in [t_1, t_2].$$

Using the nondecreasing character of ψ, we get

$$v_1'(t) \le p(t)\psi(v_1(t)), \qquad t \in [t_1, t_2].$$

This implies for each $t \in [t_1, t_2]$ that

$$\int_{v_1(0)}^{v_1(t)} \frac{du}{\psi(u)} \le \int_{t_1}^{t_2} p(s)\, ds.$$

In view of (H2), we obtain

$$|v_1(t^*)| \le \Gamma_1^{-1}\left(\int_{t_1}^{t_2} p(s)\, ds\right) := M_1.$$

Since for every $t \in [t_1, t_2], |y(t)| \le \mu_1(t) \le v_1(t)$, we have

$$\sup_{t\in[t_1,t_2]} |y(t)| \le M_1.$$

We continue this process and taking into account that $y|_{[t_m,T]}$ is a solution to the problem

$$\begin{aligned} y'(t) &= f(t, y_t), \qquad \text{for a.e. } t \in (t_m, T), \\ y\left(t_m^+\right) &= I_m(y(t_m)) + y(t_m). \end{aligned} \tag{7}$$

We obtain that there exists a constant M_m, such that

$$\sup_{t\in[t_m,T]} |y(t)| \le \Gamma_{m+1}^{-1}\left(\int_{t_m}^{T} p(s)\, ds\right) := M_m.$$

Consequently, for each possible solution y to (1)–(3), we have

$$\|y\|_\Omega \le \max\{\|\phi\|, M_{k-1} : k = 1, \dots, m+1\} := b.$$

Now, we are in position to state and prove our main result.

THEOREM 3.2. *Suppose that Hypotheses (H1)–(H3) are satisfied. Then the impulsive initial value problem (1)–(3) has at least one solution on* $[-r, T]$.

PROOF. Transform the problem into a fixed-point problem. Consider the map, $G : \Omega \longrightarrow \Omega$ defined by

$$(Gy)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0) + \displaystyle\int_0^t f(s, y_s)\, ds + \sum_{0<t_k<t} I_k(y(t_k)), & t \in J. \end{cases}$$

REMARK 3.3. Clearly, from Lemma 2.5, the fixed points of G are solutions to (1)–(3).

We shall show that G satisfies the assumptions of Lemma 2.3. The proof will be given in several steps.

STEP 1. G maps bounded sets into bounded sets in $C([-r, T], E)$.

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $y \in B_q = \{y \in C([-r,T],E) : \|y\|_\infty \leq q\}$ one has $\|Gy\|_\infty \leq \ell$.

Let $y \in B_q$, then for each $t \in J$, we have

$$(Gy)(t) = \phi(0) + \int_0^t f(s, y_s)\, ds + \sum_{0<t_k<t} I_k(y(t_k)).$$

By (H1), we have for each $t \in J$

$$\begin{aligned} |(Gy)(t)| &\leq \|\phi\| + \int_0^t |f(s,y_s)|\, ds + \sum_{0<t_k<t} |I_k(y(t_k))| \\ &\leq \|\phi\| + \int_0^t |g_q(s)|\, ds + \sum_{k=1}^{m} \sup\{|I_k(|y|)| : \|y\|_\infty \leq q\}. \end{aligned}$$

Thus,

$$\|Gy\|_\infty \leq \|\phi\| + \int_0^T |g_q(s)|\, ds + \sum_{k=1}^{m} \sup\{|I_k(|y|)| : \|y\|_\infty \leq q\} = \ell.$$

STEP 2. G maps bounded set into equicontinuous sets of $C([-r,T],E)$.

Let $r_1, r_2 \in J$, $r_1 < r_2$, and $B_q = \{y \in C([-r,T],E) : \|y\|_\infty \leq q\}$ be a bounded set of $C([-r,T],E)$. Let $y \in B_q$. Then

$$|(Gy)(r_2) - (Gy)(r_1)| \leq \int_{r_1}^{r_2} |g_q(s)|\, ds + \sum_{0<t_k<r_2-r_1} |I_k(y(t_k))|.$$

As $r_2 \longrightarrow r_1$, the right-hand side of the above inequality tends to zero.

The equicontinuity for the cases $r_1 < r_2 \leq 0$ and $r_1 \leq 0 \leq r_2$ is obvious.

STEP 3. $G : C([-r,T],E) \longrightarrow C([-r,T],E)$ is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in $C([-r,T],E)$. Then there is an integer q such that $\|y_n\|_\infty \leq q$ for all $n \in \mathbb{N}$ and $\|y\|_\infty \leq q$, so $y_n \in B_q$ and $y \in B_q$.

We have then, by the dominated convergence theorem,

$$\|Gy_n - Gy\|_\infty \leq \sup_{t\in J} \left[\int_0^t |f(s,y_{ns}) - f(s,y_s)|\, ds + \sum_{0<t_k<t} |I_k(y_n(t_k)) - I_k(y(t_k))| \right] \longrightarrow 0.$$

Thus, G is continuous.

Set

$$U = \{y \in \Omega : \|y\|_\Omega < b+1\},$$

where b is defined in the proof of Theorem 3.1.

As a consequence of Step 2, Step 3, and (H3) together with the Ascoli-Arzela theorem, we can conclude that the map $G : \overline{U} \longrightarrow \Omega$ is compact.

From the choice of U there is no $y \in \partial U$ such that $y = \lambda G y$ for any $\lambda \in (0,1)$.

As a consequence of Lemma 2.3, we deduce that G has a fixed point $y \in \overline{U}$ which is a solution of (1)-(3).

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