Dynamic analysis of suspended cables carrying moving oscillators

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Abstract

An efficient procedure for analyzing in-plane vibrations of flat-sag suspended cables carrying an array of moving oscillators with arbitrarily varying velocities is presented. The cable is modelled as a mono-dimensional elastic continuum, fully accounting for geometrical nonlinearities. By eliminating the horizontal displacement component through a standard condensation procedure, the nonlinear integro-differential equation governing vertical cable vibrations is derived. Due to the dynamic interaction at the contact points with the moving oscillators, such equation is coupled to the set of ordinary differential equations ruling the response of the travelling sub-systems. An improved series representation of vertical cable displacement is proposed, which allows to overcome the inability of the traditional Galerkin method to reproduce the kinks and abrupt changes of cable configuration at the interface with the moving sub-systems. Following the philosophy of the well-known “mode-acceleration” method, the convergence of the series expansion of cable response in terms of appropriate basis functions is improved through the introduction of the so-called “quasi-static” solution. Numerical results demonstrate that, despite the basis functions are continuous, the improved series enables to capture with very few terms the abrupt changes of cable profile at the contact points between the cable and the moving oscillators.

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1. Introduction

The dynamics of suspended cables has attracted the attention of researchers since the 18th century (Irvine, 1981). Indeed, cables are used in many engineering applications that require long, flexible and lightweight structural elements, such as electrical and optical signal transmission lines, tethers and umbilicals, aerial cable-ways, cable supported bridges, etc. Some studies have focused on the vibration analysis of suspended cables with attached masses (Al-Qassab and Nair, 2004; Cheng and Perkins, 1992a,b, 1994; Lin and Perkins, 1995;
In many engineering applications, such as aerial cableways or tramways, the analysis of cable vibrations is even more difficult since the dynamics of a nonlinear continuum (the cable) coupled to moving sub-systems (cabin or payloads) has to be investigated. As well-known, several studies have been devoted in the past to the vibration analysis of elastic structures under moving loads (Frýba, 1999; Tzou and Bergman, 1998). The dynamics of stretched strings has attracted significant attention (Frýba, 1999; Gavriloj, 1999; Metrikine, 2004; Rodeman et al., 1976; Tait and Duncan, 1992; Tzou and Bergman, 1998; Wickert and Mote, 1991; Yen and Tang, 1970; Zhu et al., 1994), while, to the authors’ knowledge, comparatively much less studies have focused on the response of sagged cables to moving loads. Wu and Chen (1989) investigated the dynamic behaviour of an extensible cable suspended between level supports under a moving load. They used the Updated Lagrangian formulation and the finite element method to derive the equation of motion, properly including the convective acceleration terms. Then, the dynamic response of the cable was determined by the Newmark direct integration method together with the Newton–Raphson iterative procedure. More recently, Wang (2000) presented an analytical and numerical study of the transient vibrations of a taut inclined cable with a riding accelerating mass modelled as a rigid body. The differential equations of motion were derived by superposing small displacements on the catenary state of the cable. The spatial dependence was eliminated by the Galerkin method which, however, allows to determine only approximately the kinks and abrupt changes in cable configuration. Indeed, the displacement components are approximated as continuous functions. In order to overcome this drawback, the authors (Muscolino and Sofi, 2003, 2006) proposed an alternative series representation of cable response to moving masses, which possesses better convergence than the conventional one in terms of eigenfunctions of the associated linear problem. The underlying idea stems from the well-known “mode-acceleration” method (Maddox, 1975), which has already been exploited in the literature (see, e.g. Biondi et al., 2004; Pesterev and Bergman, 2000; Pesterev et al., 2001) to improve the convergence of the conventional series expansion in the evaluation of bending moment and shear force laws along distributed parameter systems carrying one or more moving sub-systems. The improvement is achieved by introducing the so-called “quasi-static” solution, which explicitly accounts for the contribution of the truncated high frequency modes due to the weight of the travelling sub-systems. Numerical results have demonstrated that the “improved” series allows to capture with very few terms the abrupt changes of cable configuration at mass locations.

In the present study, the approach proposed in References (Muscolino and Sofi, 2003, 2006) is properly extended to analyze in-plane vibrations of flat-sag suspended cables carrying an array of moving oscillators with arbitrarily varying velocities. Specifically, a more sophisticated representation of the travelling sub-systems, i.e. the moving oscillator model, is here adopted in order to provide a realistic description of engineering systems like tramways or aerial cableways, which may be potentially analyzed by using the “improved” series expansion.

The equations governing the motion of the coupled cable-moving oscillators system are derived, properly including the interaction forces due to convective acceleration terms. In particular, neglecting the longitudinal inertia forces and applying a standard condensation procedure, the motion equations of the cable are reduced to a unique integro-differential equation in the vertical displacement component. Due to the dynamic interaction at the contact points with the oscillators, such equation turns out to be coupled to the ordinary differential equations governing the motion of the travelling sub-systems. The spatial dependence is eliminated by the Galerkin method approximating cable displacement as a series expansion in terms of appropriate basis functions. The discretization yields a set of coupled nonlinear ordinary differential equations with time-dependent coefficients in the generalized coordinates of the cable and relative displacements of the oscillators, which can be integrated by standard step-by-step algorithms. The equations governing the so-called “moving mass” and “moving force” problems are also derived as limiting cases.

In order to assess the effectiveness of the presented procedure, numerical results concerning a suspended cable carrying two oscillators travelling with constant or varying speed are included in the paper. It is shown...
that very few terms of the proposed series expansion are enough to capture the discontinuities of slope at the contact points between the cable and the moving sub-systems. The influence of the convective acceleration terms is also investigated.

2. Problem statement

Consider a uniform elastic cable suspended between two fixed level supports carrying \( N \) moving oscillators (Fig. 1). Let \( O_{xy} \) be a Cartesian coordinate system with origin at the left-hand support of the cable and \( s \in [0, l_c] \) denote a curvilinear abscissa, \( l_c \) being the unstretched cable length.

The cable is modelled as a mono-dimensional homogeneous elastic continuum fully accounting for geometrical nonlinearities. It is assumed that expansions and contractions of the cross-section as well as flexural rigidity are negligible. It is further supposed that the cable is perfectly flexible so that it resists applied loads by developing stresses normal to the cross-section only. A Lagrangian description of the dynamic problem is adopted by referring cable motion to the initial static equilibrium configuration \( C^0 \) under self-weight, which lies in the vertical plane \((O_{xy})\) and is represented by the function \( y(s) \). The varied configuration \( C^v \) occupied by the cable under the moving oscillators is described by the displacement components \( u(s, t) \) and \( w(s, t) \) of a given point \( P(s) \), measured from the reference configuration \( C^0 \) along the coordinate axes \( x \) and \( y \), respectively (Fig. 1).

The present study focuses on flat-sag cables, namely those with small sag-to-span ratio \( d/l \) (i.e. \( d/l \leq 1/8 \) or less), which are widely employed in engineering applications including tramways or aerial cableways. For such cables, a curvilinear element \( ds \) can be approximated by \( dx \), i.e. \( ds \approx dx \), so that the static tension \( N_0(s) \) turns out to be constant over the span and equal to its horizontal component \( H \) (\( H = N_0(s)ds/dx = \text{const} \)). Therefore, the static equilibrium configuration \( C^0 \) can be adequately described by a parabolic profile:

\begin{equation}
 y(x) = 4d \left[ \frac{x}{l} - \left(\frac{x}{l} \right)^2 \right]; \quad d = \frac{mg l^2}{8H},
\end{equation}

where \( m \) is the cable mass per unit length, \( l \) is the distance between the level supports and \( g \) denotes the acceleration of gravity. Furthermore, it is assumed that (Luongo et al., 1984): (i) the gradient of the horizontal component of the dynamic displacement is negligible with respect to unity, \( \partial u(x, t)/\partial x \ll 1 \), i.e. moderately large rotations occur in the motion; (ii) the initial strain is negligible with respect to unity, which entails

\begin{equation}
 y(x) = 4d \left[ \frac{x}{l} - \left(\frac{x}{l} \right)^2 \right]; \quad d = \frac{mg l^2}{8H},
\end{equation}

Fig. 1. Suspended cable under moving oscillators.
where \( H/EA \ll 1 \), where \( E \) and \( A \) denote the modulus of elasticity and the cross-sectional area of the cable, respectively. The Lagrangian strain measure is assumed for the centreline extension

\[
\varepsilon(x, t) = \frac{\partial w(x, t)}{\partial x} + \frac{dy(x)}{dx} \frac{\partial w(x, t)}{\partial x} + \frac{1}{2} \left( \frac{\partial w(x, t)}{\partial x} \right)^2. \tag{2}
\]

As regards the moving oscillators, the following assumptions are introduced: (i) they travel with arbitrarily varying velocities; (ii) they exhibit a linear-elastic behaviour; (iii) they undergo small amplitude vertical vibrations relative to the cable without rotational motion; (iv) the friction at the interface with the cable is neglected.

Without loss of generality, homogeneous initial conditions are considered for the cable. This corresponds to \( w(x, 0) = 0 \) and \( \dot{w}(x, 0) = 0 \) for \( 0 < x < l \) and the case where the first oscillator enters the left-hand support of the cable at time \( t = 0 \) and the cable is at rest in its dead load configuration \( C^0 \) for \( t \leq 0 \).

### 3. Equations of motion

#### 3.1. Suspended cable

By applying the extended Hamilton’s principle, the following set of nonlinear coupled partial differential equations governing in-plane vibrations about the reference configuration \( C^0 \) of a suspended cable carrying \( N \) moving oscillators (see Fig. 1) is obtained

\[
m \left( \frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial}{\partial x} [EA \varepsilon(x, t)] \right) = 0, \tag{3}
\]

\[
m \left( \frac{\partial^2 w(x, t)}{\partial t^2} - \frac{\partial}{\partial x} \left( H \frac{\partial w(x, t)}{\partial x} + E \left( \frac{dy(x)}{dx} + \frac{\partial w(x, t)}{\partial x} \right) \varepsilon(x, t) \right) \right) = \sum_{i=1}^{N} f_{x,i}(t) \delta(x - x_i(t)) \gamma_i(t) \tag{4}
\]

where \( \delta(\cdot) \) denotes the Dirac delta function; \( f_{x,i}(t) \) represents the interaction force transmitted to the cable by the \( i \)th moving oscillator at the instantaneous position \( x_i(t) \) (see Fig. 1); and \( \gamma_i(t) \) is the so-called window function, defined as follows

\[
\gamma_i(t) = \begin{cases} 
1 & \text{for } 0 < x_i(t) < l, \\
0 & \text{for } x_i(t) \leq 0 \text{ or } x_i(t) \geq l.
\end{cases} \tag{5}
\]

Under the assumption of small sag-to-span ratio, the longitudinal inertia forces \( m \partial^2 u(x, t)/\partial t^2 \) in Eq. (3) can be neglected and the equations governing in-plane vibrations can be reduced to a unique integro-differential equation in the vertical displacement component \( w(x, t) \) by applying a standard condensation procedure (Luongo et al., 1984). So operating, the elongation \( \varepsilon(x, t) \) (Eq. (2)) turns out to be a function of time only which, taking into account the boundary conditions, \( u(0, t) = u(l, t) = 0 \), is given by

\[
\varepsilon^{(\text{cond.})}(t) = \frac{1}{7} \int_0^t \left[ \frac{dy(x)}{dx} \frac{\partial w(x, t)}{\partial x} + \frac{1}{2} \left( \frac{\partial w(x, t)}{\partial x} \right)^2 \right] dx \tag{6}
\]

where the superscript in parentheses stands for “condensed”. Substituting the previous relationship into Eq. (4), the following integro-differential equation in the vertical displacement component \( w(x, t) \) is recovered:

\[
m \left( \frac{\partial^2 w(x, t)}{\partial t^2} - \frac{\partial}{\partial x} \left( H \frac{\partial w(x, t)}{\partial x} + E \left( \frac{dy(x)}{dx} + \frac{\partial w(x, t)}{\partial x} \right) \varepsilon^{(\text{cond.})}(t) \right) \right) = \sum_{i=1}^{N} f_{x,i}(t) \delta(x - x_i(t)) \gamma_i(t). \tag{7}
\]

As will be outlined next in detail, by properly specifying the force \( f_{x,i}(t) \) transmitted by the \( i \)th moving sub-system, the formulations pertaining to the moving mass and moving force problems can be deduced as well.

#### 3.2. Coupled cable-moving oscillators system

Following the traditional component-mode synthesis method (Biondi and Muscolino, 2000; Hurty, 1960; Muscolino, 2002), the equations of motion of the coupled cable-moving oscillators system are obtained by
imposing the equilibrium and the compatibility conditions at the time-dependent positions \(x_i(t)\) locating the contact points.

Let \(\eta_i(t)\) denote the relative vertical displacement of the mass \(M_i\) of the \(i\)th moving oscillator with respect to the supporting structure and \(\eta_{ci}(t)\) the displacement of the contact point between the moving sub-system and the cable, located at the instantaneous position \(x_i(t)\) (Fig. 1). Then, as long as the \(i\)th oscillator moves on the cable, i.e. \(0 \leq x_i(t) \leq l\), the total displacement of the mass \(M_i\) is given by the equation:

\[
\ddot{\eta}_i(t) = \eta_{ci}(t) + \eta_i(t). \tag{8}
\]

The equations of motion of the oscillators are readily found to be:

\[
M_i(\ddot{\eta}_i(t) + \gamma_i(t)\dot{\eta}_{ci}(t)) + c_i\dot{\eta}_i(t) + k_i\eta_i(t) = M_i\ddot{g}, \quad (i = 1, 2, \ldots, N) \tag{9}
\]

where the dot over a variable denotes total derivative with respect to time \(t\); \(c_i\) and \(k_i\) are the coefficients of viscous damping and stiffness of the \(i\)th oscillator, respectively. According to the definition (5) of the window function, when the oscillator is outside the cable, the motion of the travelling sub-system is still ruled by Eq. (9) though the relative and total displacements coincide \((\gamma_i(t) = 0)\).

The differential equations (9) should be supplemented by the initial conditions \(\eta_i(0) = \eta_i^{(qs)} = k_i^{-1}M_i\ddot{g}\) and \(\dot{\eta}_i(0) = 0\), \(\eta_i^{(qs)}\) being the quasi-static response of the \(i\)th moving oscillator under self-weight.

In order to satisfy the equilibrium between the supporting structure and the moving oscillators, the interaction force \(f_{y,i}(t)\) should coincide with the reaction of the spring-dashpot system, that is:

\[
f_{y,i}(t) = c_i\dot{\eta}_i(t) + k_i\eta_i(t) = M_i[\ddot{g} - (\dot{\eta}_i(t) + \ddot{\eta}_{ci}(t))], \quad (i = 1, 2, \ldots, N). \tag{10}
\]

Notice that the previous relationship has been derived from Eq. (9) setting \(\gamma_i(t) = 1\) since the force \(f_{y,i}(t)\) acts on the cable as long as \(0 \leq x_i(t) \leq l\).

The compatibility at the interface nodes requires that \(\eta_{ci}(t) \equiv w(x, t)|_{x=x_i(t)}, \quad (i = 1, 2, \ldots, N)\). Such conditions imply that the vertical displacement \(\eta_{ci}(t)\) of the contact point between the cable and the \(i\)th oscillator depends on time \(t\) both directly and through the instantaneous position \(x_i(t)\). Hence, the acceleration \(\ddot{\eta}_{ci}(t)\) of the \(i\)th interface node takes the following form:

\[
\ddot{\eta}_{ci}(t) = \frac{d^2}{dt^2}[w(x, t)|_{x=x_i(t)}] = A_i[w(x, t)|_{x=x_i(t)}] \tag{11}
\]

where \(A_i[\cdot]\) is the convective acceleration operator defined as follows:

\[
A_i[\cdot] = \frac{\partial^2(\cdot)}{\partial t^2} + 2\ddot{x}_i(t)\frac{\partial^2(\cdot)}{\partial x\partial t} + \ddot{x}_i(t)\frac{\partial (\cdot)}{\partial x} + \dddot{x}_i(t)\frac{\partial^2(\cdot)}{\partial x^2}. \tag{12}
\]

Substituting Eq. (11) into Eq. (10), the interaction force \(f_{y,i}(t)\) transmitted by the \(i\)th oscillator moving with arbitrarily varying velocity takes the following form:

\[
f_{y,i}(t) = M_i\left\{\ddot{g} - \dddot{\eta}_i(t) - \left[\frac{\partial^2 w(x, t)}{\partial x^2} + 2\ddot{x}_i(t)\frac{\partial^2 w(x, t)}{\partial x\partial t} + \dddot{x}_i(t)\frac{\partial w(x, t)}{\partial x} + \dddot{x}_i(t)\frac{\partial^2 w(x, t)}{\partial x^2}\right] \right\}|_{x=x_i(t)}. \tag{13}
\]

Thus, once the horizontal displacement component \(u(x, t)\) has been eliminated through the condensation procedure, the dynamic response of the cable-moving oscillators system, in terms of \(w(x, t)\) and \(\eta_i(t)\) \((i = 1, 2, \ldots, N)\), is governed by the integro-differential equation (7) associated with the \(N\) ordinary differential equations (9). Notice that such equations are coupled since the evaluation of the interaction force (13) requires the knowledge of the relative acceleration \(\dddot{\eta}_i(t)\) which in turn depends on cable vibrations through the acceleration of the contact point \(\dddot{\eta}_{ci}(t)\) (see Eqs. (9) and (11)).

Furthermore, taking into account that \(\varepsilon(x, t) \equiv \varepsilon^{\text{cond.}}(t)\), the longitudinal displacement of the cable, \(u(x, t)\), can be estimated by integrating Eq. (2):

\[
u(x, t) = \varepsilon^{\text{cond.}}(t)x - \int_0^x \left[\frac{dy(x)}{dx}\frac{\partial w(x, t)}{\partial x} + \frac{1}{2} \left(\frac{\partial w(x, t)}{\partial x}\right)^2\right] dx \tag{14}
\]

where \(\varepsilon^{\text{cond.}}(t)\) is given by Eq. (6).
In the special case of oscillators moving with the same constant speed \( \dot{x}_i(t) = v \), the instantaneous position of the \( i \)th sub-system can be expressed as

\[
x_i(t) = vt - d_i
\]

where \( d_i \) denotes the distance between the \( i \)th and the first oscillator \( (d_1 = 0) \). Furthermore, Eq. (11) reduces to

\[
\ddot{\eta}_{i,j}(t) = \frac{d^2}{dt^2} \left[ w(x, t) \bigg|_{x=x_i(t)} \right] = \left[ \frac{\partial^2 w(x, t)}{\partial t^2} + 2v \frac{\partial^2 w(x, t)}{\partial x \partial t} + v^2 \frac{\partial^2 w(x, t)}{\partial x^2} \right]_{x=x_i(t)}.
\]

Eq. (16) shows that the acceleration of the contact point between the cable and the \( i \)th oscillator in uniform motion is given by the sum of three terms: the first one is the relative acceleration due to the motion of the cable; the second term is the Coriolis acceleration arising because the oscillator travels with velocity \( v \) in the longitudinal direction while the cable itself has angular velocity \( \omega \); the last contribution in Eq. (16) is represented by the centripetal acceleration which is experienced by the mass \( M_i \) as it moves along the curved profile of the cable.

### 3.3. Cable under moving masses or moving loads

When the moving sub-systems are modelled as masses \( M_i \), \( i = 1, 2, \ldots, N \), regarded as rigid bodies directly attached to the cable, that is when the elastic interaction force is neglected, the extended Hamilton’s principle in conjunction with the condensation procedure yields the integro-differential equation (7), where the interaction force \( f_{i,j}(t) \) reduces to:

\[
f_{i,j}(t) = M_i \left\{ g - A_i \left[ w(x, t) \bigg|_{x=x_i(t)} \right] \right\}
\]

\[
= M_i \left\{ g - \left[ \frac{\partial^2 w(x, t)}{\partial t^2} + 2\dot{x}_i(t) \frac{\partial^2 w(x, t)}{\partial x \partial t} + \ddot{x}_i(t) \frac{\partial w(x, t)}{\partial x} + \dot{x}_i(t)^2 \frac{\partial^2 w(x, t)}{\partial x^2} \right]_{x=x_i(t)} \right\}.
\]

It should be pointed out that the “moving mass” model can be obtained as limiting case of the “moving oscillator” problem when \( k_i \to \infty \). Indeed, for sufficiently large values of the spring stiffness \( k_i \), the moving oscillator reduces to a rigid body and the relative displacement \( \eta_i \) tends to zero. This implies that the acceleration \( \ddot{\eta}_{i,j}(t) \) of the \( i \)th contact point coincides with the vertical acceleration \( a_{w,i}(t) \) of the mass \( M_i \), that is

\[
\ddot{\eta}_{i,j}(t) = a_{w,i}(t) = A_i \left[ w(x, t) \bigg|_{x=x_i(t)} \right].
\]

It has to be mentioned that in both the moving oscillator and moving mass problems, Eq. (3) is obtained by the extended Hamilton’s principle neglecting the longitudinal inertia forces of the travelling sub-systems. Obviously, this assumption may be inadequate when relatively large masses \( M_i \) are considered. Nevertheless, numerical investigations conducted during the present study have demonstrated that in typical engineering applications, such as aerial cableways, the longitudinal inertia forces of both the cable and the moving sub-systems can be reasonably neglected.

If the cable is crossed by \( N \) moving concentrated loads of intensity \( F_{y,i}(t) \), \( i = 1, 2, \ldots, N \), acting along the \( y \) axis, then the problem (known as “moving load problem”) is still governed by the integro-differential equation (7) where, however, the external force \( f_{i,j}(t) = F_{y,i}(t) \) is known a priori. In fact, in this case there is no interaction between the suspended cable and the moving sub-systems. As well-known, the “moving load” model is the simplest one and can be obtained as limiting case of the moving oscillator problem when the spring stiffness goes to zero \( k_i \to 0 \).

### 4. Galerkin method

The presence of the moving sub-systems greatly complicates the dynamic behaviour of the cable as well as the pertinent mathematical model. Specifically, due to the moving singularities on the right-hand side of Eq. (7), the varied configuration of the cable is no longer described by a smooth continuous curve. As a result, the Galerkin method, which is widely used in the literature for analyzing cable vibrations under external excita-
tions, loses its efficiency. Indeed, the series expansion of cable response in terms of appropriate basis functions, satisfying the boundary conditions, converges poorly when moving sub-systems are involved.

According to the Galerkin method, the displacement component of the cable along the y axis, \( w(x, t) \), can be represented as follows:

\[
w_{\text{CSE}}(x, t) = \sum_{j=1}^{n_w} \phi_j(x)q_j(t) = \phi^T(x)q(t)
\]

(18)

where \( \phi(x) \) and \( q(t) \) are the vectors collecting \( n_w \) appropriate basis functions, \( \phi_j(x) \), and the associated time-dependent generalized coordinates, \( q_j(t) \), respectively. The subscript “CSE” stands for “Conventional Series Expansion”. An appropriate choice for the basis functions is represented by the eigenfunctions of the associated linear problem (Irvine, 1981), which is deduced from Eq. (7) by dropping all nonlinear terms and the contributions related to the moving sub-systems. Alternatively, any set of comparison functions may be adopted.

Substituting the conventional series expansion (18) into Eq. (6) and performing integration, the following approximate expression of the time-varying elongation is obtained

\[
\ddot{e}^{(\text{cond})}(t) = b^Tq(t) + q^TBq(t)
\]

(19)

where the superimposed hat (\(^\wedge\)) means that use has been made of Eq. (18); the \( n_w \)-vector \( b \) and the \( (n_w \times n_w) \) matrix \( B \) are defined in Appendix A.

In a similar way, substituting Eq. (18) into Eq. (11), the acceleration \( \ddot{\eta}_{c,i}(t) \) of the contact point between the cable and the \( i \)th oscillator in terms of generalized coordinates is obtained:

\[
\ddot{\eta}_{c,i}(t) = \frac{d^2}{dt^2} \left[ w_{\text{CSE}}(x, t) \right]_{x = x_i(t)}
= \phi^T(x_i(t))\dot{q}(t) + 2\ddot{x}_i(t)\phi^T(x_i(t))\dot{q}(t) + \dddot{x}_i(t)\phi^T(x_i(t))q(t) + \dddot{x}_i(t)\phi''(x_i(t))q(t),
\]

(20)

where the apex denotes total derivative with respect to the spatial variable \( x \).

In the following, the set of ordinary differential equations governing the generalized coordinates \( q(t) \) will be derived for three different models of the moving sub-system: the moving oscillator, the moving mass and the moving load.

### 4.1. Suspended cable carrying moving oscillators

As outlined in the previous section, the moving oscillator problem is governed by the integro-differential equation (7), with \( f_{r,e}(t) \) given by Eq. (13), coupled to the set of \( N \) ordinary differential equations (9).

Inserting Eqs. (18) and (19) into Eq. (7), pre-multiplying both sides of the resulting equation by \( \phi_j(x) \), \( (r = 1, 2, \ldots, n_w) \), and integrating with respect to \( x \) from 0 to \( l \), the following set of nonlinear coupled ordinary differential equations governing the generalized coordinates \( q_i(t) \) is recovered

\[
M_i\ddot{q}_i(t) + r(q(t)) = \Phi(t)M_{\text{CSE}}(t)\left[ \mathbf{g} - \ddot{\eta}(t) - \Phi^T(t)\ddot{q}(t) - 2\dddot{X}(t)\Phi^T(t)q(t) - \dddot{X}(t)\Phi''(t)q(t) \right]
\]

(21)
in which \( r(q(t)) \) is the vector of the nonlinear restoring forces whose \( k \)-th element is given by

\[
r_k(q(t)) = (a_k^{(1)} + a_k^{(2)}b)^Tq(t) + q^T(t)(a_k^{(2)}B + a_k^{(3)}b^TB)q(t) + a_k^{(3)}q(t)q^T(t)Bq(t)
\]

(22)
where \( a_k^{(1)}, a_k^{(2)} \) and \( a_k^{(3)} \) are defined in Appendix A; \( \mathbf{r} \) is a \( N \)-vector whose elements are all equal to 1; and \( \eta(t) \) is a vector of order \( N \) listing the relative displacements of the moving oscillators. Furthermore, the matrices appearing in Eq. (21) are defined as
\[ M_c = m \int_0^1 \phi(x)\phi^T(x) \, dx; \]
\[ M_m = \text{Diag}[M_1, M_2, \ldots, M_N]; \]
\[ \Gamma(t) = \text{Diag}[\gamma_1(t), \gamma_2(t), \ldots, \gamma_N(t)]; \]
\[ X(t) = \text{Diag}[x_1(t), x_2(t), \ldots, x_N(t)]; \]
\[ \ddot{X}(t) = \text{Diag}[\ddot{x}_1(t), \ddot{x}_2(t), \ldots, \ddot{x}_N(t)]; \]
\[ \Phi(t) = [\phi(x_1(t)), \phi(x_2(t)), \ldots, \phi(x_N(t))]; \]
\[ \Phi'(t) = [\phi'(x_1(t)), \phi'(x_2(t)), \ldots, \phi'(x_N(t))]; \]
\[ \Phi''(t) = [\phi''(x_1(t)), \phi''(x_2(t)), \ldots, \phi''(x_N(t))]. \]

(23a–i)

Substituting the discretized expression (20) of the acceleration \( \ddot{\eta}_{ci}(t) \) into Eq. (9) and rearranging terms, the reduced equations of motion of the coupled cable-moving oscillators system can be written in the following form

\[ M^{(o)}(t)\ddot{y}(t) + C^{(o)}(t)\dot{y}(t) + K^{(o)}(t)y(t) + r^{(o)}(t; y(t)) = F^{(o)}(t) \]  

(24)

where the superscript in parentheses stands for “moving oscillator problem” and:

\[ \Delta M_c(t) = \Phi(t)M_m\Gamma(t)\Phi^T(t); \quad \Delta C_c(t) = 2\Phi(t)M_m\Gamma(t)\ddot{X}(t)\Phi^T(t); \]
\[ \Delta K_c(t) = \Phi(t)M_m\Gamma(t)[\dddot{X}(t)\Phi^T(t) + \ddot{X}^2(t)\Phi^T(t)]; \]
\[ M_{cm}^I(t) = \Phi(t)M_m\Gamma(t); \quad C_{cm}(t) = 2M_m\Gamma(t)\ddot{X}(t)\Phi^T(t); \]
\[ K_{cm}(t) = \Gamma(t)M_m[\dddot{X}(t)\Phi^T(t) + \ddot{X}^2(t)\Phi^T(t)]; \]
\[ K_m = \text{Diag}[k_1, k_2, \ldots, k_N]; \quad C_m = \text{Diag}[c_1, c_2, \ldots, c_N]. \]

(26a–h)

Thus, the Galerkin method provides a set of \( n_w + N \) nonlinear coupled ordinary differential equations with time-dependent coefficients (Eq. (24)), governing the generalized coordinates \( q(t) \) of the cable and the relative displacements \( \eta(t) \) of the oscillators. It can be observed that the inertia, damping and restoring forces depend on time as a consequence of the time dependence of the instantaneous positions \( x_i(t) \) of the moving oscillators along the cable. In particular, \( \Delta M_c(t) \), \( \Delta C_c(t) \) and \( \Delta K_c(t) \) (see Equations (26a–c)) may be regarded as modifications of the mass, damping and stiffness matrices of the cable, respectively, due to the acceleration \( \ddot{\eta}_{ci}(t) \) of the contact points between the supporting structure and the moving oscillators (see Eq. (20)); \( M_{cm}(t), C_{cm}(t) \) and \( K_{cm}(t) \) are the corresponding modifications of the mass, damping and stiffness matrices of the moving oscillators. Some authors, dealing either with the moving oscillator or moving mass problem, retain just the first term in the expression (12) of the convective acceleration operator, so that the damping and stiffness matrices of the discretized model turn out to be independent of time. As will be shown next through numerical applications, this assumption may lead to erroneous results especially when relatively large masses \( M_i \) or high velocities \( \dot{x}_i(t) \) are considered.
4.2. Suspended cable under moving masses or moving loads

As shown in Section 3, nonlinear in-plane vibrations of a suspended cable under $N$ moving masses are governed by the integro-differential equation (7) where $\mathbf{f}_{\text{gy}}(t)$ is given by Eq. (17). It can be readily verified that by applying the Galerkin method, Eq. (7) is reduced to the following set of $n_w$ nonlinear coupled ordinary differential equations with time-dependent coefficients governing the generalized coordinates $\mathbf{q}(t)$:

$$
(\mathbf{M}_c + \Delta \mathbf{M}_c(t))\ddot{\mathbf{q}}(t) + \Delta \mathbf{C}_c(t)\dot{\mathbf{q}}(t) + \Delta \mathbf{K}_c(t)\mathbf{q}(t) + \mathbf{r}(t; \mathbf{q}(t)) = g\mathbf{\Phi}(t)\mathbf{M}_m\mathbf{\Gamma}(t)\tau,
$$

(27)

where, as outlined above, $\Delta \mathbf{M}_c(t)$, $\Delta \mathbf{C}_c(t)$ and $\Delta \mathbf{K}_c(t)$ (see Eqs. (26a–c)) may be viewed as modifications of the mass, damping and stiffness matrices, respectively, due to the acceleration $\mathbf{\ddot{\mathbf{h}}}_{\text{c}}(t)$ of the moving masses (see Eq. (20)).

In the case of a suspended cable crossed by $N$ moving loads $\mathbf{F}_{\text{gy}}(t)$, since the interaction force $\mathbf{f}_{\text{gy}}(t)$ is known a priori, say $\mathbf{f}_{\text{gy}}(t) = \mathbf{F}_{\text{gy}}(t)$, substituting Eqs. (18) and (19) into Eq. (7) and applying the Galerkin method, the following set of $n_w$ nonlinear coupled ordinary differential equations with constant coefficients is obtained:

$$
\mathbf{M}_c\ddot{\mathbf{q}}(t) + \mathbf{r}(t; \mathbf{q}(t)) = \mathbf{\Phi}(t)\mathbf{\Gamma}(t)\mathbf{F}_{\text{gy}}(t)
$$

(28)

where $\mathbf{F}_{\text{gy}}(t)$ is the $N$-vector collecting the moving loads.

Numerical integration of Eqs. (24), (27) or (28) with the associated initial conditions provides an approximate solution to the problem of a suspended cable carrying $N$ moving oscillators, masses or loads, respectively. For this purpose, standard step-by-step algorithms, such as Newmark-$\beta$ method ($\beta = 1/4, \gamma = 1/2$) associated with full (or modified) Newton–Raphson iterative procedure may be used.

Once the solution in terms of generalized coordinates $\mathbf{q}(t)$ has been obtained, the horizontal displacement component of the cable can be approximated by introducing Eqs. (18) and (19) into Eq. (14) and performing integration, that is:

$$
u_{\text{CSE}}(x, t) = \hat{\mathbf{e}}^{(\text{cond})}(t)x - \left[\mathbf{e}^T\mathbf{q}(t) + \frac{1}{2}\mathbf{q}^T(t)\mathbf{E}\mathbf{q}(t)\right]
$$

(29)

where the $n_w$-vector $\mathbf{e}$ and the $(n_w \times n_w)$ matrix $\mathbf{E}$ are defined in Appendix A.

5. Proposed series expansion

The conventional series expansion (18), either in terms of the eigenfunctions of the associated linear problem or of arbitrary comparisons functions, possesses poor convergence because it represents the solution $\mathbf{w}(x, t)$ as a continuous function, while cable deformed profile exhibits slope discontinuities at the abscissas $x_i(t)$ defining the positions of the moving sub-systems. In order to overcome these limits, an alternative representation of the solution is derived herein. The underlying idea stems from the well-known “mode-acceleration” method (Maddox, 1975) proposed in the literature to improve the convergence of the modal series for discretized structural systems. The method accounts for the pseudo-static contribution of the truncated high frequency modes by adding to the modal series a correction term. This is given by the difference of two different representations of the so-called “quasi-static” solution: the closed-form and the modal series. Then, following the philosophy behind the mode-acceleration method, the solution of the integro-differential equation (7), governing cable vibrations under moving sub-systems, can be approximated as follows

$$
w_{\text{PQC}}(x, t) = w_{\text{CSE}}(x, t) + w_{\text{qs}}(x, t) = \phi^T(x)\mathbf{q}(t) + [w_q(x, t) - \phi^T(x)\mathbf{q}_s(t)],
$$

(30)

where the subscript “PQC” stands for “Proposed Quasi-Static Correction”. In the previous relation $w_{\text{qs}}(x, t)$ denotes the “quasi-static” contribution of the truncated modes, which is expressed as the difference of two terms: the first one, $w_q(x, t)$, is the quasi-static response of the cable obtained as solution of Eq. (7) neglecting the inertial and damping effects; the second contribution is represented by the series expansion of the quasi-static response in terms of appropriate basis functions $\phi_j(x)$ and associated time-dependent generalized coordinates $q_j(t)$.

The quasi-static solutions $w_q(x, t)$ and $\mathbf{q}_s(t)$ are obtained considering the suspended cable subject to the quasi-static interaction forces $\mathbf{f}_{\text{gy}}^{(\text{qs})}(t)$, $(i = 1, 2, \ldots, N)$, applied at the instantaneous positions $x_i(t)$. In the case of moving oscillators, masses and loads, such forces are given, respectively, by
\[ f_{y,i}(t) = k_i y_i^{(qs)} = M_i g; \]
\[ f_{x,i}(t) = M_i g; \]
\[ f_{y,i}(t) = F_{y,i}, \]

where, as already mentioned, \( y_i^{(qs)} = k_i^{-1} M_i g \) denotes the quasi-static response of the \( i \)th moving oscillator. Then, it can be readily verified that for both the moving oscillator and moving mass problems the generalized coordinates \( q_s(t) \) are governed by the following set of \( n_w \) nonlinear equations

\[ \mathbf{r}(\mathbf{q}, t) = g \mathbf{\Phi}(t) \mathbf{M}_m \mathbf{\Gamma}(t) \mathbf{\tau}. \]

Conversely, when the moving load model is adopted, the previous equations take the form

\[ \mathbf{r}(\mathbf{q}, t) = \mathbf{\Phi}(t) \mathbf{\Gamma}(t) \mathbf{F}_s(t). \]

Notice that the proposed approach is conceptually analogous to the procedure presented in References (Biondi et al., 2004; Fryba, 1999; Pesterev and Bergman, 2000; Pesterev et al., 2001) to accelerate the convergence of the conventional series expansion in the determination of bending moment and shear force laws along linear continuous systems carrying one or more moving sub-systems. The main difference consists in the evaluation of the first contribution to the quasi-static correction (Eq. (30)), \( w_s(x, t) \), which cannot be expressed through the Green’s function due to the nonlinearity of cable behaviour. Indeed, the quasi-static response \( w_s(x, t) \) should be computed solving the equilibrium equations of the suspended cable under a series of point loads of intensity \( f_{y,i}^{(qs)}(t) \) applied at the time-varying positions \( x_i(t) \), recalling that the effects cannot be superposed since the problem is nonlinear.

As outlined by the authors in Reference (Muscolino and Sofi, 2003), in the case of a single moving sub-system, \( w_s(x, t) \) can be easily evaluated using the closed-form expression for the static response of a suspended cable under a point load \( P = f_{y,i}^{(qs)} \) applied at a distance \( x_1 \) from the left-hand support (Irvine, 1981), provided that \( x_1 \) is let to vary with time, that is \( x_1 = x_i(t) \). This solution can be generalized to the case of an arbitrary number \( N \) of moving sub-systems. As an example, in Appendix B the quasi-static solution \( w_s(x, t) \) pertaining to the case \( N = 2 \) is derived.

6. Numerical application

The presented procedure has been applied to a suspended cable characterized by the following geometrical and mechanical parameters: \( A = 1.175 \times 10^{-3} \) m\(^2\), \( m = 9.7 \) Kg/m, \( E = 147 \) GPa, \( l = 500 \) m and \( d = 13 \) m. The cable carries two identical moving oscillators with \( M_1 = M_2 = 1000 \) Kg, \( k_1 = k_2 = 75 \) kN/m, \( c_1 = c_2 = 2 \) kN s/m and spacing \( d_2 = 200 \) m. Cable vibrations due to oscillators moving either with constant or variable speed have been analyzed.

First, the validity of the basic assumptions introduced to derive the cable-moving oscillators model has been assessed through appropriate numerical studies. In particular, the influence of the longitudinal inertia forces of both the cable and the moving sub-systems has been investigated in order to verify that Eq. (7) adequately describes both the horizontal and vertical displacement components \( u(x, t) \) and \( w(x, t) \). For simplicity’s sake, the analysis has been carried out modelling the moving sub-systems as two masses \( (M_1 = M_2 = 1000 \) Kg) travelling at constant speed \( v = 5 \) m/s. As outlined in Section 3, if the longitudinal inertia forces are neglected, cable in-plane vibrations under \( N \) moving masses are governed by Eq. (7) with \( f_{y,i}(t) \) given by Eq. (17). In the context of the Galerkin method, an approximate solution can be pursued by integrating the set of \( n_w \) nonlinear coupled ordinary differential equations with time-dependent coefficients (27). Then, the horizontal displacement component can be estimated by means of Eq. (29). Conversely, when the longitudinal inertia forces are included, the extended Hamilton’s principle yields the following set of nonlinear coupled partial differential equations

\[ m \frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial}{\partial x} [EA \varepsilon(x, t)] = - \sum_{i=1}^{N} M_i a_{u,i}(t) \delta(x - x_i(t)) \gamma_i(t); \]
\[ m \frac{\partial^2 w(x, t)}{\partial t^2} - \frac{\partial}{\partial x} \left( H \frac{\partial w(x, t)}{\partial x} + E A \left[ \frac{\partial^2 w(x, t)}{\partial x^2} + \frac{\partial w(x, t)}{\partial x} \right] \right) \varepsilon(x, t) \right]_i = \sum_{i=1}^{N_i} f_{y,i}(t) \delta(x - x_i(t)) \gamma_i(t), \]  

(35)

where \( a_{x,i}(t) = A_i \|u(x, t)\|_{x=x_i(t)} \) denotes the horizontal component of the acceleration of the \( i \)th moving mass. The spatial dependence can be eliminated from the previous equations by the Galerkin method approximating both the horizontal and vertical displacement components as a linear combination of appropriate basis functions and generalized coordinates. All the numerical results to be presented in the paper have been obtained adopting the basis functions \( \phi_j(x) = \sin(j \pi x / l) \). Such choice leads to an easier implementation of the Galerkin procedure since it allows to avoid the solution of the transcendental equation governing the natural frequencies associated with the symmetric mode shapes of the linear problem (Irvine, 1981). Furthermore, the evaluation of the integrals involved in the discretization of Eq. (7) is more expeditious.

In Fig. 2, the time-histories of the horizontal and vertical cable displacement components at mid-span, obtained by the Galerkin method retaining 8 series terms, are plotted. Notice that the solutions computed neglecting (Eqs. (7) and (14)) and including (Eqs. (34) and (35)) the longitudinal inertia forces are almost coincident. Furthermore, as expected, the horizontal displacement component is much smaller than the vertical one. Numerical results, here omitted for brevity’s sake, have also shown that the effects of the longitudinal inertia forces become more pronounced as the sag-to-span ratio increases. This implies that the problem under
Fig. 3. Different approximations of cable response under two moving oscillators obtained by the CSE retaining an increasing number of series terms: (a) mid-span vertical displacement; (b) vertical displacement of the contact point between the cable and the first oscillator; (c) slope at mid-span ($v = 5$ m/s).
consideration, which concerns flat-sag cables, can be conveniently reduced to the study of vertical vibrations ruled by the integro-differential equation (7).

Once the negligible influence of the longitudinal inertia forces has been demonstrated, cable response under two moving oscillators has been analyzed. First the case of oscillators moving at constant speed \( v = 5 \text{ m/s} \) has been considered.

Fig. 3 displays different approximations of mid-span cable deflection, \( w(l/2,t) \), of the displacement at the contact point with the first moving oscillator, \( w(x_1(t),t) \), and of the slope at mid-span, \( \partial w(x,t)/\partial x \big|_{x=l/2} \), obtained as solution of Eq. (24) retaining an increasing number of terms \( n_w \) in the conventional series expansion (18). Hereinafter, the solution pertaining to \( n_w = 180 \) will be improperly denoted as “exact” since only slight improvement of accuracy is achieved for \( n_w > 180 \). It can be seen that many terms of the conventional series expansion are required to capture the discontinuities of cable displacement and slope at the time instants \( t_1 = 50 \text{ s} \) and \( t_2 = 90 \text{ s} \) in which the first and the second oscillator, respectively, reaches the mid-span section.

Another aspect that has been investigated is the influence of the second and third term in the expression (16) of the acceleration \( \ddot{\eta}_{c;i}(t) \) at the contact point between the cable and the generic oscillator in uniform motion. As shown in Eqs. (25) and (26), these terms affect the damping and stiffness of the discretized model, respectively. In Figs. 4a and b, the time-history of cable deflection at mid-span, evaluated using the correct expression of the acceleration at the cable/moving sub-systems interface (Eq. (16)), is compared to the solutions derived assuming different approximations of \( \ddot{\eta}_{c;i}(t) \). Figs. 4c and d display an analogous comparison in terms of vertical displacement at the contact point between the cable and the first oscillator. In both cases, the
Fig. 5. Time-histories of (a) mid-span vertical displacement, (b) of the vertical displacement at the contact point between the cable and the first oscillator, and (c) of slope at mid-span, obtained by the proposed approach (PQC) and the conventional series expansion (CSE), ($v = 5$ m/s).
conventional series expansion with \( n_w = 180 \) terms has been adopted and two different values of velocity have been considered. It can be seen that the contribution of the convective acceleration terms is not negligible, especially when the velocity of the moving sub-systems increases. As can be inferred by inspection of Eq. (13), the Coriolis and centripetal terms play an important role also when the masses \( M_i \) of the moving sub-systems are relatively large.

The effectiveness of the proposed quasi-static correction (Eq. (30)) is demonstrated in Fig. 5, where the approximations of mid-span cable deflection, of the displacement at the contact point between the cable and the first oscillator, and of the slope at mid-span, obtained by the conventional and the improved series expansions are compared to the “exact” solutions (CSE with \( n_w = 180 \)). In particular, Fig. 5 clearly shows that just one term of the improved series allows to capture with good accuracy the discontinuities of cable deflection at the time instants in which the moving oscillators reach the mid-span section. Conversely, at least two terms are required to reproduce the corresponding discontinuities of the slope at mid-span since \( \phi'_1(1/2) = (\pi/2) \cos(\pi/2) = 0 \). Therefore, the addition of the quasi-static correction greatly improves the convergence of the conventional series expansion, by explicitly accounting for the gravitational effects associated with the moving oscillators.

Fig. 6 displays cable deformed configuration \( y(x) + w(x,t) \) and slope \( \partial w(x,t) / \partial x \) along the cable at the time instant \( t = 70 \) s in which the first oscillator is located at the abscissa \( x_1 = 350 \) m. Notice that also in this case the approximation obtained retaining just one term of the improved series expansion is very close to the “exact” solution and it is able to capture the discontinuities of slope at the instantaneous locations of the oscillators. It is worth mentioning that a different cable-moving oscillators system may require a larger number of series terms. Nevertheless, numerical investigations conducted for various scenarios have demonstrated that very few terms of the improved series are enough to achieve good accuracy.

A main feature of the proposed procedure is that it allows to analyze cable vibrations due to sub-systems moving with arbitrarily varying velocity. Indeed, the instantaneous position of the \( i \)th moving oscillator can be represented by an arbitrary function of time \( x_i(t) \). As an example, the following time-histories have been assumed:

\[
\begin{align*}
  x_i(t) &= v_0 t - d_i, & \text{for } t \leq t_0 \\
  x_i(t) &= v_0 t_0 + v_0 (t - t_0) + \frac{1}{2} a_0 (t - t_0)^2 - d_i, & \text{for } t > t_0
\end{align*}
\]

(a)  

(b)

Fig. 6. Comparison between the proposed solution (PQC) and the conventional series expansion (CSE): (a) deformed configuration and (b) slope along the cable at the time instant \( t = 70 \) s in which the first oscillator is located at the abscissa \( x_1 = 350 \) m (\( v = 5 \) m/s).
Fig. 7. Time-histories of (a) mid-span vertical displacement, (b) of the vertical displacement at the contact point between the cable and the second oscillator, and (c) of slope at mid-span, obtained by the proposed approach (PQC) and the conventional series expansion (CSE), ($v = v_0 = 5$ m/s for $t \leq t_0 = 90$ s and $a_0 = -0.5$ m/s$^2$ for $t > t_0 = 90$ s).
where \( i = 1, 2, v_0 = 5 \text{ m/s}, t_0 = 90 \text{ s} \) and \( a_0 = -0.5 \text{ m/s}^2 \) \((d_1 = 0 \text{ and } d_2 = 200 \text{ m})\). According to Eq. (36), the oscillators start moving at constant speed \( v = v_0 \) and then, at the time instant \( t_0 \), they are subject to a uniform deceleration \( a_0 \) which leads to stop within 10 s, at time \( t_f = 100 \text{ s} \).

In Fig. 7, the time-histories of mid-span deflection, of cable displacement at the contact point with the second oscillator and of slope at mid-span are plotted. It is noted that, also in the case of variable speed, the proposed quasi-static correction allows to obtain accurate estimates of cable response retaining just few series terms. As expected, for \( t > t_f = 100 \text{ s} \) when the oscillators are at rest in the positions \( x_1(t_f) \) and \( x_3(t_f) \), cable response consists of free vibrations around the equilibrium configuration attained at the time instant \( t_f \). These results attest for the versatility of the presented approach which enables to handle different models and arbitrary displacement time-histories of the moving sub-systems.

7. Concluding remarks

The dynamics of suspended cables with small sag-to-span ratios carrying an array of moving oscillators with arbitrarily varying speed has been studied. The equations governing in-plane vibrations of the coupled cable-moving oscillators system have been derived taking into account the interaction effects due to convective acceleration. Specifically, by applying a standard condensation procedure, the motion equations of the cable have been reduced to a unique integro-differential equation in the vertical displacement component. Due to the dynamic interaction at the contact points with the moving sub-systems, such equation is coupled to the set of ordinary differential equations governing the vertical vibrations of the moving oscillators. The spatial dependence has been eliminated by the Galerkin method representing cable response as a series expansion in terms of appropriate basis functions and generalized coordinates. As expected, it has been found that such series exhibits poor convergence because of the singularities associated with the moving oscillators. Following the underlying idea of the well-known “mode-acceleration” method, an improved series has been derived by adding a correction term representing the “quasi-static” contribution of the truncated high frequency modes. Numerical results concerning a suspended cable carrying two oscillators moving either with constant or variable speed have demonstrated that very few terms of the proposed series are enough to capture the discontinuities of the deformed cable profile at the contact points between the continuum and the traveling sub-systems. In particular, for the selected case study very accurate results have been obtained even by a one-term approximation using sinusoidal basis functions. In view of the relative small amount of calculations involved, the presented method may be potentially employed to perform parametric analyses which are useful for design purposes.

Appendix A

The generic elements of the vector \( b \) and of the matrix \( B \) in Eq. (19) are given, respectively, by

\[
B_{kj} = \frac{I_{kj}^{(2)}}{2l} \quad \text{and} \quad B_{kj} = \frac{I_{kj}^{(2)}}{2l}
\]

where

\[
I_{kj}^{(1)} = \int_0^l \phi_j(x)dx; \quad I_{kj}^{(2)} = \int_0^l \phi'_k(x)\phi'_j(x)dx.
\]  
(A.2)

Furthermore, the \( j \)-th element of the vectors \( a^{(1)}_k \) and \( a^{(3)}_k \), and the coefficient \( a^{(2)}_k \) in Eq. (22) are defined as:

\[
a^{(1)}_{kj} = HI_{kj}^{(2)}; \quad a^{(3)}_{kj} = EA I_{kj}^{(2)}; \quad a^{(2)}_k = \frac{8EA d}{l^2} I_{kj}^{(1)}
\]  
(A.3)

where the integrals (A.2) appear.

Finally, the generic elements of the vector \( e \) and of the matrix \( E \) in Eq. (29) are given, respectively, by

\[
e_j = \frac{8d}{l^2} \int_0^x \phi_j(x)dx; \quad E_{kj} = \int_0^x \phi'_k(x)\phi'_j(x)dx.
\]  
(A.4)
Appendix B

Consider a suspended cable crossed by two moving loads $P_1 = M_1 g$ and $P_2 = M_2 g$ whose instantaneous positions are $x_1(t)$ and $x_2(t)$, respectively. By extending the procedure described in Reference (Irvine, 1981), first the vertical equilibrium of the cable at a generic cross-section is imposed:

\[
(H + h(t)) \frac{\partial}{\partial x} (y(x) + w_z(x, t)) = P_1 \left(1 - \frac{x_1(t)}{l}\right) + P_2 \left(1 - \frac{x_2(t)}{l}\right) + \frac{mgl}{2} \left(1 - \frac{2x}{l}\right), \quad 0 \leq x \leq x_2(t);
\]

\[
(H + h(t)) \frac{\partial}{\partial x} (y(x) + w_z(x, t)) = P_1 \left(1 - \frac{x_1(t)}{l}\right) - P_2 \frac{x_2(t)}{l} + \frac{mgl}{2} \left(1 - \frac{2x}{l}\right), \quad x_2(t) \leq x \leq x_1(t);
\]

\[
(H + h(t)) \frac{\partial}{\partial x} (y(x) + w_z(x, t)) = -P_1 \frac{h(t)}{l} - P_2 \frac{x_2(t)}{l} + \frac{mgl}{2} \left(1 - \frac{2x}{l}\right), \quad x_1(t) \leq x \leq l
\]

where $w_z(x, t)$ denotes the vertical component of cable displacement and $h(t)$ is the increment in the horizontal component of tension owing to the point loads $P_1$ and $P_2$. Notice that the right-hand side of Eq. (B.1) is analogous to the shear force in a simply supported beam of uniform weight under the action of two point loads. Furthermore, static equilibrium under self-weight requires that:

\[
{H} \frac{dy(x)}{dx} = \frac{mgl}{2} \left(1 - \frac{2x}{l}\right).
\]

Performing integration of Eq. (B.1), imposing the boundary conditions $w_z(0, t) = w_z(l, t) = 0$ and satisfying the continuity requirements at the abscissas $x_1(t)$ and $x_2(t)$, the following closed-form expression of the quasi-static response $w_z(x, t)$ is obtained:

\[
w_z(x, t) = \frac{1}{H + h(t)} \left[P_1 \left(1 - \frac{x_1(t)}{l}\right)x + P_2 \left(1 - \frac{x_2(t)}{l}\right)x - h(t)y(x)\right], \quad 0 \leq x \leq x_2(t);
\]

\[
w_z(x, t) = \frac{1}{H + h(t)} \left[P_1 \left(1 - \frac{x_1(t)}{l}\right)x + P_2 \frac{1 - x}{l}x_2(t) - h(t)y(x)\right], \quad x_2(t) \leq x \leq x_1(t);
\]

\[
w_z(x, t) = \frac{1}{H + h(t)} \left[P_1 \left(1 - \frac{x}{l}\right)x_1(t) + P_2 \left(1 - \frac{x}{l}\right)x_2(t) - h(t)y(x)\right], \quad x_1(t) \leq x \leq l
\]

where Eq. (B.2) has been taken into account.

In order to evaluate $h(t)$, use is made of the so-called “cable equation” which accounts for the elastic and geometric compatibility of the cable (Irvine, 1981):

\[
\frac{h(t)L_c}{EA} = \frac{mg}{H} \int_0^l w_z(x, t)dx + \frac{1}{2} \int_0^l \left(\frac{\partial w_z(x, t)}{\partial x}\right)^2 dx
\]

in which $L_c = \int_0^l (ds/dx)^3 dx \cong l[1 + 8(d/l)^2]$. Substituting Eq. (B.3) into Eq. (B.4) and performing integration, the following cubic equation for $h(t)$ is obtained

\[
h^3 + \left(2 + \frac{h^2}{24}\right)HH^2 + \left(1 + \frac{h^2}{12}\right)H^3 - \frac{h^2H^3}{2Pm^2g^2} \left\{P_1l^2 + P_1mg\right\}^3 = 0
\]

where $\lambda^2 \cong 64(Ed/H)(d/l)^2$ is the so-called Irvine parameter, which accounts for cable geometry and elasticity. Notice that, due to geometrical nonlinearity, cable response to point loads is a function of position, load, geometry and elasticity.

References