A note on the boundary of the set where the decreasingly ordered spectra of symmetric doubly stochastic matrices lie

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Abstract

In this paper, we study the region $\Theta_n^s$ of $\mathbb{R}^n$ where the decreasingly ordered spectra of all the $n \times n$ symmetric doubly stochastic matrices lie with emphasis on the boundary set of $\Theta_n^s$. As applications, we study the case $n = 4$ and in particular we solve the inverse eigenvalue problem for $4 \times 4$ symmetric doubly stochastic matrices of trace zero by using different techniques than that used in [H. Perfect, L. Mirsky, Spectral properties of doubly stochastic matrices, Monatsh. Math. 69 (1965) 35–57]. Also, we solve the same problem for $4 \times 4$ symmetric doubly stochastic matrices of trace two which serves only to illustrate this paper’s method. In addition, we describe a nonconvex region $E_f$ of $\Theta_4^s$ which corresponds to new sufficient conditions for the $4 \times 4$ symmetric doubly stochastic matrices. At the end, we conjecture that $E_f = \Theta_4^s$.

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1. Introduction

An $n \times n$ matrix with real entries is nonnegative (positive) if all of its entries are nonnegative (positive). If $A$ and $B$ are two $n \times n$ nonnegative matrices, and if there exists a permutation matrix

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$P$ such that $B = P^T A P$, then we say that $A$ and $B$ are cogredient. If $A$ is cogredient to a matrix of the form
\[
\begin{pmatrix}
A_1 & 0 \\
A_2 & A_3
\end{pmatrix},
\]
where $A_1$ and $A_3$ are square matrices, then $A$ is called reducible. Otherwise $A$ is said to be irreducible. Clearly a positive matrix is irreducible.

A doubly stochastic matrix is a nonnegative matrix such that each row and column sum is equal to 1. The theory of doubly stochastic matrices has been the object of study for a long time. This particular interest in this theory as well as the theory of nonnegative matrices, stems from the fact that it has applications in Physics, Engineering, Economics and Operations Research (see [5]).

An intriguing problem in this theory is the inverse eigenvalue problem for doubly stochastic matrices which is concerned with finding necessary and sufficient conditions that $n$ complex numbers be the eigenvalues of an $n \times n$ doubly stochastic matrix. This problem is essentially equivalent to the problem of finding the region $\Theta_n$ of $\mathbb{C}^n$ where the spectra of all $n \times n$ doubly stochastic matrices lie. A subproblem of this problem is the symmetric inverse eigenvalue problem for doubly stochastic matrices which is in turn equivalent to describing the region $\Theta_n^s$ of $\mathbb{R}^n$ where the decreasingly ordered spectra of all $n \times n$ symmetric doubly stochastic matrices lie. More precisely, $\Theta_n^s = \{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n : 1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -1 \}$ and there exists an $n \times n$ symmetric doubly stochastic with spectrum $\lambda$. For more on this subject see [9,17,15,19,20] and the references therein. For more on nonnegative matrices see [1–3,8,14].

In this paper, we are concerned with the study of the region $\Theta_n^s$ with emphasis on the case $n = 4$. Note that the case $n = 2$ is trivial (see Section 3) while the case $n = 3$ was solved in [17,20] (see also [16] for a geometric solution to this problem).

This paper is organized as follows. In Section 2, we study the boundary of the region $\Theta_n^s$. Section 3 deals with the case $n = 4$. In particular, we solve the inverse eigenvalue problems for $4 \times 4$ symmetric doubly stochastic of trace zero and that of trace two which as mentioned earlier illustrates the new techniques used in this paper for these types of inverse problems. In addition, we prove some new results including describing a nonconvex region $E_f$ of $\Theta_n^s$, and based on some numerical computations not included here, we conjecture that $E_f = \Theta_n^s$.

First we establish some notation. Let $M_n^+(n)$ be the class of all $n \times n$ symmetric nonnegative matrices. The set of all $n \times n$ doubly stochastic (resp. symmetric doubly stochastic) matrices is denoted by $D_n$ (resp. $D_n^s$). Let $I_n$ be the $n \times n$ identity matrix, $J_n$ the $n \times n$ matrix all of whose entries are $\frac{1}{n}$, and $K_n$ the $n \times n$ matrix whose diagonal entries are all zeros and whose off-diagonal entries are all equal to $\frac{1}{n-1}$. If $p_1, p_2, \ldots, p_n$ are any points in $\mathbb{R}^n$ or in $M_n^+(n)$, then their convex hull will be denoted by Conv$(p_1, p_2, \ldots, p_n)$, and $p_i p_j = [p_1, p_j]$ will denote the line-segment joining $p_i$ to $p_j$ for $i, j = 1, 2, \ldots, n$.

Next let $E : M_n^+(n) \to \mathbb{R}^n$ be the map defined by $E(X) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the decreasingly ordered set of eigenvalues of the matrix $X$. Clearly by definition $E(D_n^s) = \Theta_n^s$, and also it is easy to see that $E(I_n) = (1, 1, \ldots, 1)$. Moreover, in [15] we proved the following:

**Lemma 1.1.** $E(J_n) = (1, 0, \ldots, 0)$ and $E(K_n) = \left(1, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1}\right)$.

Now concerning $\Theta_n$ and $\Theta_n^s$, we have the following theorem for which the proof can be found in [15].
Theorem 1.2. For $n \geq 4$, $\Theta_n^s$ and $\Theta_n$ are not convex.

Recall that $A_n$ is a convex polytope of dimension $(n - 1)^2$ where its vertices are the $n \times n$ permutation matrices (see [6]). On the other hand, $A_n^s$ is a convex polytope of dimension $\frac{1}{2}n(n - 1)$, where its vertices were determined in [10] (see also [7]), and where it was proved that if $A$ is a vertex of $A_n^s$, then $A = \frac{1}{2}(P + P^T)$ for some permutation matrix $P$, although not every $\frac{1}{2}(P + P^T)$ is a vertex.

Finally, we end with the following definition which is useful for our study, and will be used in the next theorem.

Definition 1.3. A set $\Gamma_n$ of $\mathbb{R}^n$ is said to be star convex with respect to a point $p \in \mathbb{R}^n$ if the line from any point in the set to $p$ is also contained in $\Gamma_n$.

In [15], we proved the following:

Theorem 1.4. The region $\Theta_n^s$ is star convex with respect to any point of the line-segment $E(I_nK_n) = E(I_n)E(K_n) = \left[ (1, 1, \ldots, 1), (1, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1}) \right]$.

2. Boundary sets of $\Theta_n^s$

Let $T_n = \{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n : 1 = \lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq -1$ and $1 + \lambda_2 + \cdots + \lambda_n \geq 0 \}$. Then $T_n$ is a convex polytope where its vertices were determined in [13] and where it is shown that $\Theta_n^s$ is strictly contained in $T_n$. Next, we have the following definitions:

Definition 2.1. A point $\lambda = (\lambda_1, \ldots, \lambda_n)$ in $\Theta_n^s$ is said to be wall-boundary if $\lambda_i = \lambda_j$ for some $i \neq j$.

Clearly the $E$-image of any reducible matrix in $A_n^s$ is wall-boundary since the eigenvalue value 1 is repeated. Wall-boundary points for any subset $\Gamma_n$ of $T_n$ are defined analogously.

Definition 2.2. A point $\lambda = (1, \lambda_2, \ldots, \lambda_n)$ in $\Theta_n^s$ is said to be extreme-boundary if for all $a > 1$, $a\lambda + (1 - a)E(I_n)$ is not the spectrum of any $n \times n$ symmetric doubly stochastic matrix.

Roughly speaking, a point $\lambda$ in $\Theta_n^s$ is extreme-boundary if moving in the direction $\overrightarrow{E(I_n)\lambda}$, we do not stay in $\Theta_n^s$. More generally, a point $\lambda$ in a subset $\Gamma_n$ of $T_n$ is said to be extreme-boundary, if moving in the direction $\overrightarrow{E(I_n)\lambda}$, we do not stay in $\Gamma_n$.

One class of extreme-boundary points for $\Theta_n^s$ is the set $E(A_n^s(0))$, where $A_n^s(0)$ is the convex polytope of $n \times n$ trace zero symmetric doubly stochastic matrices. Indeed, if $\lambda = (1, \lambda_2, \ldots, \lambda_n) \in E(A_n^s(0))$, then for all $a > 1$, $\lambda$ is not the spectrum of any $n \times n$ symmetric doubly stochastic matrix. Since otherwise there would exist an $a > 1$ and $A \in A_n^s$ such that $E(A) = a(1, \lambda_2, \ldots, \lambda_n) + (1 - a)E(I_n)$. However the trace of the doubly stochastic matrix $A$ is equal to $n(1 - a) < 0$, which is not possible.

A point $(1, \lambda_2, \ldots, \lambda_n)$ in $\Theta_n^s$ (or in any subset $\Gamma_n$ of $T_n$) is said to be boundary if it is either wall-boundary or extreme-boundary in that set.
Recall that if $\alpha$ is an eigenvalue of a symmetric doubly stochastic matrix, then by the Perron–Frobenius Theorem $-1 \leq \alpha \leq 1$. Thus we have the following:

**Proposition 2.3.** Let $\lambda = (1, \lambda_2, \ldots, \lambda_{n-1}, -1)$ be in $\Theta_n^s$. Then $\lambda$ is boundary.

Boundary points of $\Theta_n^s$ in the hyperplane $\lambda_n = -1$ are characterized by the following theorem for which the proof can be found in [15].

**Theorem 2.4.** Let $\lambda = (1, \lambda_2, \ldots, \lambda_{n-1}, -1)$ with $1 > \lambda_2 \geq \cdots \geq \lambda_{n-1} > -1$. Then

- If $n = 2k$ even, then $\lambda \in \Theta_n^s$ if and only if $\lambda = (1, \lambda_2, \ldots, \lambda_k, -\lambda_k, \ldots, -\lambda_2, -1)$.
- If $n = 2k + 1$ odd, then $\lambda$ cannot be in $\Theta_n^s$.

Note that the interior of $\Delta_n^s$ consists of all $n \times n$ positive symmetric doubly stochastic matrices. The next proposition is concerned with the $E$-image of this class of matrices.

**Proposition 2.5.** Let $X$ be in $\Delta_n^s$ such that $E(X)$ is not wall-boundary. If all the diagonal entries of $X$ are positive then $E(X)$ is not extreme-boundary in $\Theta_n^s$. In particular, the $E$-image of every positive symmetric doubly stochastic matrix is not extreme-boundary in $\Theta_n^s$.

**Proof.** Let $X = (x_{ij})$ and $x = \min(x_{ii}) i = 1, \ldots, n$. Define $a = \frac{1}{1-x}$ then $a > 1$ as $x < 1$ and $aE(X) + (1 - a)E(I_n)$ is the spectrum of the $n \times n$ symmetric doubly stochastic $aX + (1 - a)I_n$. $\square$

Although Proposition 2.5 shows that no interior point in $\Delta_n^s$ can have an extreme-boundary $E$-image, however we could have interior points in $\Delta_n^s$ for which their $E$-images are wall-boundary such as $E(J_n) = (1, 0, \ldots, 0)$.

The importance of boundary points lies in the fact that characterizing all boundary points of $\Theta_n^s$ is equivalent to solving the inverse eigenvalue problem for $n \times n$ symmetric doubly stochastic matrices. Boundary points for the real nonnegative inverse eigenvalue problem have been studied in [4,11].

### 3. Applications

This section is devoted to the study of the case $n = 4$. The case $n = 2$ is straightforward since $\Theta_2^s$ is equal to the line-segment $[(1, 1), (1, -1)]$. For the case $n = 3$, (see [17,16,20]),

$$\Theta_3^s = \text{Conv}\left((1, 1, 1), (1, 1, -1), (1, -\frac{1}{2}, -\frac{1}{2})\right).$$

For the case $n = 4$, following [10,7], the vertices of $\Delta_4^s$ are given by

$$P_1 = I_4,$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Lemma 3.1. $P_8 = 2 \left( \frac{P_2 + P_3}{2} \right) - I_4$, $P_9 = 2 \left( \frac{P_2 + P_3}{2} \right) - I_4$, $P_{10} = 2 \left( \frac{P_2 + P_3}{2} \right) - I_4$, $P_{11} = 3 \left( \frac{P_8 + P_9 + P_6}{3} \right) - \frac{1}{2} I_4$, $P_{12} = 3 \left( \frac{P_2 + P_3 + P_4}{3} \right) - \frac{1}{2} I_4$, $P_{13} = 3 \left( \frac{P_8 + P_9 + P_7}{3} \right) - \frac{1}{2} I_4$, $P_{14} = 3 \left( \frac{P_2 + P_3 + P_4}{3} \right) - \frac{1}{2} I_4$, $J_4 = P_{11} P_{12} P_{13} P_{14}$ and $K_4 = \frac{P_8 + P_9 + P_{10}}{3}$.

Lemma 3.2. $P_{11}$ commutes with $\{P_4, P_5, P_6\}$, and so does $P_{12}$ with $\{P_2, P_3, P_4\}$. Also $P_{13}$ commutes with $\{P_3, P_5, P_7\}$, and $P_{14}$ commutes with $\{P_2, P_6, P_7\}$. Moreover $\{P_2, P_5, P_8\}$ commute and so do $\{P_3, P_6, P_9\}$ and $\{P_4, P_7, P_{10}\}$.

Next define

$$P_{15} = \frac{1}{2} (P_2 + P_5) = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix},$$

and

$$P_{16} = \frac{1}{2} (P_8 + P_9) = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}.$$
Our main results in this section are based on the fact that commuting matrices are simultaneously diagonalizable. In addition, the actual analysis here depends on the three matrices \( U \), \( V \) and \( W_a \), where

\[
U = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix},
\]

\[
V = \begin{pmatrix}
1/2 & 1/2 & 1/\sqrt{2} & 0 \\
1/2 & 1/2 & -1/\sqrt{2} & 0 \\
1/2 & -1/2 & 0 & -1/\sqrt{2} \\
1/2 & -1/2 & 0 & 1/\sqrt{2}
\end{pmatrix},
\]

and

\[
W_a = \begin{pmatrix}
1/2 & 1/2 & \frac{2a+\sqrt{5a^2-2a+1}}{2\sqrt{1-2a+5a^2+2a\sqrt{1-2a+5a^2}}} & -\frac{-1+a}{2\sqrt{1-2a+5a^2+2a\sqrt{1-2a+5a^2}}} \\
1/2 & 1/2 & -\frac{2a+\sqrt{5a^2-2a+1}}{2\sqrt{1-2a+5a^2+2a\sqrt{1-2a+5a^2}}} & -\frac{-1+a}{2\sqrt{1-2a+5a^2+2a\sqrt{1-2a+5a^2}}} \\
1/2 & -1/2 & \frac{-2a+\sqrt{5a^2-2a-1}}{2\sqrt{1-2a+5a^2+2a\sqrt{1-2a+5a^2}}} & -\frac{-1-a}{2\sqrt{1-2a+5a^2+2a\sqrt{1-2a+5a^2}}} \\
1/2 & -1/2 & \frac{2a+\sqrt{5a^2-2a-1}}{2\sqrt{1-2a+5a^2+2a\sqrt{1-2a+5a^2}}} & -\frac{-1-a}{2\sqrt{1-2a+5a^2+2a\sqrt{1-2a+5a^2}}}
\end{pmatrix},
\]

where \( 0 \leq a \leq 1 \). Note that by inspection \( U \), \( V \) and \( W_a \) are orthogonal and for \( a = 0 \), \( W_0 = U \) and for \( a = 1 \), \( W_1 = V \).

Let \( A_4^x(0) \) be the set of all \( 4 \times 4 \) trace zero symmetric doubly stochastic matrices. Then \( A_4^x(0) = \text{Conv}(P_8, P_9, P_{10}) \), and the next theorem characterizes \( E(A_4^x(0)) \) and hence solves the inverse eigenvalue problem for the \( 4 \times 4 \) trace zero symmetric doubly stochastic matrices, which had been proved before in [17]. The construction done here is the same as that used in [12,18] albeit arrived at by different means.

**Theorem 3.3.** \( E(A_4^x(0)) = \text{Conv}\left(\{(1, 0, 0, -1), (1, 1, -1, -1), (1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})\}\right) \). Also, \( E(A_4^x(0)) \) is an extreme-boundary surface in \( \Theta_4^x \).

**Proof.** It is easy to check that for \( 0 \leq a \leq 1 \), \( U^T[aP_8 + (1-a)P_{16}]U \) is equal to

\[
a \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} + (1-a) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

Therefore the \( E \)-image of the line-segment \( P_8P_{16} \) is the line-segment joining the point \((1, 1, -1, -1)\) to \((1, 0, 0, -1)\). By Theorem 1.4, \( \Theta_4^x \) is star convex with respect to \( E(K_4) = \left(1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)\), then \( \text{Conv}\left(\{(1, 0, 0, -1), (1, 1, -1, -1), (1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})\}\right) \) is contained in \( E(A_4^x(0)) \subset \Theta_4^x \). To prove the equality, it suffices to see that \( E(A_4^x(0)) \) is contained in the intersection of the plane \( 1 + x + y + z = 0 \) with \( \Gamma \) which is just \( \text{Conv}\left(\{(1, 0, 0, -1), (1, 1, -1, -1), (1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})\}\right) \). \( \square \)
Corollary 3.4 [17]. Let $1 \geq x \geq y \geq z \geq -1$. If $1 + x + y + z = 0$, then $(1, x, y, z)$ is the spectrum of a $4 \times 4$ symmetric doubly stochastic matrix of trace zero.

Next define $A_4^4(2) = \text{Conv}\{P_2, P_3, P_4, P_5, P_6, P_7\}$ as the convex hull of all $4 \times 4$ symmetric doubly stochastic matrices of trace two. The next theorem characterizes the $E$-image of this convex polytope. It is worth mentioning here that the value of this next theorem is mainly in illustrating the methods to be used in Theorem 3.10 below.

Theorem 3.5. $E(A_4^4(2)) = \text{Conv}\{(1, 1, 0, 0), (1, 1, 1, -1), \left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\} \subset \Theta_4^4$.

Proof. Then for $0 \leq a \leq 1$, an inspection shows that $V^T[aP_2 + (1-a)P_{15}]V$ is given by

$$a \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + (1-a) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

Thus the line-segment $[(1, 1, 1, -1), (1, 1, 0, 0)]$ is contained in $\Theta_4^4$. Since $\Theta_4^4$ is star convex with respect to any point of the line-segment $\left[(1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}), (1, 1, 1, 1)\right]$ and in particular it is star convex with respect to the point $\left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. As a result $\text{Conv}\{(1, 1, 0, 0), (1, 1, 1, -1), \left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\}$ is contained in $\Theta_4^4$. Since all the matrices in $A_4^4(2)$ are of trace two, therefore $E(A_4^4(2))$ is contained in the plane $x + y + z = 1$. However, it is easy to see that the intersection of $\Gamma$ with the plane $x + y + z = 1$ is just $\text{Conv}\{(1, 1, 0, 0), (1, 1, 1, -1), \left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\}$. This completes the proof. □

Corollary 3.6. Let $1 \geq x \geq y \geq z \geq -1$. If $1 + x + y + z = 2$, then $(1, x, y, z)$ is the spectrum of a $4 \times 4$ symmetric doubly stochastic matrix of trace 2.

The next theorem describes the intersection of $\Theta_4^4$ with the hyperplane $z = -1$.

Theorem 3.7. The only subsets of the plane $z = -1$ which are contained in $\Theta_4^4$, are the two line-segments joining $(1, 1, -1, -1)$ to $(1, 0, 0, -1)$ and $(1, 1, -1, -1)$ to $(1, 1, 1, -1)$.

Proof. Let $1 > x > y > -1$. By Theorem 2.4, $\lambda = (1, x, y, -1) \in \Theta_4^4$ if and only if $\lambda = (1, x, -x, -1)$. This gives the line-segment $E(P_8P_{16}) = E(P_8)E(P_{16}) = [(1, 1, -1, -1), (1, 0, 0, -1)]$; that is the $E$-image of all irreducible (except for $P_8$) symmetric doubly stochastic matrices in the plane $z = -1$. Moreover, the line-segment $E(P_2P_8) = [(1, 1, -1, -1), (1, 1, 1, -1)]$ is also contained in $\Theta_4^4$ since for $0 \leq a \leq 1$, the matrix $V^T[aP_2 + (1-a)P_8]V$ is equal to

$$a \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + (1-a) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which is the $E$-image of all the reducible symmetric doubly stochastic matrices in the plane $z = -1$. □
The following lemma will be needed to conclude the results of Theorem 3.10 below.

**Lemma 3.8.** Let $0 \leq a \leq 1$, then $E(aP_2 + (1 - a)P_{16})$ is a curve $\gamma$ in the plane $x - y - z = 1$ given by:

$$
\left(1, a, \frac{1}{2} \left(a - 1 + \sqrt{5a^2 - 2a + 1}\right), \frac{1}{2} \left(a - 1 - \sqrt{5a^2 - 2a + 1}\right)\right).
$$

**Proof.** For $0 \leq a \leq 1$, a simple check shows that $W_a^T[aP_2 + (1 - a)P_{16}]W_a$ is equal to

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a & 0 & 0 \\
0 & 0 & \frac{1}{2} \left(-1 + a + \sqrt{5a^2 - 2a + 1}\right) & 0 \\
0 & 0 & 0 & \frac{1}{2} \left(-1 + a - \sqrt{5a^2 - 2a + 1}\right)
\end{pmatrix}.
$$

Note that $\frac{1}{2} \left(-1 + a + \sqrt{5a^2 - 2a + 1}\right) \geq \frac{1}{2} \left(-1 + a - \sqrt{5a^2 - 2a + 1}\right)$ for all $a$ and $a \geq \frac{1}{2} \left(-1 + a + \sqrt{5a^2 - 2a + 1}\right)$, and only if $0 \leq a \leq 1$. Finally, it is easy to see that the point

$$
\left(1, a, \frac{1}{2} \left(a - 1 + \sqrt{5a^2 - 2a + 1}\right), \frac{1}{2} \left(a - 1 - \sqrt{5a^2 - 2a + 1}\right)\right)
$$

is in the plane $x - y - z = 1$, and the proof is complete. □

Our next theorem defines the surface $(s)$, which will be used to define the region $E_f$ of $\Theta_4^s$ that corresponds to new sufficient conditions for the $4 \times 4$ symmetric doubly stochastic matrices which are given by Theorem 3.10.

**Theorem 3.9.** The $E$-image of Conv($P_8, P_2, P_{16}$) is the surface $(s)$ obtained by joining $E(P_8) = (1, 1, -1, -1)$ to the curve $\gamma$ via straight lines.

**Proof.** The proof can be easily seen from the fact that for $0 \leq a \leq 1$, and $0 \leq b \leq 1$ then $W_a^T[bP_8 + (1 - b)[aP_2 + (1 - a)P_{16}]]W_a$ is equal to

$$
b \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} + (1 - b) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a & 0 & 0 \\
0 & 0 & \frac{1}{2} \left(-1 + a + \sqrt{5a^2 - 2a + 1}\right) & 0 \\
0 & 0 & 0 & \frac{1}{2} \left(-1 + a - \sqrt{5a^2 - 2a + 1}\right)
\end{pmatrix}.
$$

This completes the proof. □

Recall that the region $\Theta_4^s$ is star convex with respect to any point of the line-segment $[E(I_4), E(K_4)]$. Then the region $E_f$ of $\mathbb{R}^4$ obtained by joining the line-segment $[(1, 1, 1, 1), (1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})]$ to the set
is contained in $\Theta_4^s$, where $(s)$ is the surface defined in Theorem 3.9.

As a result, the region $E_f$ can roughly be described as follows. Let $(s')$ and $(s'')$ be respectively the surfaces obtained by joining $(1, 1, 1, 1)$ and $(1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$ to the curve $\gamma$. Then $(s')$ and $(s'')$ intersect along a curve $\beta$ (see Figs. 1 and 2)\(^1\) so that the part $(ps')$ of $(s')$ bounded by the curve $\beta$, the curve $\gamma$ and the line-segment $\left[(1, 0, 0, -1), (1, \frac{1}{5}, \frac{1}{5}, -\frac{3}{5})\right]$, where $\left(1, \frac{1}{5}, \frac{1}{5}, -\frac{3}{5}\right)$ is the intersection of the two line-segments $E(I_4P_{16}) = [(1, 1, 1, 1), (1, 0, 0, -1)]$ of $(s')$ and $E(K_4P_2) = \left[(1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}), (1, 1, 1, -1)\right]$ of $(s'')$, is an extreme-boundary surface for $E_f$. Also, the part $(ps'')$ of $(s'')$ bounded by the curve $\beta$, the curve $\gamma$ and the line-segment $\left[(1, 1, 1, -1), (1, \frac{1}{5}, \frac{1}{5}, -\frac{3}{5})\right]$ is another extreme-boundary surface for $E_f$. Moreover, the remaining extreme-boundary surfaces of $E_f$ are the surface $(s)$ and the plane-surface $\text{Conv}\left((1, 1, 1, 1), (1, 0, 0, -1)\right)$. However, the wall-boundary surfaces of $E_f$ are given by

\begin{align*}
\text{Conv}\left((1, 1, 1, 1), (1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}), (1, 0, 0, -1)\right), \\
\text{Conv}\left((1, 1, 1, 1), (1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}), (1, 1, 1, -1)\right), \\
\text{Conv}((1, 1, 1, 1), (1, 1, 1, -1), (1, 1, 1, -1)),
\end{align*}

\(^1\) Generated by Maple.
An alternative way of obtaining the region \( E_f \) is by joining the point \((1, 1, 1, 1)\) to the surface \((s), (s'), (s'')\) and \(\text{Conv}\left((1, 1, -1, -1), \left(1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right), (1, 0, 0, -1)\right)\) (see Fig. 2).

Fig. 2. A view of the surfaces \((s), (s')\) and \((s'')\).

\[
\text{Conv}\left((1, 1, 1, 1), \left(1, \frac{1}{5}, \frac{1}{5}, -\frac{3}{5}\right), (1, 1, 1, -1)\right).
\]

More precisely, these surfaces can be described as follows. By Theorem 3.3, the \(E\)-image of \(\text{Conv}(P_8, K_4, P_{16})\) is given by

\[
E(\text{Conv}(P_8, K_4, P_{16})) = \text{Conv}\left((1, 1, -1, -1), \left(1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right), (1, 0, 0, -1)\right)
\]

since for \(0 \leq a \leq 1\) and \(0 \leq b \leq 1\),

\[
E(bP_8 + (1 - b)[aK_4 + (1 - a)P_{16}]) = \left(1, b - \frac{a}{3}(1 - b), -b - \frac{a}{3}(1 - b), \frac{2}{3}a(1 - b) - 1\right).
\]

Now by Theorem 3.9, the surface \((s)\) which is the \(E\)-image of \(\text{Conv}(P_8, P_2, P_{16})\) is given by \(E(bP_8 + (1 - b)[aP_2 + (1 - a)P_{16}]\) and is equal to
Similarly, from (3) we obtain
\[ b(1, 1, -1, -1) + (1 - b) \]
\[ \times \left[ \left( a, \frac{1}{2} \left( a - 1 + \sqrt{5a^2 - 2a + 1} \right), \frac{1}{2} \left( a - 1 - \sqrt{5a^2 - 2a + 1} \right) \right) \right]. \] (2)

On the other hand, making use of Lemma 3.8, the surface \((s'')\) can be defined explicitly as \(E(bK_4 + (1 - b)[aP_2 + (1 - a)P_{16}])\) and is equal to
\[ b \left( 1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3} \right) + (1 - b) \]
\[ \times \left[ \left( a, \frac{1}{2} \left( a - 1 + \sqrt{5a^2 - 2a + 1} \right), \frac{1}{2} \left( a - 1 - \sqrt{5a^2 - 2a + 1} \right) \right) \right]. \] (3)

Also by Lemma 3.8, the surface \((s')\) can be described explicitly in a similar fashion by \(E(bI_4 + (1 - b)[aP_2 + (1 - a)P_{16}])\) and is equal to
\[ b(1, 1, 1, 1) + (1 - b) \]
\[ \times \left[ \left( a, \frac{1}{2} \left( a - 1 + \sqrt{5a^2 - 2a + 1} \right), \frac{1}{2} \left( a - 1 - \sqrt{5a^2 - 2a + 1} \right) \right) \right]. \] (4)

From (1), the \(E\)-image of \(\text{Conv}(P_8, K_4, P_{16})\) is the set of \((x, y, z)\) satisfying: \(1 \geq x \geq y \geq z \geq -1\) and
\[ 1 + x + y + z = 0. \] (5)

From (2), we have
\[
\begin{aligned}
x &= a(1 - b) + b, \\
y &= \frac{1}{2} \left[ a(1 - b) - b - 1 + (1 - b)\sqrt{5a^2 - 2a + 1} \right], \\
z &= \frac{1}{2} \left[ a(1 - b) - b - 1 - (1 - b)\sqrt{5a^2 - 2a + 1} \right],
\end{aligned}
\]
so that \(b = \frac{1}{3}(x - y - z - 1)\) and \(a = \frac{x + y + z + 1}{5 - x - y - z}\). Substituting these values of \(a\) and \(b\) in the equation \(y - z = (1 - b)\sqrt{5a^2 - 2a + 1}\) to obtain the surface \((s)\) as the subset of \(I'\) defined by
\[ y - z - \sqrt{2x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 2y + 2z + 2} = 0. \] (6)

Similarly, from (3) we obtain
\[
\begin{aligned}
x &= a(1 - b) - \frac{1}{3}b, \\
y &= \frac{1}{2} \left[ a(1 - b) + b + \frac{1}{3} - 1 + (1 - b)\sqrt{5a^2 - 2a + 1} \right], \\
z &= \frac{1}{2} \left[ a(1 - b) + b + \frac{1}{3} - 1 - (1 - b)\sqrt{5a^2 - 2a + 1} \right],
\end{aligned}
\]
so that \(b = \frac{3}{2}(y + z - x + 1)\) and \(a = \frac{x + y + z + 1}{3x - 3y - 3z - 1}\). Substituting these values of \(a\) and \(b\) in the equation \(y - z = (1 - b)\sqrt{5a^2 - 2a + 1}\), we obtain the surface \((s'')\) as the subset of \(I'\) defined by
\[ y - z - \sqrt{2x^2 + 5y^2 + 5z^2 - 2xy - 2xz + 10yz + 6y + 6z + 2} = 0. \] (7)
Also from (4), we have
\[
\begin{align*}
x &= a(1 - b) + b, \\
y &= \frac{1}{2} \left[ a(1 - b) + 3b - 1 + (1 - b)\sqrt{5a^2 - 2a + 1} \right], \\
z &= \frac{1}{2} \left[ a(1 - b) + 3b - 1 - (1 - b)\sqrt{5a^2 - 2a + 1} \right],
\end{align*}
\]
so that \( b = \frac{1}{2}(y + z - x + 1) \) and \( a = \frac{3x - y - z - 1}{x - y - z + 1} \). Substituting these values of \( a \) and \( b \) this time in the equation \( y - z = (1 - b)\sqrt{5a^2 - 2a + 1} \), we obtain the surface \( (s') \) as the subset of \( \Gamma \) defined by
\[
y - z - \sqrt{10x^2 + y^2 + z^2 - 6xy - 6xz + 2yz - 8x + 2y + 2z + 2} = 0.
\]

Hence the region \( E_f \) which also can be defined as the set of real 4-tuples \((x, y, z)\) where \( 1 \geq x \geq y \geq z \geq -1 \) and \((x, y, z)\) satisfies simultaneously the following conditions:
\[
\begin{align*}
x + y + z + 1 &\geq 0, \\
y - z - \sqrt{2x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 2y + 2z + 2} &\leq 0, \\
y - z - \sqrt{2x^2 + 5y^2 + 5z^2 - 2xy - 2xz + 10yz + 6y + 6z + 2} &\leq 0, \\
y - z - \sqrt{10x^2 + y^2 + z^2 - 6xy - 6xz + 2yz - 8x + 2y + 2z + 2} &\leq 0.
\end{align*}
\]
is contained in \( \Theta_4^s \). Thus we have established the following:

**Theorem 3.10.** Let \( 1 \geq x \geq y \geq z \geq -1 \). If \((x, y, z)\) satisfies simultaneously
\[
\begin{align*}
x + y + z + 1 &\geq 0, \\
y - z - \sqrt{2x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + 2y + 2z + 2} &\leq 0, \\
y - z - \sqrt{2x^2 + 5y^2 + 5z^2 - 2xy - 2xz + 10yz + 6y + 6z + 2} &\leq 0, \\
y - z - \sqrt{10x^2 + y^2 + z^2 - 6xy - 6xz + 2yz - 8x + 2y + 2z + 2} &\leq 0.
\end{align*}
\]
then \((x, y, z)\) is the spectrum of a \( 4 \times 4 \) symmetric doubly stochastic matrix.

Note that for each point \( p \) in \( E_f \) it is easy to construct the solution matrix \( X \in A_4^s \) such that \( E(X) = p \).

With extensive numerical computations (using Maple) we are unable to find any \( 4 \times 4 \) symmetric doubly stochastic matrix for which its \( E \)-image is outside the region \( E_f \), however at this stage we are not able to prove this claim. This readily suggests the following conjecture:

**Conjecture 3.11.** Let \( 1 \geq x \geq y \geq z \geq -1 \). Then \((x, y, z)\) is the spectrum of a \( 4 \times 4 \) symmetric doubly-stochastic matrix if and only if \((x, y, z)\) belongs to \( E_f \). That is \( \Theta_4^s = E_f \).

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**References**
