Vanishing Minor Conditions for Inverse Zero Patterns

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ABSTRACT

We study the relationship between the zero-nonzero pattern of an invertible matrix and the vanishing minors of the matrix and of its inverse. In particular we show how to determine when a matrix B could be the inverse of a matrix A with a given zero-nonzero pattern. In fact, there is always a set of almost principal minors of B (in one-to-one correspondence with the set of zero entries of A) whose vanishing implies that B^{-1} has zeros everywhere that A does, provided certain principal minors of B do not vanish.

1. INTRODUCTION

We study the relationships between the zero-nonzero pattern of a square, invertible matrix A and the zero minors of A and of A^{-1}. We confine our attention to the case in which A is irreducible and all diagonal entries of A

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are nonzero. As a matrix must have a generalized diagonal of all nonzero entries in order to be invertible, there is little loss of generality in our assumptions. We note that for almost every matrix A satisfying our assumptions, all entries of A^{-1} will be nonzero.

There are two primary aspects of this topic. First, we say that a minor of A is *generically zero* provided that the corresponding minor of every matrix with the same zero-nonzero pattern as A is zero. If a matrix A has several zero entries, these may imply, depending upon their location in A, that some minors of A are generically zero. Because of Jacobi's identities [8], the vanishing of a minor of A implies the vanishing of a certain minor of $B = A^{-1}$. How may we describe all the minors of B that vanish because of the zero pattern of A? Since all entries of B will typically be nonzero, the vanishing minors in B do not vanish generically. Nevertheless, when such minors in B vanish, they may collectively imply that certain entries of A must be zero. This suggests the second aspect of our problem. From which sets of vanishing minors in B can we recover the zero-nonzero pattern of A? In particular, can minimal sets of vanishing minors in B that compel A to have a specified zero-nonzero pattern be identified?

Such questions have previously been studied in several highly structured cases:

(a) the matrix A is tridiagonal (e.g., [6, 3]),
(b) the matrix A is banded (e.g., [1, 4]),
(c) the matrix A is in "comrade" form [2].

(The comrade matrices are tridiagonal, except that the last row is also allowed to be nonzero.) The above papers do not fully address the question of identifying minimal sets of vanishing minors in B $(= A^{-1})$ that imply the zero-nonzero pattern of A. Their results on vanishing minors are included among the general results presented here. See also [11] for a long list of references about inverses of matrices with specific zero-nonzero structure.

To illustrate the scope of our results we mention two examples. First, let $A_1 = [a_{ij}^{(1)}]$ with $a_{ij}^{(1)} \neq 0$ except for $a_{i,i+1}^{(1)} = 0$, $i = 1, \dots, n-1$, and $a_{n,1}^{(1)} = 0$. Thus A_1 has exactly n zero entries. The second example is $A_2 = [a_{ij}^{(2)}]$ in which, except along the principal diagonal, $a_{ij}^{(2)} \neq 0$ if and only if $a_{ij}^{(1)} = 0$. Thus A_2 has exactly n nonzero entries not on its principal diagonal. Both of these matrices are irreducible and satisfy $a_{ii} \neq 0$, $i = 1, 2, \dots, n$. Our results fully address the two aspects of our problem for both of these examples.

In Section 2 we show how to find all of the generically zero minors of A when its zero-nonzero pattern is given. We then identify the corresponding vanishing minors of $B = A^{-1}$.

In Section 3 we explain a fundamental relationship that must exist between the location of zero entries of the matrix A and sets of vanishing minors of B.

In Section 4 we give one solution to the minimal-set-of-vanishing-minors problem mentioned above. We prove, in fact, that if certain principal minors of $B = A^{-1}$ do *not* vanish and if the matrix A has exactly μ zeros among its entries, then there is always a set of μ vanishing almost principal minors of B that imply that B^{-1} has zeros where A does.

Finally, in Section 5 we present several examples (including the matrices A_1 and A_2 above) that illustrate our results.

2. GENERICALLY VANISHING MINORS

Let $N = \{1, 2, \dots, n\}$, and for $\alpha \subseteq N$ let α^c be the complement of α relative to N. We use $A[\alpha|\beta]$ to denote the submatrix of A in rows α and columns β. We write $A[\alpha|\beta] = 0$ to indicate that the submatrix in rows α and columns β is a zero submatrix. When $\alpha = \beta$ we let $A[\alpha] = A[\alpha|\alpha]$, as is customary. If $|\alpha| = |\beta|$ and $|\alpha \cap \beta| = |\alpha| - 1$, then $A[\alpha|\beta]$ is called an *almost principal submatrix*. In this event $A[\alpha|\beta]$ is square, and $\det A[\alpha|\beta]$ is called an *almost principal minor*.

If C is a $k \times k$ matrix, the condition that $\det C = 0$ generically is classical and given by the following well-known result.

THEOREM A (Frobenius-König). *Let C be a $k \times k$ matrix. Then* $\det C = 0$ *generically if and only if there is an $r \times s$ submatrix of zero entries of C with $r + s \geqslant k + 1$.*

This theorem first appeared in [5]; an interesting discussion of the result as well as a proof using Hall's theorem may be found in [10].

Theorem A makes it clear that we must concern ourselves with submatrices $A[\alpha|\beta]$ of A satisfying $A[\alpha|\beta] = 0$. Note that for a zero submatrix consisting of a single zero entry of A we have $r + s = 2$, so that no minors of A of size greater than or equal to two will be generically zero because of the vanishing of a single entry. Therefore we will call a zero submatrix of A *proper* if $r + s \geqslant 3$. If a zero entry of A is not contained in any proper zero submatrix of A, we call it an *isolated zero* of A. Observe that the matrix A_1 introduced above has exactly n isolated zeros and no proper zero submatrices. Notice that a proper zero submatrix is not the same thing as a proper submatrix.

Because of our assumption that A is irreducible, the following concept is basic. Let $\alpha \subseteq N$ properly. We call α a *separator of* A if there exists at least one partition of α^c into (nonempty) parts β, γ such that $A[\beta|\gamma] = 0$. Thus to every separator of A there corresponds at least one zero submatrix of A. Observe that several zero submatrices may result from a separator of A by virtue of the fact that there may be several different partitions of α^c meeting the stated requirements.

LEMMA 1. *Let A be an irreducible $n \times n$ matrix satisfying $a_{ii} \neq 0$ $(i = 1, 2, \dots, n)$. Then A has a proper zero submatrix if and only if A has a separator α with $|\alpha| \leqslant n - 3$.*

Proof. We have already seen that if A has a separator, it has a zero submatrix. If the zero submatrix is proper, then $|\beta| + |\gamma| \geqslant 3$, and hence $|\alpha| \leqslant n - 3$. Thus, the "if" part of the lemma is established. For the converse, suppose A has a proper zero submatrix $A[\beta|\gamma] = 0$ with $|\beta| + |\gamma| \geqslant 3$. Then, since $\beta \cap \gamma = \varnothing$ (because of the assumption about diagonal entries), we define $\alpha^c = \beta \cup \gamma$, so that $|\alpha^c| \geqslant 3$. This implies that $|\alpha| \leqslant n - 3$ and α is a separator of A, thus proving the "only if" statement. ■

A zero submatrix $A[\alpha|\beta]$ is called *maximal* if it is not a proper submatrix of any other zero submatrix of A. Thus $A[\alpha|\beta] = 0$ is maximal if and only if no column of $A[\alpha|\beta^c]$ is zero and no row of $A[\alpha^c|\beta] = 0$. If α is a separator of A, we call it a *maximal separator* if it induces at least one maximal zero submatrix of A.

Suppose $A[\beta|\gamma] = 0$. Then $\det A[\delta|\varepsilon] = 0$ generically if $\beta \subseteq \delta$, $\gamma \subseteq \varepsilon$, $|\delta| = |\varepsilon|$, and $|\delta| = |\beta| + |\gamma| - 1$. That these are the largest minors of A required to be generically zero by the existence of the zero submatrix $A[\beta|\gamma]$ of A follows from Theorem A. Also Theorem A implies that every generically zero minor (of size $\geqslant 2$) of A is a maximal (w.r.t. index sets) minor of A required to be zero by some proper zero submatrix of A. We call the largest generically zero minors of A corresponding to a proper zero submatrix of A the generically zero minors of A *induced* by the zero submatrix.

THEOREM 2. *A minor of A is generically zero if and only if it is a zero entry or a zero minor induced by a proper zero submatrix of A.*

Theorem 2 shows that to find all of the generically zero minors of A we must find all of the zero minors induced by all proper zero submatrices of A. The matrix A_1, introduced above, has no proper zero submatrices; hence, by Theorem 2, it has no generically zero minors (of size greater than 1). On the other hand, suppose for the matrix A_2 that $n \geqslant 5$. Then $\det A_2[1, 2|4, 5] = 0$ generically, and it is induced by all of the zero submatrices $A_2[1|4, 5]$, $A_2[2|4, 5]$, $A_2[1, 2|4]$, $A_2[1, 2|5]$. Of course, $A_2[1, 2|4, 5]$ is itself a zero submatrix; hence the corresponding minor is generically zero. Thus there can be a great deal of redundancy in finding the set of all generically zero minors of A by using induced minors of proper zero submatrices. Nevertheless the condition of Theorem 2 seems to provide the simplest means of identifying all generically zero minors of A.

Next suppose that A is invertible, and set $B = A^{-1}$. By the Jacobi identities the following is true: If $\det A[\delta|\varepsilon] = 0$ (generically or not), then $\det B[\varepsilon^c|\delta^c] = 0$.

COROLLARY 3. *If the minor* $A[\delta|\varepsilon]$ *of* A *vanishes generically, then* $\det B[\varepsilon^c|\delta^c] = 0$.

We come next to a fundamental result.

THEOREM 4. *Let* $A[\beta|\gamma]$ *be a zero submatrix of* A *belonging to the separator* α, *and let* $B = A^{-1}$. *Then* $\det B[\alpha \cup \{i\}|\alpha \cup \{j\}] = 0$ *for all* $i \in \beta$ *and all* $j \in \gamma$.

Proof. Suppose that $i \in \beta$ and $j \in \gamma$. Let $\delta = \beta \cup \gamma - \{i\}$ and $\varepsilon = \beta \cup \gamma - \{j\}$. Then $A[\beta|\gamma]$ is a zero submatrix of $A[\varepsilon|\delta]$. Theorem A implies that $\det A[\varepsilon|\delta]$ vanishes generically. Hence, from the Jacobi identities we conclude that $\det B[\alpha \cup \{i\}|\alpha \cup \{j\}] = 0$. ∎

One final concept is required. Suppose row i of the matrix A has a nonvoid set of zero entries. This set does not include a_{ii}, since we are assuming that $a_{ii} \neq 0$, $i = 1, 2, \dots, n$. We call α the *row separator of row* i if the set α contains exactly the set of indices k for which $a_{ik} \neq 0$ excluding the index i itself. Similarly, when column i of A has a nonvoid set of zero entries, we call α the *column separator of column* i if the set α contains exactly the set of indices k for which $a_{ki} \neq 0$ excluding the index i itself. If a row (or column) of A contains no zero entries, it does not have a row (column) separator. For the matrix A_1, row 2 has the separator $\alpha = \{1, 4, 5, \dots, n\}$, while for A_2 row 2 has the separator $\alpha = \{3\}$.

Observe that if α is a row separator for row i and we let

$$\alpha = \{k \in N \setminus \{i\} \,|\, a_{ik} \neq 0\}, \qquad \beta = \{i\}, \qquad \gamma = \{k \in N | a_{ik} = 0\},$$

or if α is a column separator for column i and we let

$$\alpha = \{k \in N \setminus \{i\} \,|\, a_{ki} \neq 0\}, \qquad \beta = \{i\}, \qquad \gamma = \{k \in N | a_{ki} = 0\},$$

then these partitions give $\alpha^c = \beta \cup \gamma$, showing that row and column separators are separators.

3. ZEROS AND INVERSES

Let the invertible matrix $B = [b_{ij}]$ have inverse $A = [a_{ij}]$. Use i^c as short for $\{i\}^c$. Then $B[i^c|j^c]$ is the submatrix of B obtained by deleting row i and column j of B. Set $B_{ij} = \det B[j^c|i^c]$. Obviously $a_{ij} = 0$ if and only if $B_{ij} = 0$.

Let $\{x_1, x_2, \ldots, x_p\}$ be a linearly dependent set of nonzero vectors in R^k, $k + 1 \geqslant p$. We call this a *minimally dependent set* if any proper subset of these vectors is a linearly independent set.

For given $i \in N$ we denote by $r_k(i)$ the row vector $r_k(i) = (b_{k1}, \ldots, b_{k,i-1}, b_{k,i+1}, \ldots, b_{kn})$, and by $c_k(i)$ the column vector $c_k(i) = (b_{1k}, \ldots, b_{i-1,k}, b_{i+1,k}, \ldots, b_{nk})^T$. We call $r_k(i)$ a punctured row of B and $c_k(i)$ a punctured column.

THEOREM 5. *Let B be an $n \times n$ matrix satisfying* $\det B[k^c] \neq 0$ *for $k = 1, 2, \ldots, n$. Suppose $A = B^{-1}$ exists. Then the set of positions of zero entries in row i of A is the nonempty set $\{j_1, \ldots, j_p\}$ if and only if $\{r_k(i)$: $k \neq j_1, \ldots, j_p\}$ is a minimally dependent subset of $\{r_k(i) | k = 1, 2, \ldots, n\}$.*

Proof. Since $\det B[i^c] \neq 0$, the set $\{r_k(i) | k \neq i\}$ is a linearly independent set of $n - 1$ vectors in R^{n-1}.Therefore $r_i(i)$ is a linear combination of these vectors. Suppose that $r_i(i) = \sum_{k \neq i} \lambda_k r_k(i)$. Then the only nontrivial dependencies of $\{r_k(i) | 1 \leqslant k \leqslant n\}$ have the form $\sum_{k=1}^{n} a_k r_k(i) = 0$, where $a_k = -a_i \lambda_k$ for $k \neq i$. It follows that $\{r_k(i) | \lambda_k \neq 0\} \cup \{r_i(i)\}$ is the unique minimally dependent subset of $\{r_k(i)\}$. Thus $a_{ij} = \det B[j^c|i^c] = 0$ if and only if $j \in \{k | \lambda_k = 0\}$. ∎

It is clear that a similar relationship holds between the punctured columns of B and the nonzero entries in the columns of A.

Notice that if the matrix A has an isolated zero as the i, j entry, we have $a_{ik} \neq 0$ if $k \neq j$. It follows that the set $\{r_k(i) | k \neq j\}$ forms a minimally linearly dependent subset and every proper subset is independent. Thus the matrix $B[j^c|i^c]$ has rank $n - 2$. Therefore we obtain the same unique vanishing minor of B when A has an isolated zero, no matter whether we use Theorem 5 or the corresponding result for punctured columns.

We remind the reader that $\{r_{j_1}(i), \ldots, r_{j_p}(i)\}$ is a dependent set of vectors in R^{n-1} if and only if every minor of order p of the $(n - 1) \times p$ matrix with these vectors as row vectors is zero. Therefore we have the following corollary of Theorem 5.

COROLLARY 6. *Let B be an $n \times n$ irreducible matrix satisfying $b_{kk} \neq 0$ and* $\det B[k^c] \neq 0$ *for $k = 1, 2, \ldots, n$. Suppose $A = B^{-1}$ exists. Then the set of positions of zero entries in row i of A is the nonempty set $\{j_1, \ldots, j_p\}$ if and* *only if* $\det B[j_1, \ldots, j_p | k_1, \ldots, k_p] = 0$ *for every set* $\{k_1, k_2, \ldots, k_p\} \subseteq N \setminus \{i\}$, *and if* $\{l_1, \ldots, l_q\}$ *is any set of distinct integers contained in* N *with* $q < p$, *then* $\det B[l_1, \ldots, l_q | k_1, \ldots, k_q] \neq 0$ *for at least one set* $\{k_1, \ldots, k_q\} \subseteq N \setminus \{i\}$.

Obviously a similar Corollary 6′ holds relative to Theorem 5′. Thus we have for each i such that row i of B^{-1} has p nonzero entries a family of at least $\binom{n-1}{p}$ vanishing minors of B. These theorems and corollaries do not determine how many minors in these families are *required* to account for all of the zero entries of A.

However, some further insight is obtained in the following way. We call two zero entries of the matrix A *independent* if they lie in different rows and columns. Define the *zero-term rank of* A to be the minimum number of rows and/or columns needed to cover the zero entries of A. The following Menger-type theorem is easy to prove (see Harary [7, Chapter 5] for example).

THEOREM 7. *The zero-term rank of a square matrix* A *equals the maximum number of pairwise independent zero entries of* A.

This result tells us that the number of families of vanishing minors of B as specified in Corollary 6 does not exceed the maximum number of pairwise independent zero entries of A.

The matrices A_1 and A_2 have zero-term rank n, as do the tridiagonal matrices. On the other hand, the comrade matrices studied in [2] have zero-term rank $n - 1$.

4. MINIMAL SETS OF ZERO MINORS

Although the zero-term rank tells us the smallest number of sets of linearly independent punctured rows of B and/or sets of linearly independent punctured columns of B needed to identify the zero-nonzero pattern of $A = B^{-1}$, we can in fact do better by judiciously selecting the vanishing minors of B. The key to doing this is to make use of Theorem 4 and the next result.

THEOREM 8. *Let* B *be irreducible, and let* $\det B[i^c] \neq 0$, $i = 1, 2, \ldots, n$. *Suppose the nonempty sets* α, β, γ *form a partition of* N *such that* $\det B[\alpha] \neq 0$, $\det B[\alpha \cup \beta] \neq 0$, *and* $\det B[\alpha \cup \gamma] \neq 0$, *and that* $\det B[\alpha \cup \{i\} | \alpha \cup \{j\}] = 0$ *for* $i \in \beta$, $j \in \gamma$. *Then* $A = B^{-1}$ *exists,* $A[\beta | \gamma]$ *is a zero submatrix of* A, *and* α *is a separator of* A. *Finally, if in addition* $\det B[\alpha \cup \{i\} | \alpha \cup \{j\}] \neq 0$ *for at least one* $j \in \gamma$ *for each* $i \in \beta^c$ *and also for at least one* $i \in \beta$ *for each* $j \in \gamma^c$, *then* α *and* $A[\beta | \gamma]$ *are maximal.*

Proof. Without loss of generality we may suppose that the rows and columns of B are permuted so that $\beta = \{1, 2, \dots, k\}$, $\alpha = \{k+1, \dots, m-1\}$, $\gamma = \{m, \dots, n\}$, where $m \geqslant k+2$. Then the matrices $C_1 = [c_{ij}^{(1)}]$ where $c_{ij}^{(1)} = \det B[\alpha \cup \{i\} | \alpha \cup \{j\}]$ for $i, j \in \beta$ and $C_2 = [c_{ij}^{(2)}]$ where $c_{ij}^{(2)} = \det B[\alpha \cup \{i\} | \alpha \cup \{j\}]$ for $i, j \in \gamma$ satisfy

$$\det C_1 = (\det B[\alpha])^k \det B[\alpha \cup \beta],$$

$$\det C_2 = (\det B[\alpha])^{n-m} \det B[\alpha \cup \gamma],$$

by Sylvester's identity, and hence are nonzero. On the other hand, the matrix $C = [c_{ij}]$ with $c_{ij} = \det B[\alpha \cup \{i\} | \alpha \cup \{j\}]$ for i and j in α^c has the form

$$\det C = \begin{vmatrix} C_1 & 0 \\ C_{22} & C_2 \end{vmatrix} = (\det B[\alpha])^{n-m+k+2} \det B;$$

hence $\det C \neq 0$, so $\det B \neq 0$.

To prove $A[\beta|\gamma] = 0$ we do as follows. Now we have

$$\det B[\alpha \cup \{i\} | \alpha \cup \{j\}] = \det \begin{bmatrix} B[i|\alpha] & b_{ij} \\ B[\alpha] & B[\alpha|j] \end{bmatrix} = 0$$

$$\text{for} \quad i \in \beta, \quad j \in \gamma.$$

Since $\det B[\alpha] \neq 0$, this implies that the Schur complement of $B[\alpha]$, namely $b_{ij} - B[i|\alpha](B[a])^{-1}B[\alpha|j]$, is zero. Applying this for all $i \in \beta$, $j \in \gamma$ yields

$$B[\beta|\gamma] = B[\beta|\alpha](B[\alpha])^{-1} B[\alpha|\gamma].$$

Then we can write

$$B[\beta \cup \alpha|\gamma] = B[\beta \cup \alpha|\alpha](B[\alpha])^{-1} B[\alpha|\gamma]. \tag{1}$$

It follows that

$$A = \begin{bmatrix} B[\alpha \cup \beta] & 0 \\ 0 & I_{|\gamma|} \end{bmatrix} \begin{bmatrix} I_{|\beta|} & 0 & 0 \\ 0 & I_{|\alpha|} & (B[\alpha])^{-1} B[\alpha|\gamma] \\ B[\gamma|\beta] & B[\gamma|\alpha] & B[\gamma] \end{bmatrix} \tag{2}$$

from (1). We conclude that $A[\beta|\gamma] = 0$.

It remains to establish maximality. Since B is irreducible, so is A, and $\det B[i^c] \neq 0$, $i = 1, 2, \dots, n$, implies $a_{ii} \neq 0$. Thus Theorem 4 applies to A. Suppose for contraction that $A[\beta|\gamma]$ is not a maximal zero submatrix of A. Then for some $i \in \beta^c$ or some $j \in \gamma^c$ we have $A[i|\gamma] = 0$ or $A[\beta|j] = 0$. But then by Theorem 4 we must have $\det B[\alpha \cup \{i\}|\alpha \cup \{j\}] = 0$ either for an $i \in \beta^c$ and all $j \in \gamma$ or for a $j \in \gamma^c$ and all $i \in \beta$, contrary to the conditions on B. Thus α and $A[\beta|\gamma] = 0$ must be maximal. ■

Notice that it follows from the proof of Theorem 8 that, if B^{-1} exists, the conditions $\det B[\alpha] \neq 0$ and $\det B[\alpha \cup \{i\}|\alpha \cup \{j\}] = 0$ for $i \in \beta$, $j \in \gamma$ imply that $A[\beta|\gamma] = 0$ where $A = B^{-1}$. Therefore we may regard each zero submatrix of A as being induced by the vanishing of an appropriate set of almost principal minors of B together with the nonvanishing of an appropriate principal minor of B.

We now apply Theorem 8 to determine conditions on a matrix B that guarantee that B^{-1} exists and has zeros in a given set of entries.

We say that a finite set of partitions of N, $\{\alpha_k, \beta_k, \gamma_k\}$, $k = 1, 2, \dots, r$, $r \leqslant n$, *covers the zeros* of the matrix A if $\beta_j \cap \beta_k = \varnothing$ and $\gamma_j \neq \gamma_k$ for $j \neq k$, and the submatrices $A[\beta_k|\gamma_k] = 0$, and these are the only zeros of the matrix A.

LEMMA 9. *Let A be an irreducible matrix with $a_{ii} \neq 0$ for $i = 1, 2, \dots, n$, and with a given zero-nonzero pattern. Then there exists a finite set of partitions of N that covers the zeros of A.*

Proof. For each $i \in N$ let $C_i = \{k \in N | a_{ik} = 0\}$. Let r_{i_1} be the first row of A such that $C_{i_1} \neq \varnothing$, and set $\beta_1 = \{i_1, \dots, i_q\}$, where $i_1 < i_2 < \cdots < i_q$ and $C_{i_j} = C_{i_1}$, $j = 2, \dots, q$. Since $i_1 \in \beta_1$, $\beta_1 \neq \varnothing$. Let $\gamma_1 = C_{i_1}$ and $\alpha_1 = N \setminus \{\beta_1 \cup \gamma_1\}$. Then $\gamma_1 \neq \varnothing$ by definition, and $\alpha_i \neq \varnothing$ because A is irreducible. Thus $\{\alpha_1, \beta_1, \gamma_1\}$ is a partition of N, and $A[\beta_1|\gamma_1] = 0$. Moreover, $A[\beta_1|\gamma_1^c]$ has no nonzero entries. Now consider the set β_1^c. For C'_{i_1} the first row (if any exist) of A not in the set β_1 satisfying $C'_{i_1} \neq \varnothing$, we construct a second partition of N, $\{\alpha_2, \beta_2, \gamma_2\}$, in the same way. By the construction, $A[\beta_2|\gamma_2] = 0$, $A[\beta_2^c|\gamma_2]$ has no nonzero entries, $\beta_1 \cup \beta_2 = \varnothing$, and $\gamma_1 \neq \gamma_2$. We then consider the row set $(\beta_1 \cup \beta_2)^c$. Clearly, in a finite number $r \leqslant n$ of steps we cover all of the zeros of A with the partitions $\{\alpha_k, \beta_k, \gamma_k\}$, and all of the zeros of A belong to the sets $A[\beta_k|\gamma_k]$, $k = 1, 2, \dots, r$. ■

Observe that if we use columns in place of rows we may also cover the zeros of A using partitions of N such that $\beta_j \neq \beta_k$ and $\gamma_j \cap \gamma_k = \varnothing$ for $j \neq k$.

THEOREM 10. *Let B be irreducible, and suppose that* $\det B[i^c] \neq 0$ *for $i = 1, 2, \dots, n$. Let A be an irreducible matrix, with $a_{ii} \neq 0$ for $i =$*

$1, 2, \ldots, n$. *Suppose that* $\{\alpha_k, \beta_k, \gamma_k\}$, $k = 1, 2, \ldots, r$, *is a finite set of partitions of* N *that covers the zeros of* A. *Then* B^{-1} *exists and has a zero everywhere that* A *has a zero if and only if the following conditions are satisfied*:

(i) *For at least one value of* k, $\det B[\alpha_k \cup \beta_k] \neq 0$ *and* $\det B[\alpha_k \cup \gamma_k] \neq 0$.

(ii) *For each* k, $\det B[\alpha_k] \neq 0$ *and* $\det B[\alpha_k \cup \{i\} | \alpha_k \cup \{j\}] = 0$ *for* $i \in \beta_k$, $j \in \gamma_k$.

Proof. By Theorem 8, (i) implies that B^{-1} exists, and (ii) implies that $B^{-1}[\beta_k | \gamma_k] = 0$, $k = 1, 2, \ldots, r$. Since $\{\alpha_k, \beta_k, \gamma_k\}$ for $k = 1, 2, \ldots, r$ covers the zeros of A, the result follows. ■

The natural question to ask is the following. Can B^{-1} have zero entries where A has nonzero entries? Since B^{-1} exists, the hypothesis that $\det B[i^c] \neq 0$ for $i = 1, 2, \ldots, n$ implies that if $B^{-1} = [\tilde{b}_{ij}]$, then $\tilde{b}_{ii} \neq 0$, $i = 1, 2, \ldots, n$. Next, if we impose the conditions that $\det B[\alpha_k \cup \{i\} | \alpha_k \cup \{j\}] \neq 0$ for at least one $j \in \gamma_k$ for each $i \in \beta_k^c$ and also for at least one $i \in \beta_k$ for each $j \in \gamma_k^c$, then the zero submatrices $B^{-1}[\beta_k | \gamma_k]$ are all maximal. It follows that no column of $B^{-1}[\beta_k^c | \gamma_k]$ and no row of $B^{-1}[\beta_k | \gamma_k^c]$ can be zero. These observations do not imply that B^{-1} cannot have additional zeros to those of A in the general case. In fact, this can often happen. Observe, however, that B^{-1} cannot have an additional (generic) zero at a fixed location, i.e., a zero at a fixed location for all possible choices of entries of B such that B^{-1} exists. For if it has, we have chosen the wrong zero-nonzero pattern for A. But there is at least one important special case where B^{-1} cannot, in fact, have additional zeros.

COROLLARY 11. *Let* B *and* A *satisfy the conditions of Theorem* 10, *and suppose in addition that* A *is nearly reducible, i.e., if any nonzero entry of* A *not on the principal diagonal is changed to zero, the resulting matrix* A' *is reducible. Then* B^{-1} *has the same zero-nonzero pattern as* A.

Proof. By Theorem 10 we know that B^{-1} exists and has a zero everywhere that A has a zero. Thus B^{-1} is nearly reducible. Also, our hypotheses imply that the principal diagonal of B^{-1} has no zero entries. It follows that, if B^{-1} had a zero entry in addition to those of A, it would be reducible. This contradicts the hypothesis that B is irreducible. Hence if $B^{-1} = [\tilde{b}_{ij}]$, $\tilde{b}_{ij} = 0$ if and only if $a_{ij} = 0$. ■

Theorem 10 and Corollary 11 provide basic results that can be used in a variety of ways, but they leave open some interesting questions. Here is an example. Let B be an $n \times n$ matrix (which may or may not be invertible), and suppose $\alpha \subseteq N$ and there exists a partition β, γ of α^c such that the almost principal minors $\det B[\alpha \cup \{i\}|\alpha \cup \{j\}] = 0$ for $i \in \beta$, $j \in \gamma$. Then which additional minors of B must vanish?

We observe that Theorem 10 provides us with a minimal set of vanishing minors of B, which implies that B^{-1} has at least the zero entries of A. This set is not unique in general, and it does not usually consist only of vanishing minors of smallest order of B. When will there exist a set of vanishing minors of B of smallest order which implies that B^{-1} will have zeros everywhere that A does? If such a set exists, how small can it be?

5. EXAMPLES

First consider an irreducible tridiagonal matrix A. The zero sets for the rows are all distinct, so that a set of partitions that covers all of the zeros has n elements. We set $\beta_i = \{i\}$, $i = 1, 2, \dots, n$, $\gamma_1 = \{3, 4, \dots, n\}$; $\gamma_2 = \{4, 5, \dots, n\}$; $\gamma_k = \{1, 2, \dots, k-2, \dots, n\}$, $3 \leqslant k \leqslant n-2$; $\gamma_{n-1} = \{1, 2, \dots, n-3\}$; $\gamma_n = \{1, 2, \dots, n-2\}$. The corresponding sets α_k are $\alpha_1 = \{2\}$; $\alpha_k = \{k-1, k+1\}$, $2 \leqslant k \leqslant n-1$; $\alpha_n = \{n-1\}$. Therefore, if B is irreducible ($\det B[k^c] \neq 0$ for $k = 1, 2, \dots, n$), then B^{-1} has a zero entry everywhere A does if:

(iT) For at least one k one has $\det B[\alpha_k] \neq 0$, $\det B[\alpha_k \cup \beta_k] \neq 0$, and $\det B[\alpha_k \cup \gamma_k] \neq 0$.

(iiT) One has

$$b_{22} \neq 0 \text{ and } \det B[1,2|2,j] = 0, \qquad j \in \{3, 4, \dots, n\},$$

$$\det B[1,3] \neq 0 \text{ and } \det B[1,2,3|1,3,j] = 0, \qquad j = \{4, 5, \dots, n\},$$

$$\det B[k-1, k+1] \neq 0 \text{ and } \det B[k-1, k, k+1|k-1, k+1, j] = 0,$$

$$3 \leqslant k \leqslant n-2, \quad j \in \{1, \dots, k-2, k+2, \dots, n\},$$

$$\det B[n-2, n] \neq 0 \text{ and } \det B[n-2, n-1, n|j, n-2, n] = 0,$$

$$j \in \{1, 2, \dots, n-3\},$$

$$b_{n-1,n-2} \neq 0 \text{ and } \det B[n-1, n|j, n-1] = 0, \qquad j \in \{1, 2, \dots, n-2\}.$$

Observe also that Corollary 11 applies to the present example, since an irreducible tridiagonal matrix is nearly reducible. Therefore, when B satisfies conditions (iT) and (iiT), B^{-1} has precisely the same zero-nonzero pattern as an irreducible tridiagonal matrix. These results are based upon Theorem 10 and Corollary 11, but we can do no more in the present case. It follows from Theorem 2 that if $B = A^{-1}$ for A irreducible and tridiagonal, then all second-order minors of the form $\det B[i, j|j, k]$ for either $i < j < k$ or $i > j > k$ vanish. These are also almost principal minors of B. Now suppose $b_{33} \neq 0$, and consider, for example

$$B[1,2,3|1,3,j] = \begin{bmatrix} b_{11} & b_{13} & b_{1j} \\ b_{21} & b_{23} & b_{2j} \\ b_{31} & b_{33} & b_{3j} \end{bmatrix}, \qquad j \in \{4,5,\dots,n\}.$$

Since $\det B[1,3|3,j] = 0$ and $\det B[2,3|3,j] = 0$ for any such j, we have $b_{1j} = b_{13}b_{3j}/b_{33}$ and $b_{2j} = b_{23}b_{3j}/b_{33}$. Therefore

$$B[1,2,3|1,3,j] = \begin{bmatrix} b_{11} & b_{13} & b_{13}b_{3j}/b_{33} \\ b_{21} & b_{23} & b_{23}b_{3j}/b_{33} \\ b_{31} & b_{33} & b_{3j} \end{bmatrix}.$$

Factoring b_{3j}/b_{33} out of the last column of $\det B[1,2,3|1,3,j]$, we see that the vanishing of $\det B[1,3|3,j]$ and of $\det B[2,3|3,j]$ implies $\det B[1,2,3|1,3,j] = 0$. The same kind of argument can be used on each of the third-order determinants appearing in (iiT). Thus we can replace (iiT) by the condition

(iiiT) $b_{ii} \neq 0$, $2 \leqslant i \leqslant n-1$, and $\det B[i, j|j, k] = 0$ for either $i < j < k$ or $i > j > k$; $\det B[1,3] \neq 0$; $\det B[k-1, k+1] \neq 0$, $3 \leqslant k \leqslant n-2$; $\det B[n-2, n] \neq 0$.

Finally let us count how many vanishing minors we must have. From (iiT) we have a set of $2(n-2)$ second-order minors and $(n-2)(n-3)$ third-order minors which must vanish, for a total of $(n-1)(n-2)$, as many as there are zero entries in a tridiagonal matrix. But each of the third-order minors is required to vanish by a pair of second-order minors; hence we require $2(n-2)(n-3) + 2(n-2) = 2(n-2)^2$ (distinct) 2-by-2 minors to vanish. All of the minors are almost principal minors. Such sets of 2-by-2's need not be minimal.

For the matrix A_1 defined in the introduction, the zero sets for the rows are again all distinct. We have for our partitions

$$\alpha_i = \{1,2,\dots,i-1,i+2,\dots,n\}, \quad \beta_i = \{i\}, \quad \gamma_i = \{i+1\},$$
$$1 \leqslant i \leqslant n-1,$$
$$\alpha_n = \{2,3,\dots,n-1\}, \qquad \beta_n = \{n\}, \qquad \gamma_n = \{1\}.$$

Therefore, if B is irreducible with $\det B[k^c] \neq 0$ for $k = 1, 2, \dots, n$, then B has a zero entry everywhere A_1 does if (iT) holds and

(ii1) $\det B[\alpha_i] \neq 0$ and $\det B[i^c|(i+1 \bmod n)^c] = 0$ for $i = 1, 2, \dots, n$.

For the matrix A_2 we have

$$\alpha_1 = \{2\}, \qquad \beta_1 = \{1\}, \qquad \gamma_1 = \{3, \dots, n\},$$

$$\alpha_i = \{i+1\}, \quad \beta_i = \{i\}, \quad \gamma_i = \{1, 2, \dots, i-1, i+2, \dots, n\},$$

$$2 \leqslant i \leqslant n-2,$$

$$\alpha_{n-1} = \{n\}, \qquad \beta_{n-1} = \{n-1\}, \qquad \gamma_{n-1} = \{1, 2, \dots, n-2\},$$

$$\alpha_n = \{1\}, \qquad \beta_n = \{n\}, \qquad \gamma_n = \{2, \dots, n-1\}.$$

Hence if (iT) holds and if

$$\begin{aligned} \det B[1,2|2,j] &= 0, \qquad j \in \{3, \dots, n\}, \\ \det B[i,i+1|i+1,j] &= 0, \qquad j \in \{1, 2, \dots, i-1, i+2, \dots, \dots, n\}, \\ &\qquad 2 \leqslant i \leqslant n-1, \\ \det B[n-1,n|j,n-1] &= 0, \qquad j \in \{1, 2, \dots, n-2\}, \end{aligned} \tag{ii2}$$

then B^{-1} has the same zero-nonzero pattern as A_2. Here, as in the tridiagonal case, Corollary 11 applies.

In each of our first three examples the zero sets for different rows are different, so that, in Theorem 8, $r = n$. We present next a quite different example. Consider the 10×10 matrix with the zero-nonzero pattern shown below:

$$A = \begin{bmatrix} x & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & x & 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & 0 & x & 0 & 0 & 0 & 0 \\ x & x & x & 0 & x & x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & x & x & x & 0 & x & x & x \\ 0 & 0 & 0 & 0 & x & 0 & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x & x \end{bmatrix}.$$

For this matrix we have

$$\begin{aligned}
\alpha_1 &= \{4,5\}, & \beta_1 &= \{1,2,3\}, & \gamma_1 &= \{6,7,8,9,10\},\\
\alpha_2 &= \{1,2,3,6\}, & \beta_2 &= \{4\}, & \gamma_2 &= \{5,7,8,9,10\},\\
\alpha_3 &= \{1,2,3,6,7\}, & \beta_3 &= \{5\}, & \gamma_3 &= \{4,8,9,10\},\\
\alpha_4 &= \{4,5,8,9,10\}, & \beta_4 &= \{6\}, & \gamma_4 &= \{1,2,3,7\},\\
\alpha_5 &= \{5,8,9,10\}, & \beta_5 &= \{7\}, & \gamma_5 &= \{1,2,3,4,6\},\\
\alpha_6 &= \{6,7\}, & \beta_6 &= \{8,9,10\}, & \gamma_6 &= \{1,2,3,4,5\}.
\end{aligned}$$

Therefore, if (iT) holds and if

(iiS) $\det B[4, 5] \neq 0$, $\det B[1, 2, 3, 6] \neq 0$, $\det B[1, 2, 3, 6, 7] \neq 0$, $\det B[4,5,8,9,10] \neq 0$, $\det B[5,8,9,10] \neq 0$, $\det B[6,7] \neq 0$,

while

$$\begin{aligned}
\det B[i,4,5|4,5,j] &= 0, & & i\in\{1,2,3\},\ j\in\{6,7,8,9,10\},\\
\det B[1,2,3,4,6|1,2,3,6,j] &= 0, & & j\in\{5,7,8,9,10\},\\
\det B[1,2,3,5,6,7|1,2,3,6,7,j] &= 0, & & j\in\{4,8,9,10\},\\
\det B[4,5,6,8,9,10|j,4,5,6,8,9] &= 0, & & j\in\{1,2,3,7\},\\
\det B[5,7,8,9,10|j,5,8,9,10] &= 0, & & j\in\{1,2,3,4,5\},\\
\det B[6,7,i|j,6,7] &= 0, & & i\in\{8,9,10\},\ j\in\{1,2,3,4,5\},
\end{aligned}$$

then the matrix B^{-1} has zeros everywhere A does.

The matrix A has a maximal zero submatrix $A[\beta|\gamma] = 0$, where $\beta = \{1,2,3,4\}$, $\gamma = \{7,8,9,10\}$. Thus every minor of A_2 of the form $A[\delta|\varepsilon]$ equals zero generically if $\delta = \{1,2,3,4,i,i_2,i_3\}$ and $\varepsilon = \{j_1,j_2,j_3,7,8,9,10\}$. It follows that all third-order minors of $B = A^{-1}$ having the form $\det B[i_1,i_2,i_3|j_1,j_2,j_3]$, where $\{i_1,i_2,i_3\} \subseteq \{1,2,3,4,5,6\}$, $\{j_1,j_2,j_3\} \subseteq \{5,6,7,8,9,10\}$, vanish. These are almost principal minors if and only if $\{5,6\} \subseteq \{i_1,i_2,i_3\}$ and $\{5,6\} \subseteq \{j_1,j_2,j_3\}$.

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