ORIGINAL ARTICLE

Analysis of thin plates with holes by using exact geometrical representation within XFEM

Logah Perumal *, C.P. Tso, Lim Thong Leng

Faculty of Engineering and Technology, Multimedia University, Jalan Ayer Keroh Lama, Bukit Beruang, 75450 Melaka, Malaysia

GRAPHICAL ABSTRACT

ARTICLE INFO

Article history:
Received 16 November 2015
Received in revised form 2 February 2016

ABSTRACT

This paper presents analysis of thin plates with holes within the context of XFEM. New integration techniques are developed for exact geometrical representation of the holes. Numerical and exact integration techniques are presented, with some limitations for the exact integration technique. Simulation results show that the proposed techniques help to reduce the solution error, due to the exact geometrical representation of the holes and utilization of appropriate...
Introduction

Holes can be found in many thin walled structures. For example, holes are found in buildings’ steel structural studs to enable installation of plumbing, electrical and heating conduits in the walls or ceilings, flange or web of steel box girders in bridges is equipped with holes to ease inspection duties, and ribs attached to the main spar of an airplane’s wing are often come with holes. These holes or discontinuities within the domain (thin plate) cause changes in elastic stiffness [1]. Conventional finite element method (FEM) requires meshing strategies to track these discontinuities and capture singularities within the domain. For these cases, the element edges need to be aligned with the boundary discontinuities, and mesh refinement is needed near singularities. These are accomplished in conventional FEM by utilizing abrupt re-meshing strategies.

Extended finite element method (XFEM) is a numerical method which was initially developed to avoid re-meshing strategy to locate discontinuities over a boundary [2,3]. In XFEM, the boundaries with discontinuities are tracked through utilization of appropriate level-set functions and regions with singularities are modeled/enhanced by utilizing enrichment functions. Fig. 1 shows both conventional FEM and XFEM techniques in simulation of a domain with a circular hole. Proper meshing strategy is needed to capture the boundary discontinuities in conventional FEM (Fig. 1(a)). Re-meshing strategies are needed in case of moving interfaces (splitting elements), such as in crack propagation. In XFEM, the domain is meshed by utilizing mapped mesh with square (Fig. 1(b)) or triangular elements, with enrichment functions near singularities. Elements that are enhanced by utilizing enrichment functions (elements that are cut by the discontinuities) and the enriched nodes are highlighted in Fig. 1(c).

One of the challenges faced in XFEM method is the numerical integration (to obtain the stiffness matrices, $k$) within elements on the boundary discontinuities. For example, in case of a plate with a circular hole as shown in Fig. 1(c), the enriched elements contain both regions from the hole and the plate. Therefore, integration of the stiffness matrices for these elements is done over the region containing the plate, usually by dividing the element into several sub-elements. An example of sub-division of the element into several sub-quadrilaterals is shown in Fig. 2 for element 17 from Fig. 1(c).

Overall stiffness matrix, $k$ for element 17 is obtained by summing the integration of $k$ over the regions of quadrilaterals 1 and 2 (Fig. 2). It is seen that the actual circular boundary is simplified to be linear for the purpose of numerical integration. This introduces error in the computation.

Several techniques have been proposed to simplify the numerical integration in XFEM, such as substituting non-polynomials within the integral with approximate polynomials.
Exact geometrical representation within XFEM

447

[4], converting surface integration into equivalent boundary integration by utilizing the Green-Ostrogradsky theorem
[5,6], using conformal mapping to a unit disk through Schwarz-Christoffel mapping to avoid sub-division of the elements [7] and recently higher order accurate numerical integration is developed [8,9]. Shortages of most of the methods above are as follows:

- The domain needs to be partitioned into several sub-elements to perform the numerical integration.
- Limited to linear or fixed boundaries.
- High number of quadrature points and weights are needed to achieve the desired accuracy.

In this work, the generalized equations that were developed in previous work [10] are utilized within the context of XFEM for analysis of thin plates with holes. The methods demonstrated in this work show exact geometrical representation of the discontinuities (linear lines or curves within the enriched elements). This enables exact integration within the enriched elements (the highlighted elements in Fig. 1(c)) and shows improvement in the solution accuracy. The domain is partitioned into two sub-elements only and less number of quadrature points and weights are utilized, by selecting proper quadrature scheme.

Generalized equations for exact geometrical representation and integration

Integration of a function within a closed region can be represented analytically by utilizing Fubini’s theorem [11] given by the following:

$$I_{ix} = \int_{a}^{b} \int_{r}^{s} f(x, y) dy dx \quad \text{or} \quad I_{iy} = \int_{a}^{b} \int_{r}^{s} f(x, y) dx dy$$

where \( a, b, r \) and \( s \) are the upper and lower limits.

The domain needs to be enclosed by either of the following combinations:

- 4 constant lines
- 3 constant lines and 1 function
- 2 constant lines and 2 functions

The analytical formulas in Eq. (1) are later converted to the form required for utilization of Gauss quadrature rules (numerical integration) by using the formulas [10]:

$$I_1 = \int_{a}^{b} \int_{r}^{s} f(x, y) dy dx = \int_{U}^{L} \int_{u}^{w} f(m_x u + c_x, m_y v + c_y) m_x m_y du dv$$

or

$$I_2 = \int_{a}^{b} \int_{r}^{s} f(x, y) dx dy = \int_{L}^{U} \int_{w}^{u} f(m_x u + c_x, m_y v + c_y) m_x m_y du dv$$

where

- \( U \) is upper limit
- \( L \) is lower limit
- \( w \) and \( u \) are integration weights
- \( i, v \) are integration points
- \( i = 1, 2, 3, \ldots, n \)
- \( n \) is integration order.

For \( I_1 \):

$$m_x = \frac{a - b}{w_i}; \quad m_y = \frac{r(m_x u + c_x) - s(m_x u + c_x)}{L - U};$$

$$c_x = \frac{(b x L) - (a x U)}{L - U}; \quad c_y = \frac{(a(m_x u + c_x) + L) - (r(m_x u + c_x) + U)}{L - U}.$$  \hspace{1cm} \text{(2)}

For \( I_2 \):

$$m_x = \frac{r(m_x u + c_x) - s(m_x u + c_x)}{L - U}; \quad m_y = \frac{a - b}{w_j};$$

$$c_x = \frac{(a(m_x u + c_x) + L) - (U(m_x u + c_x) + U)}{L - U}; \quad c_y = \frac{(b x L) - (a x U)}{L - U};$$

The generalized equations \( (I_1 \text{ and } I_2) \) above utilize fully numerical method (basic four arithmetic operations) for the conversion of the integration limits. Any quadrature rules can be applied with the generalized Eq. (2), by simply changing the upper and lower limits, \( U \) and \( L \), according to the quadrature rule of choice. Therefore, Eq. (2) can be utilized to perform integration over any boundary (linear or curved boundaries, which can be represented by functions) and integrate any integrands (by selecting suitable quadrature rules, based on the nature of the integrands).

Eq. (2) can be further extended to perform exact integration of monomials within a domain enclosed by polynomial curves and/or linear lines, without involving any quadrature points and weights. This can be done by changing the upper and lower limits in Eq. (2) to 1 and 0, respectively. Then, the analytical expressions for the integration of monomials within the domain can be represented numerically as follows:

$$\int_{0}^{1} \int_{0}^{1} x^m y^n dy dx = \frac{1}{(m + 1)(n + 1)}$$

\hspace{1cm} \text{(3)}

Eq. (3) can only be utilized to perform integration of monomials within a domain enclosed by curves (which can be represented by polynomial functions) and/or linear lines. Advantages of the exact integration method are that it does not require any quadrature points and weights, provides exact solutions faster than the analytical method (which involves fully symbolic computations) and can be used as a reference to determine number of quadrature points required for the numerical integration, for problems involving higher order polynomials. Disadvantage of the exact method given in Eq. (3) is that the computational time is higher compared to the numerical method, when the integrands involve high number of terms. This is due to the fact that the integrand needs to be expanded to determine the coefficients \( m \) and \( n \).

An example is shown below to demonstrate the numerical and exact integration equations presented above. A set of functions \( f(x, y) \) are integrated using the proposed integration schemes. A domain with both curved and linear lines that are represented by polynomial functions as shown in Fig. 3 is chosen for the study, in order to make direct comparison between both (numerical and exact) methods.

The domain with coordinates as shown in Fig. 3(a) is separated into 2 regions: \( R_1 \) and \( R_2 \) according to the requirement of Fubini’s Theorem (Fig. 3(b)). Region \( R_1 \) is enclosed by two constant lines (one of them is imaginary) and two functions (linear and quadratic functions), while region \( R_2 \) is also enclosed by two constant lines (one of them is imaginary) and two functions (linear and cubic functions). Integration of a function over the entire domain can be written analytically by utilizing Fubini’s Theorem (Eq. (1)) by the following:
The integrations given by Eq. (4) are solved by utilizing the numerical integration method given by Eq. (2) and exact integration method given by Eq. (3). Both classical Gauss Legendre and generalized Gaussian quadrature rules are utilized for the numerical integration method. A sample program has been developed using the Mathematica software to carry out the integrations. The simulations are run on a computer with 2.93 GHz Dual Core CPU, 32 bit operating system and 2 GB of memory. Comparisons are made between the results obtained with the fully analytical solution, as shown in Tables 1 and 2. Percentage error is calculated based on Eq. (5).

\[
\text{Percentage Error} = \frac{|\text{Analytical solution} - \text{Numerical solution}|}{\text{Analytical solution}} \times 100\%
\]

The numerical integration technique given by Eq. (2) is utilized to perform numerical integrations using classical Gauss Legendre and generalized Gaussian quadrature. From the Table 1, it can be seen that percentage error reduces when higher number of integration points and weights are utilized. Any quadrature rules can be utilized in Eq. (2), by simply changing the upper and lower limits, \( U \) and \( L \). From results in Table 2, it is seen that the exact integration technique yields accurate solutions at lower computational time compared to the analytical solutions, without involving any integration points and weights.

**Application in XFEM: plate with circular and curved (polynomial curves) holes**

In this section, the numerical and exact integration techniques presented above are applied within the context of XFEM, to analyze plates with circular and curved (polynomial curves) holes. Mathematica software is utilized to perform the computations. For Case 1, the numerical integration technique that is given by Eq. (2) is utilized to solve for inner boundary displacements of a plate with circular hole. Both classical Gauss Legendre and generalized Gaussian quadrature rules are utilized and their performances are compared. For Case 2, the exact integration technique that is given by Eq. (3) is utilized as a reference solution to determine the integration error which appears in numerical integration technique. For this Case 2, a plate with curved (polynomial curves) hole is selected, since the exact integration technique is applicable for monomials only.

Again, both classical Gauss Legendre and generalized Gaussian quadrature rules are utilized and their performances are compared.

**Case 1: plate with circular hole**

Geometry of the problem is shown in Fig. 1(b). The external boundaries are subjected to known displacement values and the internal displacements are determined. The external boundaries are subjected to known displacement values, according to the analytical solution given by Thomas Jr and Finney [11]:

\[
u = \frac{a}{8\mu} \left[ (r^2 - \kappa + 1) \cos \theta + \frac{2a}{r} ((1 + \kappa) \cos \theta + \cos 3\theta) - \frac{2a^3}{r^3} \cos 3\theta \right]
\]

\[
u = \frac{a}{8\mu} \left[ (r^2 - \kappa - 3) \sin \theta + \frac{2a}{r} ((1 - \kappa) \sin \theta + \sin 3\theta) - \frac{2a^3}{r^3} \sin 3\theta \right]
\]

where \( a \) represents radius of the circular hole, \( \mu \) represents shear modulus of elasticity, \( r \) and \( \theta \) represent polar coordinates, \( \kappa \) represents the coefficient kappa. Plane strain conditions are assumed: \( \kappa = 3 - 4v \), \( \mu = E/2(1 + v) \), \( E = 10^7 \) Pa and radius of the circular hole, \( a = 0.4 \) m. Five different levels of mesh are considered, which are 4 by 4, 5 by 5, 6 by 6, 7 by 7, and 8 by 8, with global nodes of 25, 36, 49, 64 and 81, respectively. Fig. 1(b) and (c) show mesh level of 7 by 7, with 64 global nodes.

The level set function utilized to identify the enriched elements (elements cut by the inner boundary discontinuities), outer elements (elements that enclose the plate) and inner elements (elements that enclose the void/hole) is the equation of the circle, given by the following:

\[
\phi(x, y) = \sqrt{x^2 + y^2} - a
\]

The enrichment function utilized is the sign function of the level-set function (Heaviside-function), which is given by the following:

\[
\psi(x, y) = \text{sign}(\phi(x, y)) = \begin{cases} 
-1 & \text{if } \phi(x, y) < 0 \\
0 & \text{if } \phi(x, y) = 0 \\
1 & \text{if } \phi(x, y) > 0 
\end{cases}
\]

The curves within the enriched elements are not identical. Therefore, 12 possible combinations of inner boundary discontinuities (curves of the circle) within the enriched elements are classified, as shown in Fig. 4.

The type of combination (for the curve) for a given enriched element is identified based on the intersections of the curve.
classes of functions [10]. Generalized Gaussian quadrature rules and weights are generated based on wider quadrature rules perform better, due to the fact that the integrands are polynomials. On the other hand, generalized Gaussian Gauss Legendre rules perform very well when the integrands of the stiffness matrices consist of non-monomials. Classical integration orders tested. This is because the integrands for the conventional method. The conventional method utilizes classical Gauss Legendre rules which were obtained by projecting the 1 dimensional quadrature rules to 2 dimensions [12]. The \( L_2 \) error norm, \( e \) is determined by using the formula:

\[
e = \sqrt{\frac{\int_\Omega ((u,v)^{\text{exact}} - (u,v)^{\text{calculated}})^2 \, d\Omega}{\int_\Omega ((u,v)^{\text{exact}})^2 \, d\Omega}}
\]  

(7)

The integrations in Eq. (7) are performed numerically, by using 441 integration points and weights of classical Gauss Legendre. Results of the simulations are shown in Fig. 5 and 6.

From Fig. 5, it is seen that generalized Gaussian quadrature rules provide stable and better results for the four different integration orders tested. This is because the integrands for the stiffness matrices consist of non-monomials. Classical Gauss Legendre rules perform very well when the integrands are polynomials. On the other hand, generalized Gaussian quadrature rules perform better, due to the fact that the integration points and weights are generated based on wider classes of functions [10]. Generalized Gaussian quadrature rules are recommended for integration of non-polynomials. Fig. 6 shows comparison between the classical XFEM technique (which divides the element into several quadrilaterals) and the proposed exact geometrical representation technique (by utilizing Eq. (2) and generalized Gaussian quadrature rules). It is seen that the proposed integration technique reduces the solution error. The reduction in the error is caused by the exact geometrical representation as well as utilization of generalized Gaussian quadrature rules, which is suitable for integration of non-polynomials.

### Case 2: plate with curved (polynomial curves) hole

In this case, a plate with a hole which is represented by polynomial curves is analyzed. Geometry of the problem is shown in Fig. 7. Three levels of mesh are considered, which are 4 by 4, 6 by 6, and 8 by 8, with global nodes of 25, 49 and 81, respectively. Two level set functions are utilized, which are the equations of the curves forming the geometry (upper and lower halves of the hole). The level set functions are as follows:

\[
\varphi_1(x, y) = 2000x^3 + x^2 - 0.55 - y
\]

\[
\varphi_2(x, y) = 2000x^3 + x^2 - 0.55 + y
\]  

(8)

The enrichment function utilized is the sign function of the level-set function (Heaviside-function) given by Eq. (8). Similar to Case 1, 12 possible combinations of inner boundary discontinuities (polynomial curves) within the enriched elements are classified, as shown in Fig. 4.

Stiffness matrices for the enriched elements are determined by utilizing Eq. (2), with both classical Gauss Legendre and generalized Gaussian quadrature rules. The errors for the stiffness matrices are determined via comparison with exact solution. The exact solutions that are obtained from Eq. (3) are used as analytical/reference to calculate the percentage error, by utilizing Eq. (5). The results are given in Table 3. It is seen

### Table 1 Percentage error for the Quadrature rules used in Eq. (2).

<table>
<thead>
<tr>
<th>Function ( f(x,y) )</th>
<th>Integration order, ( n )</th>
<th>Classical Gauss Legendre ((U = 1, L = -1))</th>
<th>Generalized Gaussian quadrature ((U = 1, L = 0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 + 2y^4 )</td>
<td>5</td>
<td>4.15561 \times 10^{-5}</td>
<td>5.80713 \times 10^{-3}</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1.50106 \times 10^{-14}</td>
<td>9.2348 \times 10^{-9}</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>1.50106 \times 10^{-14}</td>
<td>6.00423 \times 10^{-14}</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>6.00423 \times 10^{-14}</td>
<td>0</td>
</tr>
<tr>
<td>( e^{1+x} )</td>
<td>5</td>
<td>2.16329 \times 10^{-8}</td>
<td>9.2595 \times 10^{-4}</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1.11906 \times 10^{-14}</td>
<td>1.02953 \times 10^{-12}</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>1.11906 \times 10^{-14}</td>
<td>4.47623 \times 10^{-14}</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.11906 \times 10^{-14}</td>
<td>3.35717 \times 10^{-14}</td>
</tr>
</tbody>
</table>

### Table 2 Results obtained for integration of the functions over the curved element using the exact integration technique (Eq. (3)) and analytical method.

<table>
<thead>
<tr>
<th>Function ( f(x,y) )</th>
<th>Solution from exact integration technique</th>
<th>Analytical solution</th>
<th>Percentage error (%)</th>
<th>Average maximum time elapsed for exact integration technique (s)</th>
<th>Average maximum time elapsed for analytical technique (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 + 2y^4 )</td>
<td>( R_1 = \frac{2 \times 57.643}{3.217} )</td>
<td>( R_1 = \frac{2 \times 57.643}{3.217} )</td>
<td>0</td>
<td>0.11 for ( R_1 )</td>
<td>0.44 for ( R_1 )</td>
</tr>
<tr>
<td></td>
<td>( R_2 = \frac{2 \times 17.489}{3.217} )</td>
<td>( R_2 = \frac{2 \times 17.489}{3.217} )</td>
<td></td>
<td>0.11 for ( R_2 )</td>
<td>0.42 for ( R_2 )</td>
</tr>
<tr>
<td>( 3x^3y^4 + 2x^2y^3 )</td>
<td>( R_1 = \frac{2 \times 36.503}{3.249} )</td>
<td>( R_1 = \frac{2 \times 36.503}{3.249} )</td>
<td>0</td>
<td>0.12 for ( R_1 )</td>
<td>0.48 for ( R_1 )</td>
</tr>
<tr>
<td></td>
<td>( R_2 = \frac{2 \times 66.645}{3.249} )</td>
<td>( R_2 = \frac{2 \times 66.645}{3.249} )</td>
<td></td>
<td>0.11 for ( R_2 )</td>
<td>0.50 for ( R_2 )</td>
</tr>
</tbody>
</table>
that for the case of stiffness matrices consisting of polynomials, the classical Gauss Legendre rules provide correct solutions at lower integration order (converge faster), compared to the generalized Gaussian quadrature rules. This is due to the fact that the classical Gauss Legendre rules were generated based on Legendre polynomials and give accurate results for polynomials.

Minimum order of integration for accuracy and convergence

The accuracy of numerical integration depends on the order of integration (that relates to the number of quadrature points and weights) utilized, as shown in Tables 1 and 3. Higher number of quadrature points and weights yield more accurate results. However, higher order of integration leads to higher computational time and data storage requirements. Therefore, it is important to know the minimum order of integration necessary to achieve the required accuracy and convergence.

The minimum order of integration, \( n \), necessary to maintain accuracy by utilizing classical Gauss Legendre rules (for polynomials) is given by the relation [13]:

\[
 n = \text{Roundup} \left( \frac{m + 1}{2} \right),
\]

Fig. 4 12 possible combinations for the circular curve within the enriched elements (a) combinations 1a to 6a and (b) combinations 1b to 6b.

Fig. 5 \( L_2 \) errors for case 1 by utilizing numerical integration technique (a) \( L_2 \) errors for mesh level 4 by 4, with 25 global nodes (b) \( L_2 \) errors for mesh level 8 by 8, with 81 global nodes.

Fig. 6 Comparison of \( L_2 \) errors between the classical integration technique and the proposed technique, by using fifth order numerical integration.
where \( m \) represents the highest polynomial power present in the integrand. For the Case 2 considered in this work, the highest polynomial power present in the integrand (for 4 by 4 mesh size) is 16 and therefore \( n = 9 \) (or 10) yields good results as shown in Table 3. Similar relation is not available for generalized Gaussian quadrature rules, since they are meant for integration of non-polynomials. However, for the Case 1 considered in this work, the minimum number of integration order required to achieve desired accuracy (by utilizing generalized Gaussian quadrature rules) is 5, as shown in Fig. 5.

Conventional finite elements in FEM (which utilize classical Gauss Legendre rules) maintain convergence toward exact solution when the integration order follows the relation [14]:

\[
n = \text{Roundup} \left[ \frac{2(p - r) + 1}{2} \right], \tag{10}
\]

where \( p \) represents highest polynomial power which occurs in the complete shape functions of the element and \( r \) represents the order of partial differentiation appearing in the calculation of stiffness matrix (\( r = 1 \), for solid mechanics). Therefore, minimum integration order, \( n \), needed to achieve convergence for linear (\( p = 1 \)), quadratic (\( p = 2 \)) and cubic (\( p = 3 \)) quadrilateral elements is 1, 2 and 3 respectively. Eq. (10) is also valid for current work (exact geometrical representation within XFEM), since the outer elements (regions that cover only the plate) are treated similar to conventional FEM. However in Case 1, the enriched elements (regions that cover both the hole and plate) are subjected to non-polynomial integrands, depending on the curvature of the discontinuity. Therefore, even though convergence would be observed for the outer elements, there will be loss in overall accuracy due to errors in integration of non-polynomials within the enriched elements, if classical Gauss Legendre rules are utilized. From the results obtained in this work (Fig. 5), it is observed that minimum integration order \( n = 5 \) is required to achieve desired accuracy and convergence for Case 1, by utilizing generalized Gaussian quadrature rules. Neither the accuracy nor convergence is improved with higher integration orders for Case 1.

Convergence is also attained when the matrices are non-singular. Singularity may occur even if the integration order satisfies Eq. (10). Singularity occurs when lesser number of independent relations (number of strains utilized in the formulation of stiffness matrix) is supplied at all the integration points compared to the number of global degree of freedom (excluding constraints) [14,15]. This can be represented by the relations:

\[
V = s \times i \times t \tag{11}
\]

\[
D = (f \times e) - c \tag{12}
\]

where \( V \) represents total independent relations, \( s \) represents number of strains utilized in the formulation of stiffness matrix (3 for the cases considered in this work), \( i \) represents number of integration points for each element (corresponds to integration order), \( t \) represents total number of elements in the domain, \( D \) represents total degree of freedom, \( f \) represents degree of freedom for each element node, \( e \) represents total number of global nodes, and \( c \) represents total number of constrained degree of freedom in the domain. Singularity occurs when \( D \) is greater than \( V \). The relation aforesaid can be rearranged to obtain minimum order of integration, \( n \), to avoid singularity:

\[
n = \text{Roundup} \left[ \frac{(f \times e) - c}{(s \times t)} \right] \tag{13}
\]

Therefore, minimum number of integration order to be utilized to achieve required accuracy and convergence within XFEM would be the maximum integration order, \( n \), obtained from Eqs. (9), (10), and (13) aforesaid. Consider 4 by 4 mesh in Case 2 as an example (linear quadrilateral elements are utilized with classical Gauss Legendre rules). All the 4 sides of the plate boundaries are not constrained. Corresponding variables

<table>
<thead>
<tr>
<th>Mesh level</th>
<th>Integration order, ( n )</th>
<th>% Error for classical Gauss Legendre</th>
<th>% Error for generalized Gaussian quadrature</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 by 4</td>
<td>5</td>
<td>3.538 \times 10^{-2}</td>
<td>2.370 \times 10^{-1}</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>3.378 \times 10^{-11}</td>
<td>1.550 \times 10^{-7}</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>3.392 \times 10^{-11}</td>
<td>3.387 \times 10^{-11}</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.396 \times 10^{-11}</td>
<td>3.326 \times 10^{-11}</td>
</tr>
<tr>
<td>6 by 6</td>
<td>5</td>
<td>1.222</td>
<td>1.403</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>7.737 \times 10^{-8}</td>
<td>4.3124 \times 10^{-5}</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>3.463 \times 10^{-9}</td>
<td>3.47157 \times 10^{-9}</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.462 \times 10^{-9}</td>
<td>3.46271 \times 10^{-9}</td>
</tr>
<tr>
<td>8 by 8</td>
<td>5</td>
<td>5.333 \times 10^{-7}</td>
<td>2.434 \times 10^{-3}</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>2.561 \times 10^{-12}</td>
<td>1.740 \times 10^{-12}</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>2.584 \times 10^{-12}</td>
<td>2.698 \times 10^{-12}</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.515 \times 10^{-12}</td>
<td>2.263 \times 10^{-12}</td>
</tr>
</tbody>
</table>
for this case are $m = 16$, $p = 1$, $r = 1$, $s = 3$, $t = 16$, $f = 2$, $e = 25$, $c = 0$. Eqs. (9), (10) and (13) yield $n = 9$, 1, and 2, respectively. Therefore, $n = 9$ (or $n = 10$) should be utilized in order to ensure accuracy and convergence of the solution.

Conclusions

In this work, two new integration techniques, which are numerical and exact integration techniques, have been demonstrated within the context of XFEM. The generalized equations (Eq. (2)) can be utilized with any quadrature rules to perform numerical integrations by simply converting the integration limits $U$ and $L$ accordingly. The techniques described in this paper can be utilized for both linear and nonlinear boundaries, with less number of quadrature points and weights (by selecting appropriate quadrature scheme), and with fewer number of sub-elements. Application of the new techniques in engineering domain (analysis of plates with holes) showed improvement in the solution accuracy. The exact integration technique given by Eq. (3) can be utilized for certain cases that involve polynomials only, and can be utilized as a reference/analytical solution. The exact geometrical representation and integration techniques that are presented help to reduce the solution error in analysis of thin plates with arbitrary holes. Optimal order of integration, $n$ for accuracy and convergence of the solution can be determined by following the guidance provided in this paper.

Conflict of Interest

The authors have declared no conflict of interest.

Compliance with Ethics Requirements

This article does not contain any studies with human or animal subjects.

Acknowledgments

The first author would like to thank Research Management Centre (RMC) of Multimedia University, Malaysia, for providing financial support through Mini Funds with grant numbers: MMUI/130070 and MMUI/160047, which enabled purchase of required software and equipment for this work. The authors would also like to express their sincere appreciation to the anonymous reviewers who have provided valuable feedbacks which helped to improve content of the paper.

References