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ORIGINAL ARTICLE

Unsteady flow of a Maxwell fluid over a stretching surface in presence of chemical reaction

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KEYWORDS

Unsteady flow; Maxwell fluid; Stretching surface; Chemical reaction **Abstract** An analysis is presented for unsteady two-dimensional flow of a Maxwell fluid over a stretching surface in presence of a first order constructive/destructive chemical reaction. Using suitable transformations, the governing partial differential equations are converted to ordinary one and are then solved numerically by shooting method. The flow fields and mass transfer are significantly influenced by the governing parameters. Fluid velocity initially decreases with increasing unsteadiness parameter and concentration decreases significantly due to unsteadiness. The effect of increasing values of the Maxwell parameter is to suppress the velocity field. But the concentration is enhanced with increasing Maxwell parameter.

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1. Introduction

During last few years the boundary layer flow behaviours of different types of fluids attracted the interest of many researchers (Hayat et al. [1]). Due to engineering applications, the boundary layer flows of non-Newtonian fluids have been given considerable attention in the recent years. The flow characteristics of non-Newtonian fluids are quite different in comparison to Newtonian fluids. In order to obtain a clear idea of non-Newtonian fluids and their various applications, it is

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necessary to study their flow behaviour. Because of the complexity of these fluids, there is not a single constitutive equation which exhibits all properties of such non-Newtonian fluids. Several models have been suggested (Hayat et al. [2,3]). Among these, the vast majority of non-Newtonian fluid models are concerned with the simple models viz. the power law and grade two or three [4-14]. These simple fluid models have some drawbacks that they are unable to provide results not having accordance with fluid flows in reality. The powerlaw model is used in modelling fluids with shear-dependent viscosity. But it cannot predict the effects of elasticity. On the other hand, though the fluids of grade two or three can calculate the effects of elasticity, the viscosity in these models is not shear dependent (Hayat et al. [15,16]). Moreover, they are unable to predict the effects of stress relaxation. Maxwell model, a subclass of rate type fluids, can predict the stress relaxation and therefore, have become more popular (Sadeghy et al. [17], Abel et al. [18]). This model excludes the complicating effects of shear-dependent viscosity from any boundary layer

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С	concentration of the species of the fluid
C_w	concentration of the wall of the surface
C_∞	free-stream concentration
D	diffusion coefficient of the diffusing species
f	non-dimensional stream function
f'	first order derivative with respect to η
f''	second order derivative with respect to η
f'''	third order derivative with respect to η
k_1	reaction rate
M	unsteadiness parameter
Sc	Schmidt number
u , v	components of velocity in x and y directions

analysis and enables one to focus solely on the effects of a fluid's elasticity on the characteristics of its boundary layer (Heyhat and Khabazi [19]). Hayat et al. [20] constructed an analytic solution for unsteady MHD flow in a rotating Maxwell fluid through a porous medium. Hayat et al. [21] studied the MHD flow of a UCM fluid over a porous stretching plate with the homotopy analysis method.

Mass transfer phenomenon is used in various scientific disciplines for different systems and mechanisms that involve molecular and convective transport of atoms and molecules.

The driving force for mass transfer is the difference in concentration (Hayat et al. [15]).

Cortell [22] discussed mass transfer with chemically reactive species for two classes of viscoelastic fluid over a porous stretching sheet. The transport of mass and momentum with chemical reactive species in the flow caused by a linear stretching sheet was discussed by Andersson et al. [23].

In all these above studies, the flow, temperature and concentration fields were considered to be at steady state. However, in some cases the flow field, heat, and mass transfer can be unsteady due to a sudden stretching of the flat sheet. When the surface is impulsively stretched with certain velocity, the inviscid flow is developed instantaneously. However, the flow in the viscous layer near the sheet is developed slowly, and it becomes a fully developed steady flow after a certain instant of time. Many authors [24-35] studied the problem for unsteady stretching surface under different conditions by using a similarity method to transform governing time-dependent boundary layer equations into a set of ordinary differential equations. Recently, Mukhopadhyay [36] analyzed the combined effects of slip and suction/blowing on unsteady mixed convection flow past a stretching sheet.

No attempt has been made so far to analyze the Maxwell fluid flow and mass transfer past an unsteady stretching surface in presence of first order constructive/destructive chemical reaction. The present work aims to fill the gap in the existing literature. With the help of suitable transformations the governing partial differential equations are converted to ordinary one and the reduced ordinary differential equations are solved numerically using shooting method. The effects of governing parameters on velocity and concentration fields are investigated and analysed with the help of their graphical representations.

β	Maxwell parameter
η	similarity variable
γ	reaction rate parameter
λ	relaxation time of the period
v	kinematic viscosity
ψ	stream function
ϕ	non-dimensional concentration
ϕ'	first order derivative with respect to η
ϕ''	second order derivative with respect to η

2. Equations of motion

We consider laminar boundary-layer two-dimensional flow and mass transfer of an incompressible non-Newtonian Maxwell fluid over an unsteady stretching sheet. Let C_w be the concentration at the sheet surface and the concentration far away from the sheet is C_{∞} . Also the reaction of the species be the first order homogeneous chemical reaction of rate k_1 which varies with time.

We assume that for time t < 0 the fluid and mass flows are steady. The unsteady fluid and mass flows start at t = 0. The sheet emerges out of a slit at origin (x = 0, y = 0) and moves with non-uniform velocity $U(x, t) = \frac{bx}{1-\alpha t}$ where b, α are positive constants with dimensions (time) $^{-1}$, b is the initial stretching rate and $\frac{b}{1-\alpha t}$ is the effective stretching rate which is increasing with time. In case of polymer extrusion, the material properties of the extruded sheet may vary with time.

The governing equations of such type of flow (Alizadeh-Pahlavan and Sadeghy [37]) and mass transfer are, in the usual notation,

$$\begin{aligned} \frac{\partial u}{\partial x} &+ \frac{\partial v}{\partial y} = 0, \end{aligned} \tag{1} \\ \frac{\partial u}{\partial t} &+ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \lambda \left(u^2 \frac{\partial^2 u}{\partial x^2} + v^2 \frac{\partial^2 u}{\partial y^2} + 2uv \frac{\partial^2 u}{\partial x \partial y} \right) \\ &= v \frac{\partial^2 u}{\partial y^2}, \end{aligned} \tag{2}$$

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2} - k_1 (C - C_\infty).$$
(3)

For Maxwell fluid, no unsteady term for the shear stress need to be included as mentioned by Alizadeh-Pahlavan and Sadeghy [37]. So no unsteady term appears in the coefficient of λ .

Here *u* and *v* are the components of velocity respectively in the *x* and *y* directions, *v* is the kinematic viscosity of the fluid, *C* is the concentration of the species of the fluid, *D* is the diffusion coefficient of the diffusing species in the fluid, $k_1(t) = \frac{k_0}{1-\alpha t}$ is the time-dependent reaction rate; $k_1 > 0$ stands for destructive reaction whereas $k_1 < 0$ stands for constructive reaction, k_0 is a constant, $\lambda = \lambda_0 (1 - \alpha t)$ is the relaxation time of the period, λ_0 is a constant.

Nomenclature

2.1. Boundary conditions

The appropriate boundary conditions for the problem are given by

$$u = U(x,t), \quad v = 0, \quad C = C_w(x,t) \text{ at } y = 0,$$
 (4)

$$u \to 0, \quad C \to C_{\infty} \text{ as } y \to \infty.$$
 (5)

where the concentration of the surface of the sheet is similarly assumed to vary both along the sheet and with time, in accordance with $C_w(x,t) = C_\infty + bx(1 - \alpha t)^{-2}$ where C_∞ is the constant free stream concentration. The wall concentration $C_w(x,t)$ represents a situation in which the sheet concentration increases (reduces) if *b* is positive (negative) in proportion to *x* and such that the amount of concentration increase (reduction) along the sheet increases with time. The expressions for U(x,t), $C_w(x,t)$, $\lambda(t)$, $k_1(t)$ are valid for time $t < \alpha^{-1}$.

2.2. Method of solution

We now introduce the following relations for u, v (Mukhopadhyay [36,38], Mukhopadhyay and Vajravelu [39]) and ϕ as,

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad \text{and} \quad \phi = \frac{C - C_{\infty}}{C_w - C_{\infty}}$$
 (6a)

where ψ is the stream function.

Let us also introduce (Mukhopadhyay [36,38])

$$\eta = \sqrt{\frac{c}{\nu(1-\alpha t)}}\nu, \quad \psi = \sqrt{\frac{\nu c}{(1-\alpha t)}}xf(\eta),$$
$$C = C_{\infty} + bx(1-\alpha t)^{-2}\phi(\eta). \tag{6b}$$

With the help of the above relations, the governing equations finally reduce to

$$M\left(\frac{\eta}{2}f''+f'\right)+f'^2-ff''+\beta(f^2f''-2ff'f'')=f''',$$
(7)

$$M\left(\frac{1}{2}\eta\phi'+2\phi\right)+f'\phi-f\phi'=\frac{1}{\mathrm{Sc}}\phi''-\gamma\phi,\tag{8}$$

where $M = \frac{\alpha}{b}$ is the unsteadiness parameter, $\beta = b\lambda_0$ is the Maxwell parameter, $Sc = \frac{\gamma}{D}$ is the Schmidt number, $\gamma = \frac{k_0}{b}$ is the reaction rate parameter. Here $\gamma > 0$ represents the destructive reaction, $\gamma = 0$ corresponds to no reaction, and $\gamma < 0$ stands for the generative reaction.

The boundary conditions (4) and (5) then become

$$f' = 1, \quad f = 0, \quad \phi = 1 \text{ at } \eta = 0$$
 (9)
and

$$f' \to 0, \quad \phi \to 0 \text{ as } \eta \to \infty.$$
 (10)

Eqs. (7) and (8) along with boundary conditions (9) and (10) were solved numerically by shooting method (Mukhopadhyay [32,34]).

Table 1 The values of f''(0) for various values of unsteadiness parameter M with $\beta = 0$.

М	Sharidan et al. [29]	Chamakha et al. [33]	Present study
0.8	-1.261042	-1.261512	-1.261479
1.2	-1.377722	-1.378052	-1.377850

3. Results and discussion

In order to validate the method used in this study and to judge the accuracy of the present analysis, comparison with available results corresponding to the skin-friction coefficient f''(0) for unsteady flow of viscous incompressible Newtonian fluid ($\beta = 0$) are compared with the available results of Sharidan et al. [29] and Chamkha et al. [33] in Table 1 and found that the results agree well.

In order to study the behaviour of velocity and concentration fields for Maxwell fluid, a comprehensive numerical computation is carried out for various values of the parameters that describe the flow characteristics, and the results are reported in terms of graphs Figs. 1–4.

Fig. 1a exhibits the velocity profiles for several values of unsteadiness parameter M. It is seen that the velocity along the sheet decreases initially with the increase of unsteadiness parameter M and this implies an accompanying reduction of the thickness of the momentum boundary layer near the wall but away from the wall fluid velocity increases with increasing unsteadiness. M = 0 indicates the steady case.

Fig. 1b represents the effects of unsteadiness parameter on the solute distribution. From this figure, it is noticed that the concentration at a particular point is found to decrease significantly with increasing unsteadiness parameter. Rate of mass transfer (from the fluid to the sheet) decreases with increasing M [see also Fig. 4]. As the unsteadiness parameter M increases, less mass is transferred from the fluid to the sheet; hence, the concentration $\phi(\eta)$ decreases (Fig. 1b). Since the fluid flow is caused solely by the stretching sheet and the sheet surface concentration is higher than free stream concentration, the velocity and concentration decrease with increasing η . Concentration overshoot is noted in this case.

Effects of Maxwell parameter β on velocity and concentration profiles for unsteady motion are clearly exhibited in Figs. 2a and 2b respectively. Here $\beta = 0$ gives the result for viscous incompressible fluid. The effect of increasing values of β is to reduce the velocity and hence the boundary layer thickness decreases (Fig. 2a). The velocity curves in Fig. 2a show that the rate of transport is considerably reduced with the increase of β . The effect of increasing β leads to enhance the concentration field (Fig. 2b). The thickening of the solute boundary layer occurs due to increase in the elasticity stress parameter. It can also be seen from Fig. 2a that the momentum



Figure 1a Velocity profiles for variable unsteadiness parameter *M*.



Figure 1b Concentration profiles for variable unsteadiness parameter *M*.



Figure 2a Velocity profiles for variable Maxwell parameter β for unsteady motion.



Figure 2b Concentration profiles for variable Maxwell parameter β for unsteady motion.

boundary layer thickness decreases as β increases, and hence induces an increase in the absolute value of the velocity gradient at the surface. The value of the concentration gradient at the surface increases with an increase in β , as shown in Fig. 2b. Thus, the mass transfer rate at the surface increases with increasing β . The concentration field ϕ also decreases for large values of β in case of generative chemical reaction ($\gamma < 0$). But the magnitude of ϕ is larger in case of ($\gamma < 0$) when compared with the case of destructive chemical reaction



Figure 3a Concentration profiles for variable values of Schmidt number Sc.



Figure 3b Concentration profiles for variable values of reaction rate parameter γ .



Figure 4 Variation of mass transfer coefficient with Maxwell parameter β for two values of unsteadiness parameter *M*.

 $(\gamma > 0)$. It is also found that the concentration field is decreased for several values of β in all cases $(\gamma = 0, \gamma > 0 \text{ and } \gamma < 0)$.

An interesting behaviour of the concentration profiles for the variation of Schmidt number can be found from Fig. 3a. It is noted that initially (near the wall) the concentration increases but away from the wall it decreases with increasing Sc (Fig. 3a). Moreover, the solute boundary layer thickness decreases with increasing Schmidt number. Fig. 3b illustrates the effects of reaction rate parameter γ on concentration profiles. Concentration increases with the increase of reaction rate parameter γ . Concentration overshoot is noted for all values of γ considered in this study (Fig. 3b). It is seen that concentration field increases for generative chemical reaction ($\gamma < 0$) where as it decreases for destructive chemical reaction ($\gamma < 0$). Note that the change in case of generative chemical reaction ($\gamma < 0$) is larger in comparison to the case of destructive chemical reaction ($\gamma < 0$) is larger in comparison to the case of destructive chemical reaction ($\gamma < 0$). The concentration boundary layer decreases in case of destructive chemical reaction.

Furthermore, the effects of unsteadiness parameter M and Maxwell parameter β on mass transfer coefficient is presented in Fig. 4. Mass transfer rate at the surface $[\phi'(0)]$ increases for β , decreases with M.

4. Conclusions

In fine, the following conclusions can be drawn:

- (i) Fluid velocity decreases initially due to increase in unsteadiness parameter. The concentration also decreases significantly in this case.
- (ii) The effect of increasing values of the Maxwell parameter is to suppress the velocity field. The concentration is enhanced with increasing Maxwell parameter.
- (iii) Mass transfer rate at the surface decrease with unsteadiness.

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