Linear preservers between matrix modules over connected commutative rings

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Abstract

Let R be a connected commutative ring with identity 1 (R contains no idempotents except 0 and 1), and let Mn(R) be the R-module of all n × n matrices over R. R is said to be idempotence-diagonalizable if every idempotent matrix over R is similar to a diagonal matrix. For two arbitrary positive integers n and m, we characterize (a) linear maps from Mn(R) to Mm(R) preserving tripotence when R is any idempotence-diagonalizable ring with the units 2 and 3, and (b) linear maps from Mn(R) to Mm(R) preserving inverses (respectively, Drazin inverses, group inverses, {1}-inverses, {2}-inverses and {1,2}-inverses) when R is either any idempotence-diagonalizable ring with the units 2 and 3, or any commutative principal ideal domain with at least one unit except for 1 and 2. These characterizations are completed by using an idempotence-preserving result obtained by Cao [Linear maps preserving idempotence on matrix modules over some rings, J. Natur. Sci. Heilongjiang Univ. 16 (1) (1999) 1–4]. Moreover, we also give a simple proof of Cao’s result.

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1. Introduction

Suppose \( R \) is a connected commutative ring with 1 (\( R \) contains no idempotents except 0 and 1 [1]). We say \( R \) is idempotence-diagonalizable if every idempotent matrix over \( R \) is similar to a diagonal matrix. Obviously, the category of the idempotence-diagonalizable rings contains all commutative principal ideal domains, all commutative local rings, etc. [12]. We will hereafter assume that \( n \) and \( m \) be two arbitrary positive integers with \( n \geq 2 \), and that \( R \) is any idempotence-diagonalizable ring, unless otherwise specified.

Let \( R^\times \) denote the subset of \( R \) consisting of all units. We denote by \( M_{m \times n}(R) \) the \( R \)-module of all \( m \times n \) matrices, and we write simply \( M_{n \times n}(R) \) as \( M_n(R) \). Let \( I_n(R) \) (respectively, \( T_n(R) \) and \( GL_n(R) \)) be the subset of \( M_n(R) \) consisting of all idempotent (respectively, tripotent and nonsingular) matrices. Recall that a linear map \( L : M_n(R) \rightarrow M_m(R) \) is said to preserve idempotence (respectively, tripotence) if \( L(I_n(R)) \subseteq I_m(R) \) (respectively, \( L(T_n(R)) \subseteq T_m(R) \)).

For a matrix \( A \in M_n(R) \), consider the following matrix equations with unknown \( X \in M_n(R) \).

\[
AX =XA, \quad (1)
\]
\[
XAX =X, \quad (2)
\]
\[
A^kXA = A^k \quad \text{for some positive integer} \ k. \quad (3)
\]

When \( k = 1 \), (3) turns into

\[
AXA = A. \quad (4)
\]

We say that \( X \) is a \( [1] \)-inverse of \( A \) if \( X \) satisfies (4), \( X \) is a \( [2] \)-inverse of \( A \) if \( X \) satisfies (2), \( X \) is a \( [1,2] \)-inverse of \( A \) if \( X \) satisfies (2) and (4), \( X \) is a Drazin inverse of \( A \) if \( X \) satisfies (1), (2) and (3), and \( X \) is a group inverse of \( A \) if \( X \) satisfies (1), (2) and (4). For more knowledge about these generalized inverses see [2]. A linear map \( L : M_n(R) \rightarrow M_m(R) \) is said to preserve Drazin (respectively, group, \( [1] \)-, \( [2] \)- and \( [1,2] \)-) inverses if \( L(B) \) is a Drazin (respectively, group, \( [1] \)-, \( [2] \)- and \( [1,2] \)-) inverse of \( L(A) \) whenever \( B \) is a Drazin (respectively, group, \( [1] \)-, \( [2] \)- and \( [1,2] \)-) inverse of \( A \in M_n(R) \).

For a pair of real numbers \( a \) and \( b \) with \( a \leq b \), let \( [a,b] \) be the set of all integers between \( a \) and \( b \). Let \( O \) be the zero matrix. For an integer \( k \), we denote by \( I_k \) the \( k \times k \) identity matrix if \( k > 0 \) and the \( 0 \times 0 \) empty matrix if \( k = 0 \). Denote by \( \oplus \) and \( \otimes \) the usual Kronecker product and direct sum of matrices, respectively.

When \( n \geq m \), many researchers have characterized linear preservers from \( M_n(R) \) to \( M_m(R) \) (see [8,10,14] and the references therein). Here we mention only those which are close related to this paper. Chan and Lim [6] characterized linear maps from \( M_n(R) \) to itself preserving tripotence when \( R \) is any field of \( \text{Ch} R \neq 2, 3 \). Cao and Zhang [4] characterized additive maps from \( M_n(R) \) to itself preserving tripotence (respectively, inverses and group inverses) when \( R \) is the same as that in [6].
Zhang and Liu [13] characterized linear maps from $M_n(\mathbb{R})$ to $M_m(\mathbb{R})$ preserving tripotence when $\mathbb{R}$ is any commutative principal ideal domain with $2 \in \mathbb{R}^*$ and $\mathbb{R} \neq \{1, -1, 0\}$. Cao [3] studied the linear maps from $M_n(\mathbb{R})$ to itself preserving group inverses when $\mathbb{R}$ is any commutative local ring with $2, 3 \in \mathbb{R}^*$.

However, when $n < m$, only recently there appeared works concerning structure of linear preservers from $M_n(\mathbb{R})$ to $M_m(\mathbb{R})$. As far as we know, there are only [9] (on rank preserving maps when $\mathbb{R}$ is any field with some restriction), [11] (on preservers of adjoint matrices when $\mathbb{R}$ is any field), [5] (on idempotence preserving maps when $\mathbb{R}$ is any idempotence-diagonalizable ring), and [7] (on preservers of unitary matrices, norms, numerical ranges, and other related properties when $\mathbb{R}$ is the complex number field).

Inspired by these works mentioned above, in this paper we characterize

(a) linear maps from $M_n(\mathbb{R})$ to $M_m(\mathbb{R})$ preserving tripotence when $\mathbb{R}$ is any idempotence-diagonalizable ring with $2, 3 \in \mathbb{R}^*$, and

(b) linear maps from $M_n(\mathbb{R})$ to $M_m(\mathbb{R})$ preserving inverses (respectively, Drazin inverses, group inverses, $[1]$-inverses, $[2]$-inverses and $[1,2]$-inverses) when $\mathbb{R}$ is either any idempotence-diagonalizable ring with $2, 3 \in \mathbb{R}^*$, or any commutative principal ideal domain with $2 \in \mathbb{R}^*$ and $|\mathbb{R}^*| \neq 2$.

The characterizations of (a) and (b) are obtained by using an idempotence-preserving result proposed by Cao [5], which avoids to determine the images of those matrices in the form of $E_{ij}$, and hence the proofs are simpler than those in [3,4,6,13]. Moreover, we also give a simple proof of Cao’s result.

This paper is organized as follows. In the next section, main results of this paper are described. In Section 3, we provide some preliminary results which will be used to prove our main results. We will prove main results in Section 4.

2. Main results

Main results of this paper are the following three theorems.

**Theorem 1.** Let $\mathbb{R}$ be any idempotence-diagonalizable ring with $2, 3 \in \mathbb{R}^*$. Then $L : M_n(\mathbb{R}) \to M_m(\mathbb{R})$ is a linear map preserving tripotence if and only if either

(i) $L = 0$, or

(ii) $n \leq m$ and $L(A) = Q[(A \otimes I_{\delta_1}) \oplus (A^T \otimes I_{r_1-\delta_1}) \oplus (A \otimes -I_{\delta_2}) \oplus (A^T \otimes -I_{r_2-\delta_2}) \oplus O]Q^{-1}$ for every $A \in M_n(\mathbb{R})$, where $Q \in GL_m(\mathbb{R})$ and $r_1, r_2, \delta_1, \delta_2$ are integers satisfying $r_1 + r_2 \in [1, m], \delta_1 \in [0, r_1]$ and $\delta_2 \in [0, r_2]$.

**Theorem 2.** Let $\mathbb{R}$ be either any idempotence-diagonalizable ring with $2, 3 \in \mathbb{R}^*$, or any commutative principal ideal domain with $2 \in \mathbb{R}^*$ and $|\mathbb{R}^*| \neq 2$. Then $L :
$M_n(R) \rightarrow M_m(R)$ is a linear map preserving inverses (i.e., $L(A)^{-1} = L(A^{-1})$ for every $A \in GL_n(R)$) if and only if $m = rn$ for some positive integer $r$, and $L(A) = Q[(A \otimes I_{\delta_1}) \oplus (A^T \otimes I_{r_1 - \delta_1}) \oplus (A \otimes -I_{\delta_2}) \oplus (A^T \otimes -I_{r_2 - \delta_2})]Q^{-1}$ for any $A \in M_n(R)$, where $Q \in GL_m(R)$ and $r_1, r_2, \delta_1, \delta_2$ are integers satisfying $r_1 + r_2 = r$, $\delta_1 \in [0, r_1]$ and $\delta_2 \in [0, r_2]$.

Theorem 3. Let $R$ be either any idempotence-diagonalizable ring with $2, 3 \in R^*$, or any commutative principal ideal domain with $2 \in R^*$ and $|R^*|/2 \neq 2$. Then $L : M_n(R) \rightarrow M_m(R)$ is a linear map preserving Drazin inverses (respectively, group inverses, $\{1, 2\}$-inverses, $\{1\}$-inverses and $\{2\}$-inverses) if and only if $L$ is of one of the forms in Theorem 1.

Regarding our main results, the following items should be pointed out.

1. For any connected commutative ring $R$ with the identity 1, it is well-known that $2, 3 \in R^*$ is a sufficient condition (not necessary) of $2 \in R^*$ and $|R^*| \neq 2$.
2. When $R$ is a commutative principal ideal domain, the condition $2, 3 \in R^*$ in Theorem 1 can be weakened to $2 \in R^*$ and $Ch(R) \neq 3$ (for details see the proof of Theorem 1 below);
3. If $R$ is a field, then $2 \in R^*$ is equivalent to $Ch(R) \neq 2$, and $|R^*| \neq 2$ is equivalent to $R \neq \{1, 0, -1\}$;
4. Theorems 1–3 are also true when $n = 1$.

Due to the above items, our main results can be viewed as a supplementary version of [4,13]. Moreover, Theorems 3 and 1 generalize the corresponding results in [3,6], respectively.

3. Preliminary results

This section provides some lemmas which will be used to prove our main results.

Lemma 1. Suppose $R$ is a commutative principal ideal domain with $2 \in R^*$. Then $|R^*| \geq 4$ if and only if $|R^*| \geq 3$.

Proof. The “only if” part is obvious. Now we prove the “if” part.

It follows from $2 \in R^*$ that 1 and $-1$ are a pair of distinct elements in $R^*$. Since $|R^*| \geq 3$, there exists $x \in R^*$ such that $x^{-1} \in R^* \setminus \{1, -1, x\}$. Therefore, $1, -1, x$ and $x^{-1}$ are four mutually distinct elements in $R^*$, i.e., $|R^*| \geq 4$. The proof is completed. \[ \Box \]

Remark 1. When $R$ is a field such that $Ch(R) \neq 2$, these three conditions $R \neq \{1, 0, -1\}$, $|R^*| \geq 4$ and $|R^*| \geq 3$ are equivalent. However, this is not true for general
Lemma 2 [5]. Suppose \( R \) is any idempotence-diagonalizable ring. If \( A \in S_n(R) \), then there exists \( P \in GL_n(R) \) such that \( P^{-1}AP = I_r \oplus O \), where \( r = \text{rank } A \).

Lemma 3. Suppose \( R \) is any idempotence-diagonalizable ring with \( 2 \in R^* \). If \( A \in S_n(R) \), then there exists \( P \in GL_n(R) \) such that \( P^{-1}AP = I_p \oplus -I_q \oplus O \), where \( p + q = \text{rank } A \).

\[ P^{-1}AP_1 = X \oplus O, \]  
\[ \text{with } X^2 = I_r. \]

Proof. It follows from \( A \in S_n(R) \) that \( A^2 \in S_n(R) \). By Lemma 2, there exists \( P_1 \in GL_n(R) \) such that \( P_1^{-1}A^2P_1 = I_r \oplus O \), where \( r = \text{rank } A^2 \). Using \( A^2A = AA^2 \), we have

\[ P_1^{-1}AP_1 = X \oplus O, \]

with \( X^2 = I_r. \) Noting that \( 2^{-1}(X + I_r) \in S_r(R) \), one can assume from Lemma 2 that

\[ 2^{-1}(X + I_r) = P_2(I_p \oplus O)P_2^{-1}, \]

where \( P_2 \in GL_r(R) \) and \( p = \text{rank}(X + I_r) \). Let \( P = P_1(P_2 \oplus I_{n-r}) \) and \( q = r - p \). Then, by (5) and (6), the proof is completed. \( \square \)

Obviously, Lemma 3 generalizes [13–Lemma 1]. In the rest of this section we will assume that \( E_{ij}^{(n)} \) is the \( n \times n \) matrix with 1 in the \((i, j)\)-entry and 0 elsewhere.

Lemma 4. Suppose \( R \) is any idempotence-diagonalizable ring satisfying \( \text{Char } R \neq 2 \), and \( L : M_n(R) \to M_m(R) \) is a linear map preserving idempotence. Then \( L(I_n) = O \) if and only if \( L(E_{kk}^{(n)}) = O \) for some \( k \in [1, n] \).

Proof. The “only if” part. It follows from \( E_{11}^{(n)} \), \( I_n - E_{11}^{(n)} \in S_n(R) \) that \( L(E_{11}^{(n)}) \), \( L(I_n) - L(E_{11}^{(n)}) \in S_m(R) \). This, together with \( L(I_n) = O \), implies \( L(E_{11}^{(n)}) = O \).

The “if” part. For any \( i \in [1, n]\{k\} \), it follows from \( L(E_{kk}^{(n)}) = O \) and \( E_{kk}^{(n)} \in S_n(R) \) that \( \pm L(E_{ik}^{(n)}) \in S_m(R) \), which implies \( L(E_{ik}^{(n)}) = O \). Similarly, \( L(E_{ki}^{(n)}) = O \). Therefore, \( 2L(E_{ii}^{(n)}) = L(2E_{ii}^{(n)} - E_{kk}^{(n)} + 2E_{ik}^{(n)} - E_{ki}^{(n)}) \in S_m(R) \). This, together with \( L(E_{ii}^{(n)}) \in S_m(R) \), gives \( L(E_{ii}^{(n)}) = O \). In summary, \( L(I_n) = \sum_{i=1}^n L(E_{ii}^{(n)}) = O. \) \( \square \)

Lemma 5 [5]. Suppose \( R \) is an arbitrary idempotence-diagonalizable ring with \( 2 \in R^* \), and \( L : M_n(R) \to M_m(R) \) is a linear map which preserves idempotence and satisfies \( L(I_n) = I_m \). Then \( m = pn \) for some positive integer \( p \), and there exist \( P \in GL_m(R) \) and \( q \in [0, p] \) such that \( L(A) = P((A \oplus I_q) \oplus (A^T \oplus I_{p-q}))P^{-1} \) for any \( A \in M_n(R) \).
A proof of Lemma 5 was given by Cao [5]. However, we prefer to give a new proof (see below) because (i) the new proof is simpler and more readable than that in [5], (ii) [5] was written in Chinese, which restricts the readers to a narrow range, and (iii) due to the new proof, the condition 2 ∈ R* can be relaxed to ChR ≠ 2 when R is a commutative principal ideal domain.

The proof of Lemma 5. It follows from $E_{nn}^{(n)} \in \mathfrak{S}_n(R)$ that $L(E_{nn}^{(n)}) \in \mathfrak{S}_m(R)$. Without loss of generality, we can assume from $L(I_n) = I_m$, Lemmas 2 and 4 that

$$L(E_{nn}^{(n)}) = O \oplus I_p,$$

where $p \in [1, m]$. If $p = m$, then $L(E_{nn}^{(n)}) = I_m$. Thus, $I_m + L(E_{11}^{(n)}) = L(E_{nn}^{(n)} + E_{11}^{(n)}) \in \mathfrak{S}_m(R)$. This, together with $L(E_{11}^{(n)}) \in \mathfrak{S}_m(R)$ and $2 ∈ R^*$, gives $L(E_{11}^{(n)}) = O$. Using Lemma 4, we have $L(I_n) = O$, which contradicts the hypothesis $L(I_n) = I_m$. Therefore, $p \in [1, m − 1]$.

Now we prove the lemma by induction on $n$. If $n = 2$, it follows from $L(I_2) = I_m$ and (7) that

$$L(E_{11}^{(2)}) = I_{m−p} \oplus O, \quad L(E_{22}^{(2)}) = O \oplus I_p.$$  

Since $E_{11}^{(2)} = E_{12}^{(2)} = E_{21}^{(2)} = E_{22}^{(2)} \in \mathfrak{S}_2(F)$, we have $L(E_{11}^{(2)}) = L(E_{12}^{(2)}) = L(E_{21}^{(2)}) = L(E_{22}^{(2)}) \in \mathfrak{S}_m(F)$. This, together with (8), implies that

$$L(E_{12}^{(2)}) = \begin{bmatrix} O & G_1 \\ G_2 & O \end{bmatrix}, \quad L(E_{21}^{(2)}) = \begin{bmatrix} O & H_1 \\ H_2 & O \end{bmatrix},$$  

with

$$G_1G_2 = O, \quad G_2G_1 = O, \quad H_1H_2 = O, \quad H_2H_1 = O,$$

where $G_1, H_1 \in M_{(m−p)×p}(R)$ and $G_2, H_2 \in M_{p×(m−p)}(R)$. Combining (8), (9) and $2E_{11}^{(2)} + 2E_{22}^{(2)} = E_{12}^{(2)} + E_{21}^{(2)} \in \mathfrak{S}_2(R)$, one can obtain that

$$\begin{bmatrix} 2I_{m−p} & 2G_1 − H_1 \\ 2G_2 − H_2 & −I_p \end{bmatrix} \in \mathfrak{S}_m(R),$$

or equivalently,

$$(2G_1 − H_1)(2G_2 − H_2) = −2I_{m−p}, \quad (2G_2 − H_2)(2G_1 − H_1) = −2I_p.$$  

(11)

Thus, $m = 2p = pn$. This, together with (10) and (11), implies that

$$G_1 = U(I_q \oplus O)V, \quad G_2 = V^{-1}(O \oplus I_{p−q})U^{-1},$$

(12)

$$H_1 = U \begin{bmatrix} X & Y \\ Y & I_{p−q} \end{bmatrix}V, \quad H_2 = V^{-1} \begin{bmatrix} I_q & −X \\ −Y & YX \end{bmatrix}U^{-1},$$

(13)

where $q = \text{rank} \ G_1, \ X ∈ M_{q×(p−q)}(R), \ Y ∈ M_{(p−q)×q}(R)$ and $U, V ∈ GL_p(R)$. Denote

$$P = (U \oplus V^{-1}) \begin{bmatrix} I_q & O & X & O \\ O & I_{p−q} & O & O \\ O & I_p & O & O \\ −Y & O & I_{p−q} \end{bmatrix}.$$
Then, by (8), (9), (12) and (13), it is easily verified that \( L(E_{ij}^{(2)}) = P[(E_{ij}^{(2)} \otimes I_q) \oplus (E_{ij}^{(2)} \otimes I_{p-q})]P^{-1} \) for any \( i, j \in [1, 2] \). This, together the linearity of \( L \), implies that the lemma is true when \( n = 2 \).

Suppose the lemma is true for \( n = t-1 \), where \( t \geq 3 \); we will prove it is true when \( n = t \). For any \( B \in \mathfrak{J}_{t-1}(R) \), because of \( B \oplus 0, E_{ii}^{(t)} + (B \oplus 0) \in \mathfrak{J}_{t}(R) \), we have from the definition of \( L \) that \( L(B \oplus 0), L(E_{tt}^{(t)}) + L(B \oplus 0) \in \mathfrak{J}_{t}(R) \). This, together with (7), implies that

\[
L(B \oplus 0) = f(B) \oplus O, \quad \forall B \in \mathfrak{J}_{t-1}(R),
\]

where \( f(B) \in \mathfrak{J}_{m-p}(R) \) satisfies \( f(I_{t-1}) = I_{m-p} \). Since any matrix in \( M_{t-1}(R) \) can be represented as a linear combination of finitely many matrices in \( \mathfrak{J}_{t-1}(R) \), one can obtain from (14) that

\[
L(C \oplus 0) = f(C) \oplus O, \quad \forall C \in M_{t-1}(R),
\]

where \( f \) is a linear map from \( M_{t-1}(R) \) to \( M_{m-p}(R) \) which preserves idempotence and satisfies \( f(I_{t-1}) = I_{m-p} \). By the induction hypothesis, we obtain that

\[
m - p = r(t - 1) \quad \text{for some positive integer } r,
\]

and there exist \( P_1 \in GL_{m-p}(R) \) and \( q \in [0, r] \) such that \( f(C) = P_1[(C \otimes I_q) \oplus (C^T \otimes I_{r-q})]P_1^{-1} \) for any \( C \in M_{t-1}(R) \). Let \( P_2 = P_1 \oplus I_p \). Then (7) and (15) imply

\[
\begin{align*}
L(E_{tt}^{(t)}) &= P_2(O \oplus I_p)P_2^{-1}, \\
L(C \oplus 0) &= P_2[(C \otimes I_q) \oplus (C^T \otimes I_{r-q})] \oplus O]P_2^{-1}, \quad \forall C \in M_{t-1}(R).
\end{align*}
\]

(17)

Let \( i \in [1, t-1] \) be arbitrary but fixed. By a similar argument to (9), we can assume that

\[
L(E_{ii}^{(t)}) = P_2 \left[ \begin{array}{cc} O & A_i \\ B_i & O \end{array} \right] P_2^{-1},
\]

where \( A_i \in M_{(t-1)\times p}(R) \) and \( B_i \in M_{p \times (t-1)\times r}(R) \). Furthermore, noting that \( E_{ii}^{(t)} + E_{jj}^{(t)} \) and \( E_{ii}^{(t)} + E_{jj}^{(t)} \in \mathfrak{J}_r(R) \) for any \( j \in [1, t-1] \setminus \{i\} \), we have from (17) and (18) that

\[
\left[ (E_{ii}^{(t-1)} \otimes I_q) \oplus (E_{ii}^{(t-1)} \otimes I_{r-q}) \right] = \mathfrak{J}_m(R)
\]

and

\[
\left[ (E_{ii}^{(t-1)} + E_{jj}^{(t-1)} \otimes I_q) \oplus (E_{ii}^{(t-1)} + E_{jj}^{(t-1)} \otimes I_{r-q}) \right] = \mathfrak{J}_m(R),
\]

\forall j \in [1, t-1] \setminus \{i\}.

(20)
The combination of (19) and (20) gives that
\[ A_i = \left[ (e_1 \otimes X_i)^T \ O \right], \quad B_i = \left[ O \ e_i^T \otimes Y_i \right], \] (21)
where \( X_i \in M_{q \times p}(\mathbb{R}) \), \( Y_i \in M_{p \times (r-q)}(\mathbb{R}) \) and \( e_i \) is the \( t-1 \) dimensional column vector with 1 in the \( i \)th entry and 0 elsewhere. For any \( s \in [2, t-1] \), because of \( E_{11}^{(t)} + E_{1s}^{(t)} + E_{st}^{(t)} + E_{tt}^{(t)} \in \mathfrak{F}_t(\mathbb{R}) \), it follows from (17), (18) and (21)
\[
\begin{bmatrix}
(E_{11}^{(t-1)} + E_{1s}^{(t-1)}) \otimes I_q \\
O \quad (E_{11}^{(t-1)} + E_{1s}^{(t-1)}) \otimes I_{r-q} \\
O \quad e_i^T \otimes Y_i - e_i^T \otimes Y_i \\
O \quad I_p
\end{bmatrix}
\in \mathfrak{F}_m(\mathbb{R}).
\]
By a direct computation, we have \( X_s = X_1 \) and \( Y_s = Y_1 \) for any \( s \in [2, t-1] \). This, together with (18) and (21), implies that
\[ L(E_{tt}^{(t)}) = P_2 \begin{bmatrix}
O & O & e_i \otimes G \\
O & O & O \\
e_i^T \otimes H & O \\
O & O & O
\end{bmatrix} P_2^{-1}, \quad \forall \ i \in [1, t-1], \] (22)
with \( G = X_1 \) and \( H = Y_1 \). By a similar argument to (22), we can obtain that
\[ L(E_{ii}^{(t)}) = P_2 \begin{bmatrix}
O & O & e_i \otimes E \\
O & O & O \\
e_i^T \otimes F & O & O
\end{bmatrix} P_2^{-1}, \quad \forall \ i \in [1, t-1], \] (23)
where \( E \in M_{(r-q) \times p}(\mathbb{R}) \) and \( F \in M_{p \times q}(\mathbb{R}) \).
Because of \( 2E_{ii}^{(t)} - E_{ii}^{(t)} + 2E_{iti}^{(t)} - E_{tt}^{(t)} \in \mathfrak{F}_t(\mathbb{R}) \), it follows from (17), (22) and (23) that
\[
\begin{bmatrix}
2E_{ii}^{(t-1)} \otimes I_q \\
O \\
2E_{iti}^{(t-1)} \otimes I_{r-q} \quad 2e_i \otimes G \\
O \\
-e_i^T \otimes F \\
2e_i^T \otimes H \\
O \\
O \\
-e_i^T \otimes E \\
-I_p
\end{bmatrix}
\in \mathfrak{F}_m(\mathbb{R}).
\]
By a direct computation, we have \( GF = I_q, \ GH = O, \ EF = O, \ EH = I_{r-q} \) and \( FG + HE = I_p \). This implies that there exists \( W \in GL_{p \times p}(\mathbb{R}) \) such that
\[ G = \begin{bmatrix} I_q & O \end{bmatrix} W^{-1}, \quad F = W \begin{bmatrix} I_q & O \end{bmatrix}^T, \]
\[ H = W \begin{bmatrix} O & \hat{H}^T \end{bmatrix}^T, \quad E = \begin{bmatrix} O & \hat{E} \end{bmatrix} W^{-1}, \] (24)
with
\[ \hat{E} = I_{r-q}, \quad \hat{H} \in M_{p \times q}(\mathbb{R}). \]
Using (25), one can obtain that
\[ p = r, \quad \tilde{E} = \tilde{H}^{-1}. \] (26)
This, together with (16), implies $m = pt$. Let

$$P = P_2 \begin{bmatrix} I_{(t-1)p} & O \\ O & W \end{bmatrix} \begin{bmatrix} I_{(t-1)q} & O & O & O \\ O & O & I_{(t-1)(p-q)} & O \\ O & I_q & O & O \\ O & O & O & I_{p-q} \end{bmatrix} .$$

Then, by (17), (22)–(24) and (26), it is easily verified that

$$L(E(t)_{ij}) = P[(E(t)_{ij} \otimes I_q) \oplus (E(t)_{ji} \otimes I_{p-q})]P^{-1} \text{ for any } i,j \in [1,t].$$

Using the linearity of $L$, we obtain that $L(A) = P[(A \otimes I_q) \oplus (A^T \otimes I_{p-q})]P^{-1}$ for any $A \in M_t(\mathbb{R})$, i.e., this lemma is true when $n = t$. □

4. Proofs of main results

Based on Lemmas 1, 3 and 5, we can give proofs of Theorems 1–3 as follows.

The proof of Theorem 1. The “if” part is obvious. Now we prove the “only if” part. It follows from $I_n \in T_n(\mathbb{R})$ and the definition of $L$ that

$$L(I_n) \in \mathfrak{T}_m(\mathbb{R}).$$

(27)

For any $B \in \mathfrak{T}_n(\mathbb{R})$, it is obvious that $B, I_n - B, I_n - 2B \in T_n(\mathbb{R})$. This, together with the definition of $L$, gives

$$L(B), L(I_n) - L(B), L(I_n) - 2L(B) \in \mathfrak{T}_m(\mathbb{R}).$$

(28)

By a direct computation, we have from $2 \in \mathbb{R}^*$, (27) and (28) that

$$3L(B) = L(I_n)^2L(B) + L(I_n)L(B)L(I_n) + L(B)L(I_n)^2$$

$$= L(I_n)L(B)^2 + L(B)L(I_n)L(B) + L(B)^2L(I_n), \quad \forall B \in \mathfrak{T}_n(\mathbb{R}).$$

(29)

Case 1. Suppose $L(I_n) = O$. Then, by (29) and $3 \in \mathbb{R}^*$, we have

$$L(B) = O, \quad \forall B \in \mathfrak{T}_n(\mathbb{R}).$$

(30)

Since any matrix in $M_n(\mathbb{R})$ can be represented as a linear combination of finitely many matrices in $\mathfrak{T}_n(\mathbb{R})$, we can obtain from (30) and the linearity of $L$ that $L$ is of the form (i).

Case 2. Suppose $L(I_n) \neq O$. By Lemma 3 and (27), there exists $T \in GL_m(\mathbb{R})$ such that $L(I_n) = T(I_{p_1} \oplus -I_{p_2} \oplus O)T^{-1}$ with $p_1 + p_2 = \text{rank } L(I_n) > 0$. This, together with (29) and $2, 3 \in \mathbb{R}^*$, gives

$$T^{-1}L(B)T = f_1(B) \oplus -f_2(B) \oplus O, \quad \forall B \in \mathfrak{T}_n(\mathbb{R}),$$

(31)
where $f_i(B) \in \mathfrak{N}_n(R)$ satisfies $f_i(I_n) = I_{p_i}$ for $i = 1, 2$. Since any matrix in $M_n(F)$ can be represented as a linear combination of finitely many matrices in $\mathfrak{N}_n(R)$, we deduce from (31) and the linearity of $L$ that

$$T^{-1}L(A)T = f_1(A) \oplus -f_2(A) \oplus O, \quad \forall A \in M_n(R),$$

(32)

where $f_i : M_n(R) \to M_{p_i}(R)$, $i = 1, 2$, are a pair of linear maps which preserve idempotence and satisfy $f_i(I_n) = I_{p_i}$ for $i = 1, 2$. Combining (32) and Lemma 5, we obtain that $L$ is of the form (ii). □

The proof of Theorem 2. The “if” part is obvious. Now we prove the “only if” part. Using $I_n^{-1} = I_n$ and the definition of $L$, we have $L(I_n)^{-1} = L(I_n)$, i.e.,

$$L(I_n)^2 = I_m.$$  (33)

For any $B \in \mathfrak{N}_n(R)$ and $a \in R^*$, it is obvious that $[I_n - (a + 1)B]^{-1} = I_n - (a^{-1} + 1)B$. This, together with the definition of $L$, gives $[L(I_n) - (a + 1)L(B)]^{-1} = L(I_n) - (a^{-1} + 1)L(B)$. Thus,

$$[L(I_n) - (a + 1)L(B)][L(I_n) - (a^{-1} + 1)L(B)] = I_m.$$  (34)

By a direct computation, we have from (33) and (34) that

$$a^2[L(B)L(I_n) - L(B)^2] + a[L(I_n)L(B) - 2L(B)^2] + L(B)L(I_n)] + L(I_n)L(B) - L(B)^2 = O.$$  (35)

When $R$ is an idempotence-diagonalizable ring with 2, 3 $\in R^*$, let $a = 1, -1$ and 2 in (35), respectively, we have

$$L(B)L(I_n) = L(B)^2 = L(I_n)L(B).$$  (36)

When $R$ is a commutative principal ideal domain with 2 $\in R^*$ and $|R^*| \neq 2$, then $|R^*| \geq 3$. This, together with (35), gives (36).

By Lemma 3 and (33), there exists $T \in GL_m(R)$ such that $L(I_n) = T(I_{p_1} \oplus -I_{p_2})T^{-1}$ with $p_1 + p_2 = m$. This, together with (36), gives

$$T^{-1}L(B)T = f_1(B) \oplus -f_2(B), \quad \forall B \in \mathfrak{N}_n(R),$$

(37)

where $f_i(B) \in \mathfrak{N}_n(R)$ satisfies $f_i(I_n) = I_{p_i}$ for $i = 1, 2$. Since any matrix in $M_n(F)$ can be represented as a linear combination of finitely many matrices in $\mathfrak{N}_n(R)$, we deduce from (37) and the linearity of $L$ that

$$T^{-1}L(A)T = f_1(A) \oplus -f_2(A), \quad \forall A \in M_n(R),$$

(38)

where $f_i : M_n(R) \to M_{p_i}(R)$, $i = 1, 2$, are a pair of linear maps which preserve idempotence and satisfy $f_i(I_n) = I_{p_i}$ for $i = 1, 2$. Combining (38) and Lemma 5, we complete the proof. □

The proof of Theorem 3. The “if” part is obviously. Now we prove the “only if” part.
For any $D \in \mathcal{T}_n(\mathbb{R})$, it is obvious that $D$ is a Drazin (respectively, group, $\{1,2\}$-, $\{1\}$- and $\{2\}$-) inverse of $D$. This, together with the definition of $L$, gives that $L(D)$ is a Drazin (respectively, group, $\{1,2\}$-, $\{1\}$- and $\{2\}$-) inverse of $L(D)$. Furthermore, $L(D) \in \mathcal{T}_m(\mathbb{R})$. Therefore, $L$ is a linear map preserving tripotence.

When $\mathbb{R}$ is an idempotence-diagonalizable ring with $2, 3 \in \mathbb{R}^\ast$, by Theorem 1, the proof is completed.

When $\mathbb{R}$ is a commutative principal ideal domain with $2 \in \mathbb{R}^\ast$ and $|\mathbb{R}^\ast|/2 = 2$, following the argument in the proof of Theorem 1, we can assume that (27)–(29) hold.

For any $B \in \mathcal{I}_n(\mathbb{R})$ and $a \in \mathbb{R}^\ast$, it is obvious that $\text{In} - (a + 1)B$ is a Drazin (respectively, group, $\{1,2\}$-, $\{1\}$- and $\{2\}$-) inverse of $\text{In} - (a - 1 + 1)B$, and $\text{In} - (a - 1 + 1)B$ is a Drazin (respectively, group, $\{1,2\}$-, $\{1\}$- and $\{2\}$-) inverse of $\text{In} - (a + 1)B$. This, together with the definition of $L$, gives that $L(\text{In}) - (a + 1)L(B)$ is a Drazin (respectively, group, $\{1,2\}$-, $\{1\}$- and $\{2\}$-) inverse of $L(\text{In}) - (a - 1 + 1)L(B)$, and $L(\text{In}) - (a - 1 + 1)L(B)$ is a Drazin (respectively, group, $\{1,2\}$-, $\{1\}$- and $\{2\}$-) inverse of $L(\text{In}) - (a + 1)L(B)$. Furthermore,

$$[L(\text{In}) - (a + 1)L(B)][L(\text{In}) - (a - 1 + 1)L(B)]$$
$$\times [L(\text{In}) - (a + 1)L(B)] = L(\text{In}) - (a + 1)L(B).$$

(39)

By a direct computation, we have from (27), (28) and (39) that

$$a^3Y_B + a^2[2X_B - Z_B + 2Y_B] + a[2X_B - Z_B + Y_B - U_B] + X_B - U_B = O,$$

(40)

with

$$X_B = L(B)^2L(I_n) + L(I_n)L(B)^2,$$
$$Y_B = L(B)L(I_n)L(B) - L(B),$$
$$Z_B = L(I_n)^2L(B) + L(B)L(I_n)^2,$$
$$U_B = L(I_n)L(B)L(I_n) + L(B).$$

(41)

Using $|\mathbb{R}^\ast| \neq 2$, Lemma 1 and (40), one can obtain that $Y_B = O$ and $X_B = U_B = Z_B$. This, together with $2 \in \mathbb{R}^\ast$, (29) and (41), implies

$$\{L(\text{In})L(B)L(I_n) = L(B)$$
$$L(B)^2L(I_n) + L(I_n)L(B)^2 = 2L(B)\}, \forall B \in \mathcal{I}_n(\mathbb{R}).$$

(42)

Case 1. Suppose $L(\text{In}) = O$. Then, by (42), we have

$$L(B) = O, \forall B \in \mathcal{I}_n(\mathbb{R}).$$

(43)

Since any matrix in $\mathcal{M}_n(\mathbb{R})$ can be represented as a linear combination of finitely many matrices in $\mathcal{I}_n(\mathbb{R})$, we deduce from (43) and the linearity of $L$ that $L$ is of the form (i) of Theorem 1.

Case 2. Suppose $L(\text{In}) \neq O$. By Lemma 3 and (27), there exists $T \in GL_m(\mathbb{R})$ such that $L(\text{In}) = T(I_{p_1} \oplus I_{p_2} \oplus O)T^{-1}$ with $p_1 + p_2 = \text{rank } L(\text{In}) > 0$. Substituting $L(\text{In})$ into (42) and using $2 \in \mathbb{R}^\ast$, we have...
\[ T^{-1}L(B)T = f_1(B) \oplus f_2(B) \oplus O, \quad \forall B \in \mathcal{S}_n(R), \quad (44) \]

where \( f_i(B) \in \mathcal{S}_p(R) \) satisfies \( f_i(I_n) = I_p \) for \( i = 1, 2 \). Since any matrix in \( M_n(F) \) can be written as a linear combination of finitely many matrices in \( \mathcal{S}_p(R) \), we deduce from (44) and the linearity of \( L \) that

\[ T^{-1}L(A)T = f_1(A) \oplus f_2(A) \oplus O, \quad \forall A \in M_n(R), \quad (45) \]

where \( f_i : M_n(R) \to M_p(R), \; i = 1, 2, \) are a pair of linear maps which preserve idempotence and satisfy \( f_i(I_n) = I_p \) for \( i = 1, 2 \). Combining (45) and Lemma 5, we obtain that \( L \) is of the form (ii) of Theorem 1. \( \square \)

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