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Approximation of General Smooth Convex Bodies

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1. INTRODUCTION

Let K be a convex body. In the theory of polytopal approximation, one considers the polytope P_n with n vertices, or the polytope $P_{(n)}$ with n facets

The topology on the space of convex bodies is always the same but the actual metric used depends on the nature of the problem. In this paper, we consider the most commonly used notions of distance of two convex bodies C and M. The support function of C is defined as

$$h_C(u) = \max_{x \in C} \langle u, x \rangle.$$

Symmetric Difference Metric. $\delta_S(C, M)$ is the volume of the symmetric difference $C \Delta M$ of C and M.

L_1 Metric. $\delta_1(C, M) = \int_{S^{d-1}} |h_C(u) - h_M(u)| \, du.$

Hausdorff Metric. $\delta_H(C, M)$ is the maximum of all the distances of points in C from M, and all the distances of points in M from C.

Banach–Mazur Metric. If C and M are o-symmetric then $\delta_{BM}(C, M)$ is the minimum of $\ln \lambda$ such that for certain linear transformation T, we have

$$T(C) \subset M \subset \lambda T(C).$$

Schneider's Distance. If $M \subset C$ then $\delta_{SCH}(C, M)$ is the maximum volume of caps of C cut off by supporting planes of M.

If $M \subset C$ then $\delta_1(C, M)$ is proportional with the deviation of the mean width. In addition, if $M \subset C$, ∂C is smooth, M is a polytope and ∂M is close to ∂C then $\delta_{SCH}(C, M)$ is the maximum volume of caps of C cut off by the affine hull of a facet of M.

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Assume that K has C^2 boundary, and denote by Q_x the second fundamental form at an $x \in \partial K$ (see [30]). Then the sectional curvature at the direction of the unit vector u tangent at x is $Q_x(u)$, and the Gauß curvature $\kappa(x)$ is the determinant of Q_x. Note that Q_x is positive semi-definite (and hence $\kappa(x)$ is non-negative) because of the convexity of K.

An accelerating progress has been taking place in understanding how well a polytope can approximate K if ∂K is C^2_+; namely, if the Gauß curvature is positive everywhere. After sporadic results in the plane, the book [9] of L. Fejes Tóth provided for the first time a large number of results in polytopal approximation, concentrating on the extremal properties of regular polytopes in dimensions two and three. In higher dimensional spaces, the first result about the asymptotic order of approximation is due to R. Schneider (see [27]), considering the Hausdorff metric and assuming that the boundary is C^3_+. Later P. M. Gruber managed to extend the method of L. Fejes Tóth (the so called "Momentum Lemma") to higher dimensional spaces, at least in an asymptotic sense (see [12, 13]). This way he could handle various other metrics and could relax the differentiability assumption to C^2_+. An idea of S. Glasauer (see [10]) bringing the polar body in the picture made it possible to obtain results about the L_1 metric. Finally, M. Ludwig [22] tackled the case of general approximation for the symmetric difference and the L_1 metric. For comprehensive surveys about the subject, consult the articles [14, 17] of P. M. Gruber.

Next we list the asymptotic formulae, since these are the formulae which we prove in a more general setting. We always denote by P_n ($P_{(n)}$) the best approximating polytope with at most n vertices (with at most n facets). It is also natural to consider the problem where the polytope is assumed to be inscribed in K, or circumscribed around K.

First, assume that K is o-symmetric, and denote by $\nu(x)$ the unit exterior normal at $x \in \partial K$. The center affine surface area

$$A_0(K) = \int_{\partial K} \frac{\kappa(x)^{1/2}}{h_K(\nu(x))^{(d-1)/2}}\, dx$$

of K is known to be affinely invariant (see the classical book [2] of W. Blascke, or the monograph [21] of K. Leichtweiß). Denote the density of the thinnest covering of $\mathbb{R}^{d-1}$ with congruent balls by ϑ_{d-1}, and the volume of the unit $(d-1)$-ball by κ_{d-1}. If P_{2n} ($P_{(2n)}$) is the best approximating o-symmetric polytope with $2n$ vertices ($2n$ facets) then (see [15])

$$\delta_{BM}(K, P_{2n}),\, \delta_{BM}(K, P_{(2n)}) \sim \frac{1}{2}\left(\frac{\vartheta_{d-1}}{\kappa_{d-1}}\right)^{2/(d-1)} \cdot A_0(K)^{2/(d-1)} \cdot \frac{1}{(2n)^{2/(d-1)}}. \tag{1}$$

Since the symmetric difference metric and Schneider's notion of distance are invariant under volume preserving affine transformation, it is no surprise that the corresponding asymptotic formulae contains the affine surface area

$$A(C) = \int_{\partial K} \kappa(x)^{1/(d+1)}\, dx.$$

Those interested in the main properties of the affine surface area (which we actually do not need) should consult the books [2] of W. Blaschke or [21] of K. Leichtweiß, or the paper [24] of E. Lutwak.

If $P_{(n)} \subset K$ ($P_n \subset K$) is the best approximating polytope with at most n facets (with at most n vertices) with respect to Schneider's notion of distance then

$$\delta_{SCH}(K, P_n), \delta_{SCH}(K, P_{(n)}) \sim \frac{(\vartheta_{d-1})^{(d+1)/(d-1)}}{(d+1)(\kappa_{d-1})^{2/(d-1)}} \cdot A(K)^{(d+1)/(d-1)} \cdot \frac{1}{n^{(d+1)/(d-1)}}. \tag{2}$$

This result was established by R. Schneider in [28] for $P_{(n)}$ if ∂K is C^3_+, and the general result is verified in [3].

If P_n or $P_{(n)}$ is the best approximating with respect to the symmetric difference metric then M. Ludwig [22] proved that

$$\delta_S(K, P_n) \sim \frac{ldel_{d-1}}{2} \cdot A(K)^{(d+1)/(d-1)} \cdot \frac{1}{n^{2/(d-1)}} \tag{3}$$

$$\delta_S(K, P_{(n)}) \sim \frac{ldiv_{d-1}}{2} \cdot A(K)^{(d+1)/(d-1)} \cdot \frac{1}{n^{2/(d-1)}}, \tag{4}$$

where $ldel_{d-1}$ and $ldiv_{d-1}$ are certain constants depending on the dimension. Under the additional assumption the polytope is inscribed or circumscribed, the corresponding asymptotic formulae were determined in [12, 13] by P. M. Gruber, and in [4].

Note that M. Ludwig also considered a generalization of the symmetric difference metric: Let w be a positive and continuous function in $\mathbf{R}^d$, and define

$$\delta_w(M, C) = \int_{M \Delta C} w(x)\, dx$$

for convex bodies M and C. Then (see [22])

$$\delta_w(K, P_n) \sim \frac{ldel_{d-1}}{2} \cdot \left(\int_{\partial K} w(x)^{(d+1)/(d-1)} \kappa(x)^{1/(d+1)} \, dx \right)^{(d+1)/(d-1)} \cdot \frac{1}{n^{2/(d-1)}} \tag{5}$$

$$\delta_w(K, P_{(n)}) \sim \frac{ldiv_{d-1}}{2} \cdot \left(\int_{\partial K} w(x)^{(d+1)/(d-1)} \kappa(x)^{1/(d+1)} \, dx \right)^{(d+1)/(d-1)} \cdot \frac{1}{n^{2/(d-1)}}. \tag{6}$$

Now we turn to the L_1 metric. M. Ludwig [22] proved (using polarity and formulae (5) and (6)) that

$$\delta_1(K, P_n) \sim \frac{ldiv_{d-1}}{2} \cdot \left(\int_{\partial K} \kappa(x)^{d/(d+1)} \, dx \right)^{(d+1)/(d-1)} \cdot \frac{1}{n^{2/(d-1)}} \tag{7}$$

$$\delta_1(K, P_{(n)}) \sim \frac{ldel_{d-1}}{2} \cdot \left(\int_{\partial K} \kappa(x)^{d/(d+1)} \, dx \right)^{(d+1)/(d-1)} \cdot \frac{1}{n^{2/(d-1)}}. \tag{8}$$

The cases of circumscribed or inscribed polytopes are handled in the paper [10] of S. Glasauer and P. M. Gruber and in [4].

Finally, consider the Hausdorff distance. R. Schneider [27, 29] proved in the C^3_+ case, and P. M. Gruber [15] in the C^2_+ case that

$$\delta_H(K, P_n), \delta_H(K, P_{(n)}) \sim \frac{1}{4} \left(\frac{\vartheta_{d-1}}{\kappa_{d-1}} \right)^{2/(d-1)} \cdot \left(\int_{\partial K} \kappa(x)^{1/2} \, dx \right)^{2/(d-1)} \cdot \frac{1}{n^{2/(d-1)}}. \tag{9}$$

Under the additional assumption that the polytope is inscribed or circumscribed, the optimal approximation is twice the value in (9).

The main goal of this paper is to verify the conjecture of P. M. Gruber (see [17]), namely, to remove the positivity condition on the curvature.

THEOREM A. *The formulae* (1), ..., (9) *hold for any convex body K with C^2 boundary even if the Gauß curvature is allowed to be zero.*

Note that the definition of the affine surface area can be extended to any convex body (see the book [21] of K. Leichtweiß, or the paper [24] of E. Lutwak). Therefore it is natural to ask whether (3) and (4) hold for any convex body whose affine surface area is positive. M. Ludwig has already made the first step in this direction; namely, in [23], she handles the case when $d = 2$ and the polygon is inscribed.

On the other hand, Theorem A is in some sense optimal; namely, one can not relax further the differentiability requirement on ∂K. This was verified in [17] by showing that no asymptotic formula exists for the best approximation if ∂K is only assumed to be C^1.

Next we consider how well the facets or vertices of the best approximating polytope are distributed. Let $\mathcal{F}_n$ be a family of objects on ∂K for any n. We say that $\mathcal{F}_n$ uniformly distributed on ∂K with respect to the density function ϱ if for any Jordan measurable subset X of ∂K,

$$\lim_{n\to\infty} \frac{\#\{F\in\mathcal{F}_n \mid F\cap X\neq\emptyset\}}{\#\mathcal{F}_n} = \frac{\int_X \varrho(x)\,dx}{\int_{\partial K}\varrho(x)\,dx}.$$

Note that by Theorem A, P_n ($P_{(n)}$) has asymptotically n vertices (n facets). In Theorem B below, any reasonable meaning can be attached to the phrase "projection into ∂K," like the closest point map, radial projection from a fixed interior point of K, etc. The corresponding result was proved in [11] by S. Glasauer and R. Schneider for most metrics if the curvature is everywhere positive.

THEOREM B. *If ∂K is C^2 then the projections onto ∂K of the vertices of P_n (the facets of $P_{(n)}$) are uniformly distributed with respect to the density function $\varrho(x)$ which appears in the asymptotic formulae* (1), ..., (9).

In the three dimensional case, even the asymptotic shape of most faces of the best approximating polytope can be determined (see [5], or for the C^2_+ case with more exact estimates, see [8, 18, and 19]).

Finally, assume that K is a convex body with C^2 boundary, and X is a Jordan measurable open subset of ∂K such that $\kappa(x)>0$ at some $x\in X$. Then polytopal hypersurfaces approximating X can be defined as the corresponding parts of the boundary of some polytopes (see Section 2), and the analogues of Theorem A and Theorem B holds for X (see Corollaries 1 and 2).

2. APPROXIMATING THE BOUNDARY

We use the terms open and Jordan measurable relative to the intrinsic structure of convex surfaces in $\mathbb{R}^d$ as well, the meaning will be always clear from the context. Denote the $(d-1)$-dimensional Hausdorff measure by $|\cdot|$ (see [7, 26] for the main properties).

In this section, we fix a convex body K with C^2 boundary in $\mathbb{R}^d$. Our arguments are based on the approximation of certain parts of ∂K, so let X be a Jordan measurable open subset of ∂K.

A Jordan measurable open subset of the boundary of a convex polytope Q is called a *polytopal hypersurface*. Next we define the polytopal hypersurface Y in ∂Q associated to X. In case of the symmetric difference metric or

δ_w, we assume that the origin is contained in the interior of both of K and of Q, and Y is the radial projection of X onto ∂K. For the other metrics, define Y as the set of points y of ∂Q such that there exists an exterior normal to Q at y which is the exterior normal to K at some $x \in X$. Note that in case of the Banach–Mazur distance, we assume that K and Q are o-symmetric.

The vertices of Y are the vertices of Q lying in Y, and the facets of Y are the sets of the form $F \cap Y$ where F is a facet of Q and $|F \cap Y| > 0$.

Then a polytopal hypersurface Y is inscribed (or circumscribed) with respect to X if Q can be chosen inscribed in K (or circumscribed around K).

The analogues of the notions of distances above are defined as follows: $\delta_S(X, Y)$ is the volume of the part between X and Y, and $\delta_w(X, Y)$ is the integral of w on the part between X and Y. For $x \in X$, denote by $v(x)$ the exterior unit normal to K at x, and let $y(x)$ be a point of Y with the same exterior normal. Then set

$$\delta_1(X, Y) = \int_X |\langle y(x) - x, v(x) \rangle| \, \kappa(x) \, dx,$$

$$\delta_H(X, Y) = \sup_{x \in X} |\langle y(x) - x, v(x) \rangle|.$$

If Y is inscribed then define $\delta_{SCH}(X, Y)$ as the maximal volume of any caps cut out from K by the affine hull of a facet of Y.

For any metric δ above, if $X = \partial K$ and $Y_n = \partial P_n$ ($Y_{(n)} = \partial P_{(n)}$) for the best approximating polytope P_n ($P_{(n)}$) with at most n vertices (facets) then

$$\delta(X, Y_n) = \delta(K, P_n) \qquad \text{and} \qquad \delta(X, Y_{(n)}) = \delta(K, P_{(n)}).$$

Finally, if Y is inscribed then the definition of δ_{BM} can be extended only in a restricted sense; namely,

$$\delta_{BM}(X, Y) = \sup_{x \in X} \ln \left(1 + \frac{|\langle y(x) - x, v(x) \rangle|}{|\langle x, v(x) \rangle|} \right).$$

Since affine maps keep the number of faces, this definition suffices for our purposes. More precisely, assume that K is o-symmetric and P_{2n} ($P_{(2n)}$) is the best approximating o-symmetric polytope with respect to the Banach–Mazur distance, P_{2n} ($P_{(2n)}$) is a contained in K, and the linear transformation can be chosen to be the identity. If $X = \partial K$ and $Y_{2n} = \partial P_{2n}$ ($Y_{(2n)} = \partial P_{(2n)}$) then

$$\delta_{BM}(X, Y_{2n}) = \delta_{BM}(K, P_{2n}) \qquad \text{and} \qquad \delta_{BM}(X, Y_{(2n)}) = \delta_{BM}(K, P_{(2n)}).$$

2.1. *Approximating the Flat Part*

If $\mu > 0$ then denote by $\Sigma(\mu)$ the set of points on ∂K where the minimal sectional curvature is less than μ. In the arguments below, we frequently need a small open neighborhood of some closed set σ to be Jordan measurable. The existence of such an open neighborhood always follows from the fact that any uncountable family of level sets of a continuous function vanishing exactly on σ contains a member with zero measure. For example, $\Sigma(\mu)$ is Jordan measurable for all but countably many μ.

We use the Landau symbol $O(\cdot)$ meaning that the implied constant depends only on K.

LEMMA 1. *Let δ be one of the metrics above, and let $\varepsilon > 0$. Then for any small μ where $\Sigma(\mu)$ is Jordan measurable, $\Sigma(\mu)$ satisfies the following property: For large m, there exists a polytopal hypersurface Y_m ($Y_{(m)}$) with at most m vertices (m facets) approximating $\Sigma(\mu)$ such that*

$$\delta(\Sigma(\mu), Y_m), \delta(\Sigma(\mu), Y_{(m)}) = \frac{O(\varepsilon)}{m^{\gamma}},$$

where $\gamma = \frac{d+1}{d-1}$ for δ_{SCH} and $\gamma = \frac{2}{d-1}$ otherwise. The analogous results hold for inscribed and circumscribed hypersurfaces.

Proof. Completely different arguments are needed for Scheider's notion of distance and for the other metrics. The reason is that a hyperplane close to a flat point may cut out a cap with relatively large volume, while being close in the sense of Hausdorff metric automatically yields the closeness with respect to δ_{BM}, δ_w, and δ_1.

Case I. *Schneider's Notion of Distance.* Here we use the method of cap covering developed originally by A. M. Macbeath (see [25]). If $y \in K$ and $\lambda > 0$ then a Macbeath region is

$$M(y, \lambda) = y + \lambda((K - y) \cap (y - K)).$$

These regions have the surprising property (see [6])

$$M(y_1, \tfrac{1}{2}) \cap M(y_2, \tfrac{1}{2}) \neq \emptyset \text{ yields that } M(y_1, 1) \subset M(y_2, 5). \tag{10}$$

On the other hand, consider a hyperplane H intersecting K which is parallel to the tangent hyperplane at an $x \in \partial K$ and have distance h from x. Then the cap cut out by H from K is denoted by $C(x, h)$.

Consider $\Sigma(\mu)$ where we fix μ later. For large m and $x \in \Sigma(\mu)$, define $h(x)$ so that

$$V(C(x, h(x))) = \frac{\varepsilon}{m^{(d+1)/(d-1)}}.$$

Since the sectional curvatures are bounded on ∂K, we deduce by $x \in \Sigma(\mu)$ that

$$h(x) = \frac{O(\mu^{1/(d+1)} \varepsilon^{2/(d+1)})}{m^{2/(d-1)}},$$

which in turn yields the estimate

$$V\left(\bigcup_{x \in \Sigma(\mu)} C(x, h(x)) \right) = \frac{O(\mu^{1/(d+1)} \varepsilon^{2/(d+1)})}{m^{2/(d-1)}}. \tag{11}$$

Now consider the hyperplane bounding $C(x, \frac{1}{10} h(x))$, and denote by $y(x)$ the center of mass of the section of K by this hyperplane. The definition of $h(x)$ yields the existence of a $c_1 > 0$ depending only on d such that

$$V(M(y(x), \tfrac{1}{2})) > c_1 \cdot \frac{\varepsilon}{m^{(d+1)/(d-1)}}. \tag{12}$$

Let $\{x_1, ..., x_k\}$ be a maximal family on $\Sigma(\mu)$ such that the interiors of the sets $M(y(x_i), \frac{1}{2})$ are disjoint. We deduce by (11) and by (12) that

$$k < c_2 \cdot \frac{\mu^{1/(d+1)}}{\varepsilon^{(d-1)/(d+1)}} \cdot m$$

for a c_2 depending only on K. Therefore $k < m$ if we choose μ initially small enough.

Now to construct $Y_{(m)}$, take the polytopal hypersurface determined by the hyperplanes bounding the caps $C(x_i, h(x_i))$. What is left to prove that $Y_{(m)}$ is inscribed; namely, that any $x \in \Sigma(\mu)$ is contained in some $C(x_i, h(x_i))$. The x_i we need is the one satisfying

$$M(y(x), \tfrac{1}{2}) \cap M(y(x_i), \tfrac{1}{2}) \neq \emptyset.$$

We deduce by (10) that $M(y(x), 1) \subset M(y(x_i), 5)$, which in turn yields by $x \in M(y(x), 1)$ that $x \in C(x_i, h(x_i))$.

On the other hand, Y_m is defined using the convex hull of the x_i's. Here we should verify that for any $x \in \Sigma(\mu)$, the cap $C(x, h(x))$ contains some x_i, which can be done as above.

Case II. *The Other Metrics.* First for small μ we construct a covering of $\Sigma(\mu)$ by a finite family $\{\Sigma_\alpha\}$ of open, Jordan measurable sets and corresponding hyperplanes H_α with the following properties:

(i) Denote by Φ_α the projection of Σ_α onto H_α. Then a neighborhood of $\operatorname{cl} \Sigma_\alpha$ on ∂K is the graph of a C^2 function f_α defined in a neighborhood of $\operatorname{cl} \Phi_\alpha$ in H_α.

(ii) For large m, there exists a covering $\mathscr{F}_\alpha$ of $\operatorname{cl} \Phi_\alpha$ by non-overlapping translates of a rectangular box $\Pi_\alpha \subset H_\alpha$ whose diameter tends to zero as m tends to infinity, and there exists a function φ_α linear on each $\Pi \in \mathscr{F}_\alpha$ such that if $z \in \Pi$ then

$$\varphi_\alpha(z) \leqslant f_\alpha(z) < \varphi_\alpha(z) + \frac{\varepsilon}{m^{2/(d-1)}}.$$

(iii) $\sum |\Phi_\alpha| < |\partial K|$.

(iv) $\sum \# \mathscr{F}_\alpha < m$.

Denote by Ω the maximum of $2Q_x(u)$ for $x \in \partial K$ and $\|u\| = 1$. Define $\beta > 0$ such that

$$(d-2) \cdot \Omega \cdot \beta^{2/(d-2)} \cdot |\partial K|^{2/(d-1)} < \tfrac{1}{2} \cdot \varepsilon,$$

and μ is defined so that

$$\frac{2\mu}{\beta^2} \cdot |\partial K|^{2/(d-1)} < \frac{1}{2} \cdot \varepsilon,$$

and $\Sigma(\mu)$ is Jordan measurable.

For any $x \in \operatorname{cl} \Sigma(\mu)$, denote by H_x the tangent hyperplane, and by f_x the C^2 function on H_x parameterizing ∂K around x. For $z \in H_x$, let q_z be the quadratic form representing the second derivative of f_x at z. Since $Q_x = q_x$, there exist a Jordan measurable open neighborhood U_x of x on ∂K and a suitable system of coordinates such that if z is in the projection of U_x into H_x and $\|u\| = 1$ then

$$q_z((u_1, ..., u_{d-1})) < 2\mu u_1^2 + \Omega u_2^2 + \cdots + \Omega u_{d-1}^2. \tag{13}$$

Let $U_{x_1}, ..., U_{x_k}$ be a finite covering of $\Sigma(\mu)$, and define $H_\alpha = H_{x_\alpha}$ and

$$V_\alpha = (U_{x_\alpha} \cap \Sigma(\mu)) \Big\backslash \bigcup_{i=1}^{\alpha-1} \operatorname{cl} U_{x_i}.$$

We may assume that each V_α is non-empty. Now each V_α is Jordan measurable, the sets V_α are pairwise disjoint, and the union of their closures cover $\Sigma(\mu)$. Thus define a Jordan measurable open neighborhood Σ_α of cl V_α such that in an open neighborhood in H_α of the closure of the projection Φ_α of Σ_α into H_α (13) still holds and

$$\sum |\Phi_\alpha| < |\partial K|.$$

In H_α, let Λ_α be the lattice determined by the box Π_α defined as

$$\left[0, \frac{1}{\beta}\left(\frac{|\partial K|}{m}\right)^{1/(d-1)}\right) \times \left[0, \beta^{1/(d-2)}\left(\frac{|\partial K|}{m}\right)^{1/(d-1)}\right) \times \cdots \times \left[0, \beta^{1/(d-2)}\left(\frac{|\partial K|}{m}\right)^{1/(d-1)}\right).$$

For any $z \in H_\alpha$, denote by l_z the derivative of f_α at z. Denote by $\mathscr{F}_\alpha$ the family of $w + \Pi_\alpha$ where w is a point of Λ_α intersecting Φ_α, and for $w + \Pi_\alpha \in \mathscr{F}_\alpha$ and $z \in w + \Pi_\alpha$, set

$$\varphi_\alpha(z) = f_\alpha(w) + l_w(z - w).$$

Then $f_\alpha(z) \geqslant \varphi_\alpha(z)$, and the Taylor formula and the definition of β and μ yield for any $z \in w + \Pi$ a $v \in w + \Pi$ such that $f_\alpha(z) - \varphi_\alpha(z) = \frac{1}{2} q_v(z - w)$, and hence

$$f_\alpha(z) - \varphi_\alpha(z) < \frac{2\mu}{\beta^2}\left(\frac{|\partial K|}{m}\right)^{2/(d-1)} + (d-2)\, \Omega \cdot \beta^{2/(d-2)}\left(\frac{|\partial K|}{m}\right)^{2/(d-1)} < \frac{\varepsilon}{m^{2/(d-1)}}.$$

Finally, the number of tiles intersecting $|\Phi_\alpha|$ is asymptotically $|\Phi_\alpha|/|\partial K|/m$, and hence for large m, we have that $\sum \# \mathscr{F}_\alpha < m$. At this point, all the properties (i), ..., (iv) have been established.

Now we prove the lemma for the Hausdorff metric. Call the closure of the image of a $\Pi \in \mathscr{F}_\alpha$ by φ_α a *patch*. Then $Y_{(m)}$ is simply determined by the affine hulls of the patches.

Turning to Y_m, repeat the construction above with $(1/2^d)\, m$ instead of m, and with $(1/2^{2d/(d-1)})\, \varepsilon$ instead of ε, and take the convex hull of the patches.

Observe that the Y_m and the $Y_{(m)}$ we have constructed are circumscribed. In order to have inscribed polytopal hypersurfaces, the only additional step to do is to get the patches inside K; namely, instead of being the image of φ_α, they are the image of $\varphi_\alpha + \varepsilon/m^{2/(d-1)}$ (see (ii)).

Now the polytopal hypersurfaces constructed for δ_H are also suitable for δ_{BM}, δ_w and δ_1. Q.E.D.

Note that (11) is related to the fact that the affine surface area can be defined with the help of the so called floating body (see [31]). Similar ideas about the way how to use cap covering for approximation problems are contained in [1].

2.2. *Theorem A and the Corollary*

Here we combine Corollary 1 in [3] and Corollary 2 in [4] as

LEMMA 2. *Let $\delta(\cdot, \cdot)$ be any of the metrics above. Assume that K is a convex body with C^2 boundary, and X is a non-empty, Jordan measurable open subset of ∂K such that $\kappa(x) > 0$ for $x \in \mathrm{cl}\, X$. For large m, denote by Y_m the polytopal hypersurface with at most m vertices which has minimal distance from X with respect to $\delta(\cdot, \cdot)$. Then*

$$\delta(X, Y_m) \sim c \cdot \left(\int_X \varrho(x)\, dx \right)^{\gamma_0} \cdot \frac{1}{m^\gamma},$$

where c, γ_0, and γ are the suitable constants. Similar statement holds if Y_m is assumed to be inscribed or circumscribed, or if the number of facets is bounded.

Proof of Theorem A. We use the notation of Lemma 2. First we present the proof if the number of facets is bounded.

Let $\varepsilon > 0$, and denote by $P_{(n)}$ the best approximating polytope with at most n facets. First we prove that for large n,

$$\delta(K, P_{(n)}) > (1 - O(\varepsilon)) \cdot c \cdot \left(\int_{\partial K} \varrho(x)\, dx \right)^{\gamma_0} \cdot \frac{1}{n^\gamma}. \tag{14}$$

Since $\kappa(x) = 0$ is equivalent with $\varrho(x) = 0$, there exists a Jordan measurable, open subset X of ∂K such that $\kappa(x)$ is positive on $\mathrm{cl}\, X$ and

$$\int_X \varrho(x)\, dx > (1 - \varepsilon) \cdot \int_{\partial K} \varrho(x)\, dx.$$

Denote by Y the part of $\partial P_{(n)}$ approximating X. Since Y has at most n facets, (14) follows by Lemma 2.

Next for large n, we construct a polytope $Q_{(n)}$ with at most n facets such that

$$\delta(K, Q_{(n)}) < (1 + O(\varepsilon)) \cdot c \cdot \left(\int_{\partial K} \varrho(x)\, dx \right)^{\gamma_0} \cdot \frac{1}{n^{\gamma}}. \tag{15}$$

Choose $0 < \tau,\ \varepsilon_0 < 1$ such that

$$\frac{1}{(1-\tau)^{\gamma}} < 1 + \varepsilon$$

and

$$\frac{\varepsilon_0}{\tau^{\gamma}} < \varepsilon \cdot c \cdot \left(\int_{\partial K} \varrho(x)\, dx \right)^{\gamma_0}.$$

Assume that $\Sigma = \Sigma(\mu)$ is the Jordan measurable open set on ∂K provided by Lemma 1 with this ε_0 and $m = \tau n$. Finally, choose a Jordan measurable open subset X of ∂K which covers $\partial K \backslash \Sigma$ and $\kappa(x) > 0$ for $x \in \mathrm{cl}\, X$.

For large n, consider the polytopal hypersurface $Y_{(\tau n)}$ with at most τn facets whose distance from Σ is minimal, and the polytopal hypersurface $Y_{((1-\tau)\, n)}$ with at most $(1-\tau)\, n$ facets whose distance from X is minimal. Now the $Q_{(n)}$ which is the intersection of the half spaces corresponding to $Y_{(\tau n)}$ and $Y_{((1-\tau)\, n)}$ satisfies (15) by Lemmas 1 and 2. Here no problem occurs for the distances related to the Hausdorff metric; namely, for δ_H, δ_{BM} and δ_{SCH}. In case of δ_w or δ_1, the distance of $\Sigma \cap X$ from the corresponding part of $Q_{(n)}$ is at most the sum of the distances of the corresponding parts of $Y_{\tau n}$ and $Y_{(1-\tau)\, n}$, and hence (15) still holds. Finally, (14) and (15) yield Theorem A.

If the polytope should be inscribed or circumscribed then the same argument works.

Now let us consider the problem if the number of vertices is bounded. If δ is not δ_w then the argument is the same with the obvious change of notions, only one takes the convex hull of $Y_{\tau n}$ and $Y_{(1-\tau)\, n}$ at the very end. If $\delta = \delta_1$ then still the distance of $\Sigma \cap X$ from the corresponding part of Q_n is at most the sum of the distances of the corresponding parts of $Y_{\tau n}$ and $Y_{(1-\tau)\, n}$.

So assume that $\delta = \delta_S$, and we need extra care when piecing $Y_{\tau n}$ and $Y_{(1-\tau)\, n}$. If n is large then we may assume by Lemma 2 about the part Y_0 of $Y_{(1-\tau)\, n}$ which approximates $X \cap \Sigma$ that

$$\delta_S(X \cap \Sigma, Y_0) < \varepsilon \cdot \mathrm{ldel}_{d-1} \cdot \left(\int_{\partial K} \kappa(x)^{1/(d+1)}\, dx \right)^{(d+1)/(d-1)} \cdot \frac{1}{n^{2/(d-1)}}.$$

Now Proposition 5.1 in [4] states that if Y is the part of Q_n approximating $X \cap \Sigma$ then $\delta_S(X \cap \Sigma, Y) < c_1 \cdot (\delta_S(X \cap \Sigma, Y_0) + \delta_S(\Sigma, Y_{\tau n}))$ where c_1 depends only on d. Therefore

$$\delta_S(X \cap \Sigma, Y) = O(\varepsilon) \cdot \mathrm{ldel}_{d-1} \cdot \left(\int_{\partial K} \kappa(x)^{1/(d+1)} \, dx \right)^{(d+1)/(d-1)} \cdot \frac{1}{n^{2/(d-1)}},$$

and hence the analogue of (15) holds again.

Similar arguments work if δ_S is replaced by δ_w. With this, the proof of Theorem A is complete. Q.E.D.

We have not used the fact that ∂K is closed (except for δ_{BM} where for general locally convex hypersurfaces we have to forget about the affine transformation). Therefore applying the conditions on the approximating polytopal hypersurface described at the beginning of Section 2, we have

COROLLARY 1. *Let $\delta(\cdot, \cdot)$ be any of the metrics above. Assume that K is a convex body with C^2 boundary, and X is a non-empty, Jordan measurable open subset of ∂K such that $\kappa(x) > 0$ for some $x \in X$. For large n, denote by Y_n the polytopal hypersurface with at most n vertices which has minimal distance from ∂K with respect to $\delta(\cdot, \cdot)$. Then*

$$\delta(X, Y_n) \sim c \cdot \left(\int_X \varrho(x) \, dx \right)^{\gamma_0} \cdot \frac{1}{n^{\gamma}},$$

where c, γ_0, and γ are the suitable constants. Similar statement holds if Y_n is inscribed or circumscribed, or if the number of facets is bounded.

3. PROOF OF THEOREM B

Assume that $\delta(\cdot, \cdot)$ is any of the metrics above, and P_n is the best approximating polytope with n vertices. If $X \subset \partial K$ is Jordan measurable then denote by $\mathcal{G}_n(X)$ the family of vertices v of P_n which satisfy at least one of the following properties:

(i) the projection of v along the normal to ∂K through v lands in X;

(ii) there exists an $x \in \partial K$ such that $\nu(x)$ lies in the normal cone at v;

(iii) if F is a facet of P_n containing v then $\operatorname{aff} F \cap \partial K \cap X \neq \emptyset$.

Similarly, define $\mathcal{G}_{(n)}(X)$ as the suitable family of facets F of $P_{(n)}$; namely, F satisfies (iii), or some point of F satisfies (i) or (ii).

First, we would like to eliminate the "effect" of the boundary of a Jordan measurable $X \subset \partial K$. We use the notation of Lemma 2.

PROPOSITION 1. *Let $\sigma \subset \partial K$ be closed with $|\sigma| = 0$. Then for any $\tau > 0$ there exists an open, Jordan measurable $\Sigma \subset \partial K$ containing σ such that $\kappa(x) > 0$ for $x \in \partial K \backslash \Sigma$, and*

$$\# \mathcal{G}_n(\Sigma),\ \# \mathcal{G}_{(n)}(\Sigma) < \tau \cdot n$$

holds for large n.

Proof. We present the proof for P_n, the arguments are analogous for $P_{(n)}$.

Choose some open, Jordan measurable $X \subset \partial K$ such that $\mathrm{cl}\, X \cap \sigma = \emptyset$, $\kappa(x) > 0$ for $x \in \mathrm{cl}\, X$, and

$$\left(\int_X \varrho(x)\, dx \right)^{\gamma_0} > (1-\tau)^{\gamma/3} \cdot \left(\int_{\partial K} \varrho(x)\, dx \right)^{\gamma_0}. \tag{16}$$

We claim that Σ can be chosen any open, Jordan measurable subset with $\mathrm{cl}\, \Sigma \cap \mathrm{cl}\, X = \emptyset$ and satisfying the conditions above.

Let Y be the part of P_n approximating X. If for large n, Y has at most $(1-\tau)\, n$ vertices then we deduce by Theorem A and Lemma 2 that

$$\delta(K, P_n) \geqslant \delta(X, Y) > (1-\tau)^{\gamma/3} \cdot c \left(\int_X \varrho(x)\, dx \right)^{\gamma_0} \cdot \frac{1}{(1-\tau)^{\gamma}\, n^{\gamma}}.$$

This inequality contradicts Theorem A for large n by (16). Since the diameter of the facets of P_n close to $\partial K \backslash \Sigma$ tends to zero, we conclude that $\# \mathcal{G}_n(\Sigma) < \tau \cdot n$. Q.E.D.

Observe that the size of the facets of the best approximating polytope tends to zero near the boundary points with positive curvature. Therefore Proposition 1 yields that it makes no difference how the vertices or facets are projected into ∂K in the definition of uniform distribution.

In case of the symmetric difference metric, we need some functional inequalities. Observe that the convexity of the function $s^{-2/(d-1)}$ yields for any positive t_1, t_2, s_1, s_2 the inequality

$$t_1 \cdot s_1^{-2/(d-1)} + t_2 \cdot s_2^{-2/(d-1)} \geqslant (t_1 + t_2)^{(d+1)/(d-1)}\, (t_1 s_1 + t_2 s_2)^{-2/(d-1)}.$$

Since the function $s^{-2/(d-1)}$ is actually strictly convex, we deduce

PROPOSITION 2. *Let $\varepsilon > 0$ be small, and assume that t_1, t_2, s_1, s_2 are positive numbers. Then there exists a continuous, positive function $\omega_\varepsilon(t_1, t_2)$ such that if*

$$t_1 \cdot s_1^{-2/(d-1)} + t_2 \cdot s_2^{-2/(d-1)} \leqslant (1 + \omega_\varepsilon(t_1, t_2)) \cdot (t_1 + t_2)^{(d+1)/(d-1)} (t_1 s_1 + t_2 s_2)^{-2/(d-1)}$$

then $s_2 \leqslant (1+\varepsilon) \cdot s_1$ and $s_1 \leqslant (1+\varepsilon) \cdot s_2$.

Now we are finally prepared for the

Proof of Theorem B. Let X be a Jordan measurable subset of ∂K.

If δ is one of δ_H, δ_{SCH} or δ_{BM} then cover the boundary of X by the Σ given in Proposition 1 for some small τ. Then applying Lemma 2 simultaneously to $X \cap (\partial K \backslash \mathrm{cl}\, \Sigma)$ and to $(\partial K \backslash X) \cap (\partial K \backslash \mathrm{cl}\, \Sigma)$ yields Theorem B for this δ.

Next, we present the proof for δ_S and P_n.

Assume first that X_1 and X_2 are non-empty, open, Jordan measurable subsets of ∂K such that $\mathrm{cl}(X_1 \cup X_2) = \partial K$, and set $m_i(n) = \mathscr{G}_n(X_i)$, $i = 1, 2$. Let $\varepsilon > 0$. We claim that for large n,

$$\frac{1}{1+\varepsilon} \cdot \frac{m_1(n)}{\int_{X_1} \kappa(x)^{1/(d+1)}\, dx} \leqslant \frac{m_2(n)}{\int_{X_2} \kappa(x)^{1/(d+1)}\, dx} \leqslant (1+\varepsilon) \cdot \frac{m_1(n)}{\int_{X_1} \kappa(x)^{1/(d+1)}\, dx}. \tag{17}$$

In order to prove (17), set

$$t_i = \int_{X_i} \kappa(x)^{1/(d+1)}\, dx \qquad \text{and} \qquad s_i = \frac{m_i(n)}{\int_{X_i} \kappa(x)^{1/(d+1)}\, dx}.$$

If n is large then estimating (see Corollary 1) the volume of the part between X_i and ∂P_n, $i = 1, 2$, from below and $\delta_S(K, P_n)$ from above yield for large n that

$$t_1 \cdot s_1^{-2/(d-1)} + t_2 \cdot s_2^{-2/(d-1)} \leqslant \sqrt{1 + \omega_\varepsilon(t_1, t_2)} \cdot \left(\int_{\partial K} \kappa(x)^{1/(d+1)}\, dx \right)^{(d+1)/(d-1)} \cdot \frac{1}{n^{2/(d-1)}}.$$

Now $m_1(n) + m_2(n) < (1 + \omega_\varepsilon(t_1, t_2))^{(d-1)/4} \cdot n$ can be assumed by Proposition 1. Thus Proposition 2 yields (17), and in turn Theorem B has been verified for δ_S.

Similar arguments work if the number of facets is bounded, or δ_S is replaced by δ_w or δ_1. Therefore Theorem B has been finally verified. Q.E.D.

Again, we have not used the fact that ∂K is closed (except for the obvious restriction on δ_{BM}). So replacing Theorem A with Corollary 1 in the arguments, we have

COROLLARY 2. *Let $\delta(\cdot, \cdot)$ be any of the metrics above. Assume that K is a convex body with C^2 boundary, and X is a non-empty, Jordan measurable open subset of ∂K such that $\kappa(x) > 0$ for some $x \in X$. For large n, denote by Y_n ($Y_{(n)}$) the polytopal hypersurface with at most n vertices (facets) which has minimal distance from X with respect to $\delta(\cdot, \cdot)$. Then the projections of the vertices of Y_n (the facets of $Y_{(n)}$) are uniformly distributed on X.*

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