The growth of algebroidal function✩

Sun Daochun, Kong Yinying*

School of Mathematical Sciences, South China Normal University, Guangzhou 510631, People’s Republic of China

Received 8 March 2006
Available online 1 December 2006
Submitted by A.V. Isaev

Abstract

In this paper, we investigate the growth relationship of algebroidal function and its coefficients, and then obtain a basic inequation between the maximum modulus function and Nevanlinna characteristic function. Finally, by using the inequation, we testify the order of entire algebroidal function is equal to that of its derived function.

© 2006 Elsevier Inc. All rights reserved.

Keywords: Algebroidal function; Order; Maximum modulus function

There are lots of difficulties on the research of the algebroidal function and many important theorems on meromorphic function cannot been extended to algebroidal function for its multi-valuedness and the complexity of its branch points. For instance, there is a basic and important inequation of entire function [1,3]

\[ T(r, f) \leq \log^+ M(r, f) \leq \frac{R + r}{R - r} T(R, f) \]

which still has not been extended to algebroidal function. In this paper, we study the relation between the growth of algebroidal function and that of its coefficients based on the paper [2] which extends the above inequation to the entire algebroidal function. Finally, by using the inequation, we testify that the order of an algebroidal function is equal to that of its derived function.

✩ The project is supported by NSFC (Grant No. 10471048).
* Corresponding author.
E-mail address: kongcoco@hotmail.com (Y. Kong).

0022-247X/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2006.10.031
Suppose that $A_k(z), \ldots, A_0(z)$ are analytic functions without common zeros on the complex plane $\mathbb{C}$ and the indecomposable equation
\begin{equation}
A_k(z)W^k + A_{k-1}(z)W^{k-1} + \cdots + A_0(z) = 0
\end{equation}
defines a $k$-valued algebroidal function $W(z)$ on the complex plane (if $A_k(z) = 1$, then $W(z)$ is called a $k$-valued integral algebroidal function), where $A_0(z) \not\equiv 0$, otherwise (1) is a reducible algebroidal function; where $A_k(z) \not\equiv 0$, otherwise it is $k-1$ valued or less.

$W(z)$ has Nevanlinna characteristic function $T(r, W) = m(r, W) + N(r, W)$, and its order is defined by $\rho(W)$.

Fetch $a \in \mathbb{C}$ arbitrarily, $n(r, \frac{A_t}{A_0} = a)$ denotes the number of roots, counting multiplicities, of the equation $\frac{A_t(z)}{A_0(z)} - a$ in disk $\{|z| < r\}$, and
\[ p\left(n(r, \frac{A_t}{A_0} = a)\right) := \lim_{r \to \infty} \frac{\log n(r, \frac{A_t}{A_0} = a)}{\log r} \]
denotes the convergent exponent of $n(r, \frac{A_t}{A_0} = a)$.

For any $z \in \mathbb{C}$, we set $A(z) = \max\{|A_j(z)|; \ j = 0, 1, 2, \ldots, k\} > 0$ and let
\[ \mu(r, A) := \frac{1}{2k\pi} \int_0^{2\pi} \log A(re^{i\theta}) \ d\theta. \]

In this paper, we need the following theorem:

**Theorem A.** [2] Suppose $W(z)$ is an algebroidal function defined by (1), then
\[ |T(r, W) - \mu(r, A) + \frac{1}{k} \log |c_k|| \leq \log 2, \]
where $c_k$ is the first non-zero expansion coefficient of Laurent series of $A_k(z)$ which is expanded in the neighborhood of $z = 0$.

We define $\rho(A_j)$ the order of coefficient $A_j(z)$ and choose $M \in \{0, 1, 2, \ldots, k\}$, so that
\[ \rho(A_M) = \max\{\rho(A_t); \ t = 0, 1, 2, \ldots, k\}. \]

**Lemma 1.** Suppose that $W(z)$ is a $k$-valued algebroidal function of order $\rho(W)$ defined by (1), then $\mu(r, A) \leq \frac{1}{k} \sum_{j=0}^{k} m(r, A_j) + 1$ ($\Rightarrow \rho(W) \leq \rho(A_M)$).

**Proof.** From the definition of $\mu(r, A)$, it follows that
\[ k\mu(r, A) = \frac{1}{2k\pi} \int_0^{2\pi} \log A(re^{i\theta}) \ d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \log \sum_{j=0}^{k} |A_j(re^{i\theta})| \ d\theta \]
\[ \leq \frac{1}{2\pi} \sum_{j=0}^{k} \int_0^{2\pi} \log^+ |A_j(re^{i\theta})| \ d\theta + \log(k + 1) = \sum_{j=0}^{k} m(r, A_j) + \log(k + 1). \]

Hence, when $r$ is sufficiently large, we have
\[ k\mu(r, A) \leq \sum_{j=0}^{k} m(r, A_j) + \log(k + 1). \]
But the inequation on the contrary maybe not true. For instance: 2-valued algebroidal function \(e^zW^2 - ze^z = 0\) is the algebroidal function \(W(z) = z^{1/2}\), so the order \(\rho(W) = 0\), but \(\rho(A_0) = \rho(A_2) = \rho(A_M) = 1\). \(\square\)

**Lemma 2.** Suppose that \(W(z)\) is a \(k\)-valued algebroidal function of order \(\rho(W)\) defined by (1). If there are \(t, u \in \{0, 1, 2, \ldots, k\}\), so that

\[
\rho \left( \frac{A_t}{A_t} \right) \geq \rho(A_M),
\]

then the order of \(W\) is equal to that of the coefficient function which has the highest order of (1) i.e.

\[
\rho(W) = \rho(A_M).
\]

**Proof.** We choose \(t\) and \(u\) so that both satisfy (2) and let \(f_{tu}(z) = \max\{|A_t(z)|, |A_u(z)|\} = |\frac{A_t(z)}{A_u(z)}|^{+} \cdot |A_u(z)|\), where \(|x|^{+} = \max\{1, x\}\), then

\[
k\mu(r, A) \geq \mu(r, f_{tu}) = \frac{1}{2\pi} \int_{0}^{2\pi} \log f_{tu}(re^{i\theta}) \, d\theta
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \log^{+} \left| \frac{A_t(re^{i\theta})}{A_u(re^{i\theta})} \right| \, d\theta + \frac{1}{2\pi} \int_{0}^{2\pi} \log |A_u(re^{i\theta})| \, d\theta
\]

Jensen

\[
= m \left( r, \frac{A_t}{A_u} \right) + N(r, A_u = 0) + \log |c| \geq T \left( r, \frac{A_t}{A_u} \right) + \log |c|.
\]

By Theorem A and (2), it follows that

\[
\rho(W) \geq \rho \left( \frac{A_t}{A_u} \right) \geq \rho(A_M).
\]

We obtain Lemma 2 by using Lemma 1 again. \(\square\)

**Theorem 1.** Suppose that \(W(z)\) is a \(k\)-valued algebroidal function of order \(\rho(W)\) defined by (1), then the necessary and sufficient condition supporting (3) is that \(\exists \, t \in \{1, 2, \ldots, k\}\), so that

\[
\rho \left( \frac{A_t(z)}{A_0(z)} \right) \geq \rho(A_0).
\]

**Proof.** 1) Suppose the condition (4) is true i.e. \(\exists \, t \in \{1, 2, \ldots, k\}\), so that \(\rho \left( \frac{A_t(z)}{A_0(z)} \right) \geq \rho(A_0)\).

(i) If \(\rho(A_M) = \rho(A_0)\), then (2) is true and we obtain the results by Lemma 2.

(ii) If \(\rho(A_M) > \rho(A_0)\), then \(\rho(A_M) = \rho(A_M)\) and (2) is proved, we also obtain the results by Lemma 2.

2) If (4) is not correct, i.e., for \(\forall \, t \in \{1, 2, \ldots, k\}\), we have

\[
\rho \left( \frac{A_t}{A_0} \right) < \rho(A_0)
\]
thus $p(n(r, A_0)) < \rho(A_0)$. By using the typical product theorem [3], there is an entire function $h_t(z)$, so that the poles of $\frac{A_t}{A_0}$ are the zero points of $h_t(z)$, and

$$\rho(h_t) = p(n(r, h_t = 0)) < \rho(A_0).$$

Hence $f_t(z) := \frac{A_t(z)}{A_0(z)}h_t(z)$ must be entire function. From (5) and (6), it follows that

$$\rho(f_t) < \rho(A_0). \tag{7}$$

**Note.** All zero points of $h_t(z)$ come from the zero points of $A_0(z)$ (counting multiplicities), then

$$A_t(z) = \frac{A_0(z)}{h_t(z)}f_t(z) \tag{8}$$
is an entire function.

On the other hand, if $b$ is one zero point of $A_0(z)$, it is also that (counting multiplicities) of $H(z) := h_1(z)h_2(z) \cdots h_k(z)$. Otherwise if $b$ is not the zero point of $H(z)$ nor the multiplicity of zero of $A_0(b) = 0$ is more than those of $H(b) = 0$, then by (8), $b$ is also one zero point of every $A_t(z)$. But $\{A_j(z)\}$ have no common roots, this is a contradiction.

So the zero points of $A_0(z)$ must belong to those (counting multiplicities) of $H(z)$. Since $\rho(H(z)) < \max\{\rho(h_j)\} < \rho(A_0)$, then $p(n(r, A_0 = 0)) < \rho(A_0)$. Thus we can construct an entire function $g_0(z)$, so that $g_0(z)$ and $A_0(z)$ share the common zero points (counting multiplicities), and its order satisfies

$$\rho(g_0) = p(n(r, A_0 = 0)) < \rho(A_0). \tag{9}$$

Consequently $F_0(z) := \frac{A_0(z)}{g_0(z)} = e^{f(z)}$ is an entire function with no zero points. Put $F_0(z)$ into (5), we see that

$$A_t(z) = \frac{A_0(z)}{h_t(z)}f_t(z) = F_0(z)g_t(z), \tag{10}$$
where $g_t(z) = \frac{g_0(z)}{h_t(z)}f_t(z)$ are entire functions. Combined with (6), (7) and (9), we can see that

$$\rho(g_t) = \rho\left(\frac{g_0}{h_t}f_t\right) < \rho(A_0(z)).$$

Then the equality (1) can also be written simply as

$$A_k(z)W^k + A_{k-1}(z)W^{k-1} + \cdots + A_0(z)$$

$$= F_0g_k(z)W^k + F_0g_{k-1}(z)W^{k-1} + \cdots + F_0g_0(z) = 0$$

$$\implies g_k(z)W^k + g_{k-1}(z)W^{k-1} + \cdots + g_0(z) = 0;$$

and

$$\rho(W) \leq \rho(g_M) < \rho(A_0) \leq \rho(A_M).$$

This makes out the conclusion is not right.

**Note.** From Theorem 1, we can see that if (3) is true it must meet some conditions, because the entire functions $\{A_j(z)\}$ of (1) may have a “common factor” with the higher order and without zero points. Theorem 1 gives us a necessary and sufficient condition to testify formula (3).
Is it true that if entire function (coefficient) of \( k \)-valued algebroidal function defined by (1) has no common factors, formula (3) is still true? The problem is that we have not given the definition of divisible for the entire function, yet. How can we define “factor”? For example, \( z \) dividing \( e^z \) is equal to the entire function \( ze^{-z} \), but whether it belongs to exact division? Whether \( e^z \) is one factor of \( A_j(z) \)? Fortunately we have the following theorem.

**Theorem 2.** Suppose that \( W(z) \) is a \( k \)-valued algebroidal function of order \( \rho(W) \) defined by (1). If formula (3) is not right, there must exist an entire function \( F_0 = e^{f(z)} \) of order \( \rho(A_0) \) and without zero points, so that \( A_j(z) = F_0(z)g_j(z) \), and \( \rho(g_j) < \rho(A_0) \) \( (j = 0, 1, 2, \ldots, k) \). Thus formula (1) can also be written as the following equivalent equation:

\[
g_k(z)W^k + g_{k-1}(z)W^{k-1} + \cdots + g_0(z) = 0.
\]

The above equation and (1) define the same algebroidal function \( W(z) \), and satisfy

\[
\rho(W) = \rho(g_M) = \max \{ \rho(g_j) ; j = 0, 1, 2, \ldots, k \}.
\]

**Proof.** From the proof of Theorem 1 we only need to testify the formula (11). (Note: we have proved \( \rho(g_0) = \rho(n(r, g_0 = 0)) \).

If (11) is not correct, then \( \rho(g_0) = \rho(n(r, g_0 = 0)) = \rho(n(r, W = 0)) = \rho(N(r, W = 0)) \leq \rho(T(r, W)) = \rho(W) < \rho(g_M) \) and \( \rho(\frac{g_M}{g_0}) = \rho(g_M) \), thus (2) is right and contrary to (11). \( \Box \)

**Corollary 1.** Suppose that \( W(z) \) is a \( k \)-valued algebroidal function of order \( \rho(W) \) defined by (1). If formula (3) is not right, then the growth order of the coefficient functions \( \{A_j(z)\} \) must be the same and Borel exceptional value of every \( \{A_j(z)\} \) is 0. That is to say for any \( j \in \{0, 1, 2, \ldots, k\} \), the convergence exponent of zero points of \( A_j(z) \) is lower than the order of \( W \)

\[
\rho(n(r, A_j = 0)) < \rho(A_j).
\]

**Proof.** According to (10), Lemma 1, (11), the conditions of Corollary 1 and (10) in turn, it follows that

\[
\rho(n(r, A_j = 0)) = \rho(n(r, g_j = 0)) \leq \rho(g_j) \leq \rho(W) < \rho(F_0) = \rho(A_j).
\]

So if formula (3) is not true, then Borel exceptional value of every entire function \( A_j(z) \) is 0. \( \Box \)

**Theorem 3.** Suppose that \( W(z) \) is a \( k \)-valued algebroidal function of order \( \rho(W) \) defined by (1), then we have

\[
\rho(W) = \max \{ \rho\left(\frac{A_t}{A_0}\right) ; t = 1, 2, \ldots, k \}.
\]

**Proof.** 1) Suppose that there exists \( t \in \{1, 2, \ldots, k\} \) so that \( \rho(A_0) \leq \rho\left(\frac{A_t}{A_0} \right) \). From Theorem 1, it follows that \( \rho(A_M) = \rho(W) \). Hence:

(i) If \( \rho(A_M) = \rho(A_0) \), then \( \rho\left(\frac{A_t}{A_0} \right) = \rho(A_0) = \rho(A_M) = \rho(W) \).

(ii) If \( \rho(A_M) > \rho(A_0) \), then \( \rho\left(\frac{A_M}{A_0} \right) = \rho(A_M) = \rho(W) \).

On the other hand, for any \( t \in \{1, 2, \ldots, k\} \), we have \( \rho(W) = \rho(A_M) \geq \rho\left(\frac{A_t}{A_0} \right) \), then (12) is proved.
2) Otherwise, let us assume that $\rho\left(\frac{A_t}{A_0}\right) < \rho(A_0)$ holds for any $t \in \{1, 2, \ldots, k\}$. By Theorem 2, formula (1) can be written simply as

$$A_k(z)W^k + A_{k-1}(z)W^{k-1} + \cdots + A_0(z) = g_k(z)W^k + g_{k-1}(z)W^{k-1} + \cdots + g_0(z) = 0.$$

Moreover, for $\forall t \in \{1, 2, \ldots, k\}$, we have $\frac{A_t}{A_0} = \frac{g_t}{g_0}$ and $\rho(W) = \rho(g)$, Using Theorem 1, there exists $t \in \{0, 1, 2, \ldots, k\}$ so that $\rho(g_0) \leq \rho\left(\frac{g_t}{g_0}\right)$. Combining the front part of the above argument, it follows that

$$\rho(W) = \max\left\{\rho\left(\frac{A_t}{A_0}\right); \ t = 1, 2, \ldots, k\right\} = \max\left\{\rho\left(\frac{g_t}{g_0}\right); \ t = 1, 2, \ldots, k\right\}.$$ 

Theorem 4. Suppose that $W(z)$ is a $k$-valued algebroidal function of order $\rho(W)$ defined by (1), then we have

$$\rho(W) = \max\left\{\rho\left(\frac{A_t}{A_0}\right); \ t = 0, 1, 2, \ldots, k - 1\right\}.$$ 

Proof. Let $M(z) = 1/W(z)$. It is a $k$-valued algebroidal function defined by

$$A_k(z) + A_{k-1}(z)M + \cdots + A_0(z)M^k = 0.$$ 

Since $\rho(W) = \rho(M)$, we obtain the result by using Theorem 3. □

Corollary 2. Suppose that $W(z)$ is a $k$-valued entire algebroidal function defined by

$$\Psi(z, W) = W^k + A_{k-1}(z)W^{k-1} + \cdots + A_0(z) = 0,$$ 

then $\rho(W) = \max\{\rho(A_j); \ j = 0, 1, 2, \ldots, k - 1\}.$

Because the order of the entire functions without zero points may be infinite or positive integer, it follows that:

Corollary 3. Suppose that $W(z)$ is a $k$-valued algebroidal function of order $\rho(W)$ defined by (1). If there is a coefficient function $\rho(A_j)$ whose order is fraction, we surely have $\rho(W) = \max\{\rho(A_j); \ j = 0, 1, 2, \ldots, k\}.$

Suppose that $W(z)$ is an algebroidal function defined by (13). All critical points can be linked with an acyclic polyline $H$. Cut plane $C$ along $H$, we can find $k$ analytic functions on the broken connected plane $C - H$, denoted by $\{W_j(z)\}_{j=1}^k$. Then (13) can also be written as

$$\Psi(z, W) = (W - W_1(z))(W - W_2(z)) \cdots (W - W_k(z)) = 0.$$ 

Definition 1. Suppose that $W(z)$ is the entire algebroidal function defined by (14), modulus function is defined by

$$M(r, W) := \max_j \sup\{|W_j(z)|; |z| \leq r\}.$$ 

We can easy see that modulus function has no relationship with the chosen of polyline $H$ (by the definition of paper [2], $m(r, W)$ has no relationship with the choose of polyline $H$, either).
**Theorem 5.** Suppose that \( W(z) \) is a \( k \)-valued entire algebroidal function defined by (14), then for any \( 0 < r < R \), we have
\[
T(r, W) \leq \log M(r, W) < \frac{R + r}{R - r} 2^k T(R, W) + \frac{R + r}{R - r} k \log 2.
\]

**Proof.** 1) Firstly, we will prove the first inequation:
\[
T(r, W) = m(r, W) = \frac{1}{2k\pi} \sum_{j=1}^{k} \int_{0}^{2\pi} \log^+ |W_j(re^{i\theta})| d\theta
\]
\[
\leq \frac{1}{2k\pi} \sum_{j=1}^{k} \int_{0}^{2\pi} \log M(r, W) d\theta = \log M(r, W).
\]

2) Secondly, we will verify the second inequation:

(i) Suppose that for any \( r \), \( \log M(r, W) < 2^k \), then all \( W_j(z) \) are bounded. By Viete theorem, every coefficient function \( A_j(z) \) is bounded, thus every \( A_j(z) \) is constant, and then \( \log M(r, W) \leq kT(r, W) \).

(ii) Otherwise, we can choose \( b \in \{|z| \leq r\} \) and \( j \in \{1, 2, \ldots, k\} \) so that \( |W_j(b)| = M(r, W) (> 2^k) \). Suppose that
\[
|W_1(b)| \geq |W_2(b)| \geq \cdots \geq |W_s(b)| \geq 1 \geq |W_{s+1}(b)| \geq \cdots \geq |W_k(b)|,
\]
then there at least is a \( t \in \{1, 2, \ldots, s\} \) so that \( |W_t(b)| \geq 2^k |W_{t+1}(b)| \) or \( |W_s(b)| \geq 2^j \). Otherwise
\[
|W_1(b)| \leq 2^k |W_2(b)| \leq 2^{2k} |W_3(b)| \leq \cdots \leq 2^{(s-1)k} |W_s(b)| \leq 2^{sk} \leq 2^k.
\]
Thus from the above argument it turns to the first case (i). By Viete theorem, we have
\[
A_{k-t}(b) = (-1)^t W_1(b) W_2(b) \cdots W_t(b) + \sum' (-1)^i W_{j_1}(b) W_{j_2}(b) \cdots W_{j_t}(b),
\]
where \( \sum' \) denotes the sum of all combination items except the first one. Therefore
\[
|A_{k-t}(b)| \geq |W_1(b) W_2(b) \cdots W_t(b)| - \sum' |W_{j_1}(b) W_{j_2}(b) \cdots W_{j_t}(b)|
\]
\[
\geq |W_1(b) W_2(b) \cdots W_t(b)| - 2^{-k} \sum' |W_1(b) W_2(b) \cdots W_t(b)|
\]
\[
\geq |W_1(b) W_2(b) \cdots W_t(b)| - 2^{-k} (2^k - 1) |W_1(b) W_2(b) \cdots W_t(b)|
\]
\[
= 2^{-k} |W_1(b) W_2(b) \cdots W_t(b)|
\]
\[
\implies M(r, W) = |W_1(b)| \leq \prod_{j=1}^{t} |W_j(b)| \leq 2^k |A_{k-t}(b)|.
\]

Let \( |b| = h \leq r \). Take the logarithm of the above equation and apply Poisson–Jensen formula, then
\[ \log M(r, W) - k \log 2 \leq \log |A_{k-t}(b)| \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} \log |A_{k-t}(Re^{i\phi})| \frac{R^2 - h^2}{R^2 - 2Rh \cos(\theta - \phi) + h^2} d\phi - \sum_u \log \left| \frac{R^2 - \bar{a}_u z}{R(z - a_u)} \right| \]
\[ \leq \frac{R + h}{R - h} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \sum |W_{j1}(Re^{i\phi}) W_{j2}(Re^{i\phi}) \ldots W_{jN}(Re^{i\phi})| d\phi \]
\[ \leq \frac{R + r}{R - r} \sum_{j=1}^{k} \frac{2k}{2k\pi} \int_0^{2\pi} \log^+ |W_j(Re^{i\phi})| d\phi + \frac{R + r}{R - r} k \log 2 \]
\[ < \frac{R + r}{R - r} 2^k m(R, W) + \frac{R + r}{R - r} k \log 2 \]
\[ \implies \log M(r, W) < \frac{R + r}{R - r} 2^k T(R, W) + \frac{R + r}{R - r} k \log 2. \quad \Box \]

**Theorem 6.** Suppose that \( W(z) \) is a \( k \)-valued entire algebroidal function defined by (14). If its derived function \( W'(z) \) is also the entire algebroidal function, then for \( \forall r \in (0, R) \), we have

\[ T(r, W') + o(T(r, W)) = T(r, W') - m\left( r, \frac{W'}{W} \right) \leq T(r, W) \]
\[ < \frac{R + r}{R - r} 2^k T(R, W') + \frac{R + r}{R - r} 2k \log 2 + \log \sum_{j=1}^{k} |W_j(0)| + \log(2r\pi + r). \]

**Proof.** In order to meet the demand of path of integration, we firstly construct the acyclic polyline \( H \) which cut the plan \( C \) as follows: we take \( u \in S_W \) arbitrarily. For the isolated character of the critical points, there is \( \epsilon_u > 0 \) so that no critical points are on the circular arc \( b_u := \{|u|e^{it}; \ \pi < t < \pi + \epsilon_u\} \). Let circular arch \( B_u := \{|z| = |u| - b_u\}. \) Finally, let \( H := \{\arg z = \pi\} \cup (\bigcup_{u \in S_W} B_u) \).

Set \( z = re^{i\theta} \). We can make a curve which begins with origin and runs along \( \arg z = 0 \) to \( r \), does again along \( |z| = r \) to \( re^{i\theta} \). Thus its length is less than \( (2\pi + 1)r \)

\[ T(r, W) = m(r, W) = \frac{1}{2k\pi} \sum_{j=1}^{k} \int_0^{2\pi} \log^+ W_j(re^{i\theta}) \frac{z}{0} d\theta \]
\[ = \frac{1}{2k\pi} \sum_{j=1}^{k} \int_0^{2\pi} \log^+ \frac{1}{0} W_j'(s) ds + W_j(0) \frac{z}{0} d\theta \]
\[ \leq \frac{1}{2k\pi} \sum_{j=1}^{k} \int_0^{2\pi} \log^+ \frac{1}{0} M(r, W') ds + W_j(0) \frac{z}{0} d\theta \]
\[ \leq \log M(r, W') + \log \sum_{j=1}^{k} |W_j(0)| + \log(2r\pi + r). \quad (15) \]
Combining with Theorem 5, we deduce

\[ T(r, W) < \frac{R + r}{R - r} 2^k T(R, W') + \frac{R + r}{R - r} 2k \log 2 + \log \sum_{j=1}^{k} |W_j(0)| + \log(2\pi r + r). \]

On the other hand,

\[ T(r, W') = m(r, W') \leq m(r, W) + m\left( r, \frac{W'}{W} \right) = T(r, W) + m\left( r, \frac{W'}{W} \right). \]

**Corollary 4.** Suppose that \( W(z) \) is a \( k \)-valued entire algebroidal function defined by (14). If its derived function \( W'(z) \) is also an entire algebroidal function, then \( W(z) \) and \( W'(z) \) have the same order.

**Proof.** If \( \{W_j(0)\} \) has no poles, we can obtain the results by Theorem 6. Otherwise we can choose \( b \) in the sufficiently small neighborhood of \( z = 0 \), so that \( \{W_j(b)\} \) has no poles in it. So similar to the proof of (15), we can obtain the results if we replace \( 0 \) by \( b \). □

**References**