The exponential asymptotic stability of singularly perturbed delay differential equations with a bounded lag

Hongjiong Tian

Department of Mathematics, Shanghai Teachers’ University, 100 Guilin Road, Shanghai 200234, PR China

Received 1 May 2001
Submitted by T. Gard

Abstract

This paper is concerned with the exponential stability of singularly perturbed delay differential equations with a bounded (state-independent) lag. A generalized Halanay inequality is derived in Section 2, and in Section 3 a sufficient condition will be provided to ensure that any solution of the singularly perturbed delay differential equations (DDEs) with a bounded lag is exponentially stable uniformly for sufficiently small $\varepsilon > 0$. This type of exponential asymptotic stability can obviously be applied to general delay differential equations with a bounded lag. © 2002 Elsevier Science (USA). All rights reserved.

1. Introduction

Delay differential equations arise in many areas of mathematical modelling:

$$y'(t) = f(t, y(t), y(t - \tau(t))), \quad t \in \mathbb{R}^+ \equiv [t_0, +\infty),$$  \hspace{1cm} (1)

where $y$, $f$ are $s$-dimensional vector-valued functions and $\tau(t) \geq 0$ is a (state-independent) lag. The solution (if it exists) is determined by a choice of initial function $\phi$:

$$y(t) = \phi(t), \quad t \leq t_0.$$ \hspace{1cm} (2)

E-mail address: hongjiongtian@263.net.

0022-247X/02/$ - $ see front matter © 2002 Elsevier Science (USA). All rights reserved.
PII: S0022-247X(02)00056-2
There is a substantial theory on the stability of solutions of DDEs, but most of which emphasizes linear, constant coefficient equations with a constant lag. A well-developed theory of stability, particularly in the case of state-independent bounded lags, is based on Lyapunov functions and Lyapunov functionals.

Singularly perturbed delay differential equations arise in the study of an “optically bistable device” [1] and in a variety of models for physiological processes or diseases [2]. Such a problem has also appeared to describe the so-called human pupil-light reflex [3]. For example, Ikeda [4] adopted the model

$$\varepsilon y'(t, \varepsilon) = -y(t, \varepsilon) + A^2 [1 + 2B \cos(y(t - 1, \varepsilon))]$$

to describe an optically bistable device and showed numerically that instability or chaotic behaviour occurs for small $\varepsilon$ and certain values of $A, B$. This was confirmed experimentally by Gibbs et al. [5].

In this paper, we will concentrate on the exponential stability of delay differential equations with a bounded lag. In Section 2, we obtain an exponential inequality of Halanay type. In Section 3, we will consider singularly perturbed delay differential equations and give a sufficient condition for the exponential stability uniform in sufficiently small $\varepsilon$.

2. A generalized Halanay inequality

The following lemma generalizes the famous Halanay inequality (see Halanay [6]) and will play a key role in obtaining our main result of this paper.

**Lemma 2.1** (Generalized Halanay inequality). Suppose

$$u'(t) \leq -\alpha(t)u(t) + \beta(t) \sup_{t-\tau \leq \sigma \leq t} u(\sigma)$$

for $t \geq t_0$. Here $\tau \geq 0$, $\alpha(t)$ and $\beta(t)$ are continuous with $\alpha(t) \geq \alpha_0 > 0$ and $0 < \beta(t) \leq q \alpha(t)$ for all $t \geq t_0$ with $0 \leq q < 1$. Then

$$u(t) \leq Ge^{-\gamma^*(t-t_0)} \quad \text{for } t \geq t_0.$$  \hspace{1cm} (3)

Here $G = \sup_{t_0-\tau \leq t \leq t_0} |u(t)|$, and $\gamma^* > 0$ is defined as

$$\gamma^* = \inf_{t \geq t_0} \{ \gamma(t): \gamma(t) - \alpha(t) + \beta(t)e^{\gamma(t)\tau} = 0 \}.$$

**Proof.** Note that the result is trivial if $\tau = 0$. In the following we assume that $\tau > 0$. Denote

$$H(\gamma) \equiv \gamma - \alpha(t) + \beta(t)e^{\gamma\tau}.$$  \hspace{1cm} (4)

By assumption $\alpha(t) \geq \alpha_0 > 0$, $0 \leq \beta(t) \leq q \alpha(t)$ for all $t \geq t_0$; then for any given fixed $t \geq t_0$, we see that $H(0) = -\alpha(t) + \beta(t) \leq -(1-q)\alpha(t) \leq -(1-q)\alpha_0 < 0,$
\[
\lim_{\gamma \to \infty} H(\gamma) = \infty, \quad \text{and} \quad H'(\gamma) = 1 + \beta(t) e^{\gamma \tau} > 0.
\]
Therefore for any \( t \geq t_0 \) there is a unique positive \( \gamma(t) \) such that \( \gamma(t) - \alpha(t) + \beta(t) e^{\gamma(t) \tau} = 0 \). From the definition, one has \( \gamma^* \geq 0 \). We have to prove \( \gamma^* > 0 \). Suppose this is not true. Fix \( \tilde{q} \) satisfying \( 0 \leq q < \tilde{q} < 1 \) and pick \( \epsilon < \min\{(1 - q/\tilde{q})\alpha_0, (1/\tau) \ln(1/\tilde{q})\} \).

Then there is \( t^* \geq t_0 \) such that \( \tilde{\gamma}(t^*) < \epsilon \) and
\[
\tilde{\gamma}(t^*) - \alpha(t^*) + \beta(t^*) e^{\tilde{\gamma}(t^*) \tau} = 0.
\]

Now we have
\[
0 = \tilde{\gamma}(t^*) - \alpha(t^*) + \beta(t^*) e^{\tilde{\gamma}(t^*) \tau} < \epsilon - \alpha(t^*) + \beta(t^*) e^{\epsilon \tau} < \epsilon - \alpha(t^*) + \beta(t^*) \frac{q}{\tilde{q}} \alpha(t^*) \leq \epsilon - \frac{(1 - q/\tilde{q})\alpha_0}{\tilde{q}} < 0,
\]
which is a contradiction.

For any given \( k > 1 \), set \( v(t) = k Ge^{-\gamma^*(t-t_0)} \) and \( w(t) = v(t) - u(t) \). Suppose (3) fails. Then there is a \( k > 1 \) such that
\[
k Ge^{-\gamma^*(t-t_0)} < u(t)
\]
for some \( t \geq t_0 \). Let \( \varsigma = \inf\{t \geq t_0: v(t) - u(t) \leq 0\} \). Then we have for some \( \varsigma > t_0 \) that \( w(\varsigma) = v(\varsigma) - u(\varsigma) = 0 \) and
\[
w'(\varsigma) = v'(\varsigma) - u'(\varsigma) \leq 0.
\]
Hence
\[
w'(\varsigma) = v'(\varsigma) - u'(\varsigma) \\
\geq -k G \gamma^* e^{-\gamma^*(\varsigma-t_0)} - \left[ -\alpha(\varsigma) u(\varsigma) + \beta(\varsigma) \sup_{\varsigma-t_0 \leq \sigma \leq \varsigma} u(\sigma) \right] \\
= -k G \gamma^* e^{-\gamma^*(\varsigma-t_0)} + k G \alpha(\varsigma) e^{-\gamma^*(\varsigma-t_0)} - k G \beta(\varsigma) e^{-\gamma^*(\varsigma-t_0)} \\
\geq k G e^{-\gamma^*(\varsigma-t_0)} \left[ -\gamma^* + \alpha(\varsigma) - \beta(\varsigma) e^{\gamma^* \tau} \right].
\]
Let \( \gamma(\varsigma) \) satisfy \( \gamma(\varsigma) - \alpha(\varsigma) + \beta(\varsigma) e^{\gamma(\varsigma) \tau} = 0 \). Then according to the definition of \( \gamma^* \), it follows that
\[
-\gamma^* + \alpha(\varsigma) - \beta(\varsigma) e^{\gamma^* \tau} \\
= \left[ -\gamma^* + \alpha(\varsigma) - \beta(\varsigma) e^{\gamma^* \tau} \right] + \left[ \gamma(\varsigma) - \alpha(\varsigma) + \beta(\varsigma) e^{\gamma(\varsigma) \tau} \right] \\
= (\gamma(\varsigma) - \gamma^*) + \beta(\varsigma) [e^{\gamma(\varsigma) \tau} - e^{\gamma^* \tau}] \geq 0.
\]
Therefore
\[
w'(\varsigma) = v'(\varsigma) - u'(\varsigma) \\
> k G e^{-\gamma^*(\varsigma-t_0)} \left[ -\gamma^* + \alpha(\varsigma) - \beta(\varsigma) e^{\gamma^* \tau} \right] \geq 0.
\]
This contradicts (6) and hence
\[ u(t) \leq Ge^{-\gamma^*(t-t_0)} \text{ for } t \geq t_0. \]
\[ \square \]

Remark. In the case \( \alpha(t) \equiv \alpha > \beta(t) \equiv \beta \) the result is a reformulation of Halanay’s statement in Halanay [6]. It is not possible to improve the exponential rate of decay given by Lemma 2.1 because \( u(t) = \exp(-\gamma^*(t-t_0)) \) is a solution of \( u'(t) = -\alpha u(t) + \beta \sup_{t_0 \leq \sigma \leq t} u(\sigma) \).

### 3. Exponential stability

In this section we will apply the generalized Halanay inequality to study the following exponential stability for the nonlinear singularly perturbed delay differential system.

Consider
\[ \varepsilon y'(t, \varepsilon) = f(t, y(t, \varepsilon), y(t - \tau(t), \varepsilon)), \quad t \geq t_0, \] (7)
with initial function
\[ y(t, \varepsilon) = \phi(t), \quad t \leq t_0, \] (8)
where \( f : R^+ \equiv [t_0, \infty) \times C^s \times C^s \mapsto C^s \) and \( y(t, \varepsilon) : R^+ \times R^+ \mapsto C^s \).

**Definition 3.1** (Exponentially stable uniformly for small \( \varepsilon \)). The solution \( y(t, \varepsilon) \) of Eq. (7) is said to be \( \nu \)-exponentially stable uniformly for sufficiently small \( \varepsilon \) if it is asymptotically stable and there exist finite constants \( K > 0, \nu > 0 \) and \( \delta > 0 \), which are independent of \( \varepsilon \in (0, \varepsilon_0] \) for some \( \varepsilon_0 \), such that
\[ \|y(t, \varepsilon) - z(t, \varepsilon)\| \leq Ke^{-\nu(t-t_0)} \text{ for } t \geq t_0 \] and for any initial perturbation satisfying \( \sup_{t_0 \leq s \leq t_0 + \tau_0} \|\phi(s) - \psi(s)\| < \delta \). Here \( z(t, \varepsilon) \) is the solution of Eq. (7) corresponding to the initial function \( \psi \).

**Theorem 3.2.** Consider
\[ \varepsilon y'(t, \varepsilon) = f(t, y(t, \varepsilon), y(t - \tau(t), \varepsilon)), \quad t \geq t_0, \] (9)
\[ y(t, \varepsilon) = \phi_1(t), \quad t \leq t_0, \] and
\[ \varepsilon z'(t, \varepsilon) = f(t, z(t, \varepsilon), z(t - \tau(t), \varepsilon)), \quad t \geq t_0, \] (10)
\[ z(t, \varepsilon) = \phi_2(t), \quad t \leq t_0, \] where \( f \) is sufficiently differentiable with respect to both the last two variables, \( 0 \leq \tau(t) \leq \tau^* \), where \( \tau^* \) is a constant, and the initial functions \( \phi_1(t) \) and \( \phi_2(t) \) are continuous for \( t_0 - \tau^* \leq t \leq t_0 \). Suppose
\[ \Re \left\langle f(t, y_1, u) - f(t, y_2, u), y_1 - y_2 \right\rangle \leq \eta(t) \| y_1 - y_2 \|^2, \]
\[ \forall t \in \mathbb{R}^+, \forall u, y_1, y_2 \in \mathbb{C}^s, \]
\[ \| f(t, y, u_1) - f(t, y, u_2) \| \leq \zeta(t) \| u_1 - u_2 \|, \]
\[ \forall t \in \mathbb{R}^+, \forall y, u_1, u_2 \in \mathbb{C}^s, \]  
(11)

\[ \eta(t), \zeta(t) \text{ are continuous and satisfy for } t \geq t_0 \]
\[ \eta(t) \leq -\eta_0 < 0, \quad 0 < \zeta(t) \leq -q \eta(t), \quad 0 \leq q < 1, \]  
(13)

where \( \mathbb{C}^s \) is the \( s \)-dimensional complex vector space, \( \| \cdot \| \) is the induced norm of the inner product \( \langle u, v \rangle = v^H u \), and \( \eta_0 \) is a constant. If (7) and (10) each has a unique solution, then there exists a small \( \varepsilon > 0 \) such that the solution of (7) is exponentially stable uniformly for sufficiently small \( \varepsilon \in (0, \varepsilon_0] \).

**Proof.** According to the definition of the norm on \( \mathbb{C}^s \), we have

\[ \frac{1}{2} \varepsilon \frac{d}{dt} \left( \| y(t, \varepsilon) - z(t, \varepsilon) \|^2 \right) \]
\[ = \Re \left\langle \varepsilon y'(t, \varepsilon) - \varepsilon z'(t, \varepsilon), y(t, \varepsilon) - z(t, \varepsilon) \right\rangle \]
\[ = \Re \left\{ f(t, y(t, \varepsilon), y(t - \tau(t), \varepsilon)) - f(t, z(t, \varepsilon), z(t - \tau(t), \varepsilon)), y(t, \varepsilon) - z(t, \varepsilon) \right\} \]
\[ = \Re \left\{ f(t, z(t, \varepsilon), y(t - \tau(t), \varepsilon)) - f(t, z(t, \varepsilon), z(t - \tau(t), \varepsilon)), y(t, \varepsilon) - z(t, \varepsilon) \right\} \]
\[ + \Re \left\{ f(t, z(t, \varepsilon), z(t - \tau(t), \varepsilon)), y(t, \varepsilon) - z(t, \varepsilon) \right\}. \]  
(14)

Application of Schwartz’s inequality yields

\[ \frac{1}{2} \varepsilon \frac{d}{dt} \left( \| y(t, \varepsilon) - z(t, \varepsilon) \|^2 \right) \]
\[ \leq \eta(t) \| y(t, \varepsilon) - z(t, \varepsilon) \|^2 + \zeta(t) \| y(t, \varepsilon) - z(t, \varepsilon) \| \]
\[ \times \| y(t - \tau(t), \varepsilon) - z(t - \tau(t), \varepsilon) \| \]
\[ \leq \eta(t) \| y(t, \varepsilon) - z(t, \varepsilon) \|^2 \]
\[ + \zeta(t) \| y(t, \varepsilon) - z(t, \varepsilon) \| \sup_{t - \tau^* \leq \sigma \leq t} \| y(\sigma, \varepsilon) - z(\sigma, \varepsilon) \|. \]  
(15)

Denote \( V(t, \varepsilon) = \| y(t, \varepsilon) - z(t, \varepsilon) \|^2 \). It follows from (15) with \( \varepsilon > 0 \) that

\[ V'(t, \varepsilon) \leq \frac{\eta(t)}{2\varepsilon} V(t, \varepsilon) + \frac{\zeta(t)}{2\varepsilon} \sup_{t - \tau^* \leq \sigma \leq t} V(\sigma, \varepsilon), \quad t \geq t_0. \]

Application of the inequality in Lemma 2.1 to the above equation yields

\[ V(t, \varepsilon) \leq \tilde{G} e^{-\gamma^*(\varepsilon)(t - t_0)}, \quad t \geq t_0. \]  
(16)
Here

\[
\gamma^*(\varepsilon) = \inf_{t \geq t_0} \left\{ \gamma(t) : \gamma(t) - \frac{2\eta(t)}{\varepsilon} + \frac{2\zeta(t)}{\varepsilon} e^{\gamma(t)\tau^*} = 0 \right\},
\]  

(17)

and \(\tilde{G} \geq 0\) only depends on the initial condition \(\|\phi_1(t) - \phi_2(t)\|\).

For any fixed \(t \geq t_0\), let \(\gamma(t, \varepsilon)\) be defined as the unique positive zero of

\[
\gamma - \frac{2\eta(t)}{\varepsilon} + \frac{2\zeta(t)}{\varepsilon} e^{\gamma\tau^*} = 0.
\]

(18)

It can be proven that \(\gamma(t, \varepsilon_1) \geq \gamma(t, \varepsilon_2)\) whenever \(\varepsilon_2 \geq \varepsilon_1 > 0\). This implies \(\gamma^*(\varepsilon_1) \geq \gamma^*(\varepsilon_2)\) and thus we proved that \(\gamma^*(\varepsilon)\) is monotonically decreasing with respect to the variable \(\varepsilon\). Hence we deduce that there exists a small \(\varepsilon_0 > 0\) such that the solution \(y(t, \varepsilon)\) is exponentially stable uniformly for sufficiently small \(\varepsilon \in (0, \varepsilon_0]\). This completes the proof. \(\square\)

**Remark.** From the generalized Halanay inequality and the proof above, it is obvious that the result of this theorem still holds when we set \(\varepsilon = 1\).

**Acknowledgment**

The work is partially supported by NSF of China under grant No. 10101012, Shanghai NSF Grants 00JC14057 and 01ZA14045.

**References**


