on a Question of M. Newman on the Number of Commutators*  

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M. Newman asked whether there is an absolute constant c such that every matrix in $\text{SL}_n R$ is the product of at most c commutators, where $R$ ranges over euclidean commutative rings and $n \geq 3$. We give here a negative answer. However, if for the ring $R$ every matrix in $\text{SL}_n R$ is the product of a bounded number of commutators for some fixed $m \geq 3$, then for all sufficiently large $n$, every matrix in $\text{SL}_n R$ is the product of six commutators. 

1. Introduction  

Let $R$ be a (commutative) principal ideal ring with 1. M. Newman [19, p. 45, Question (d)] asks the following question:  

Let $R$ have the property that $\text{SL}(n, R)$ is its own commutator subgroup (for example, $R$ a euclidean ring and $n > 2$). Determine whether or not an absolute constant $c$ exists such that every element of $\text{SL}(n, R)$ is the product of at most $c$ commutators. 

We show here that in general the answer is "no." Moreover we show that there is no bound on the number of commutators even if we restrict ourselves to the (euclidean) polynomial ring $\mathbb{C}[x]$ with complex coefficients and fix an arbitrary $n \geq 2$. 

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Theorem 1. Let \( F \) be a field of infinite trancendence degree over its prime field (for example, \( F = \mathbb{C} \), the complex numbers), \( n \geq 2 \) an integer. Then for every number \( c \) there is a matrix in the group \( SL_n F[x] \) which cannot be written as the product of \( c \) commutators.

For any commutative ring \( R \) with \( 1 \) we write \( SL_n R \) instead of \( SL(n, R) \) of \([19] \) for the group of \( n \times n \) matrices over \( R \) with determinant 1.

In the rest of this section we discuss situations in which the number of commutators can be uniformly bounded. For any group \( H \), let \( c(H) \) denote the smallest integer \( c \) such that every element of the commutator subgroup \([H, H]\) is the product of \( c \) commutators. Theorem 1 says that \( c(SL_n \mathbb{C}[x]) = \infty \) for every \( n \geq 2 \).

For fields \( F \), Thompson \([21]\) showed that \( c(GL_n F) = 1 \) and \( c(SL_n F) \leq 2 \) for any \( n \geq 2 \). Kurok [15] studied division rings \( D \) and proved that for any \( n \), \( c(GL_n D) \leq c(GL_1 D) + 1 \) and that \( c(GL_n \mathbb{H}) = 1 \) for \( \mathbb{H} \) the quaternions. He showed further that \( c(GL_n D) \geq 2 \) for some division rings \( D \) \([16]\).

In \([8]\) Cohn gives examples of division rings with \( c(GL_1 D) = 1 \). Draxl [11] studied the case of division rings finite dimensional over their centers. He showed that there is a function \( F \) so that \( c(GL_1 D) \leq F(i(D)) \) if the index \( i(D) \) is square-free or if the center of \( D \) satisfies a certain hypothesis (which is satisfied for local or global fields). Related material appears in \([20]\).

In the next theorem we will assume that \( R \) satisfies a stable range condition of Bass \([1]\). We use the notation \( sr(R) \) of \([24]\). By \([23]\) we have \([GL_n R, GL_n R] \subset E_n R \) for \( n \geq 1 + sr(R) \); hence \( c(GL_n F) \leq c(E_n R) \). Here \( E_n R \) is the subgroup of \( GL_n R \) generated by elementary matrices. If \( n \geq 3 \), then \( E_n R = [E_n R, E_n R] \) for any ring \( R \) with \( 1 \) \([1]\).

Theorem 2. Let \( R \) be an associative ring with \( 1 \) such that \( c(GL_m R) < \infty \) for some \( m \geq \max(sr(R) + 1, 3) \). Then

\( (a) \) \( c(GL_n R) \leq c(GL_m R) + 5 \) and \( c(E_n R) \leq c(E_m R) + 5 \) for all \( n \geq m \);

\( (b) \) \( c(E_n R) \leq c(GL_m R) + 5 \) for all \( n \geq 3m \);

\( (c) \) \( c(GL_n R) \leq c(E_n R) < \infty \) for all \( n \geq sr(R) + 1 \);

\( (d) \) \( c(GL_n R) \leq 7 \) for all \( n \geq mc(GL_m R) \); \( c(E_n R) \leq 7 \) for all \( n \geq 3mc(GL_m R) \); \( c(E_n R) \leq 6 \) for all \( n \geq 2m(c(GL_m R) + 2) \).

Example 3. When \( R \) is a principal ideal ring, \( sr(R) \leq \dim(R) + 1 \leq 2 \) by \([1]\), so \([GL_n R, GL_n R] = E_n R = [E_n R, E_n R] \) for all \( n \geq 3 \). If \( c(GL_m R) < \infty \) for some \( m \geq 3 \), then, by Theorem 2, \( c(GL_n R) \leq c(SL_n R) \leq c(E_n R) < \infty \) for all \( n \geq 3 \) and \( c(GL_n R) \leq c(SL_n R) \leq c(E_n R) \leq 6 \) for all sufficiently large \( n \).
Example 4. Let $R$ be a Dedekind ring of arithmetic type (Hasse domain), or, more generally, a subring (with 1) of a global field. Then $sr(R) \leq 2$ and, using [3], one can prove that $c(SL_n R) \leq 6$ for sufficiently large $n$. For $R = \mathbb{Z}$ the results of [4] enable one to determine such an integer $n$. That result should be far from the best possible. In fact we conjecture that for all such rings $R$ and all $n \geq 3$, $c(E_n R) \leq 6$. V. K. Murty, using techniques of [12, 13], has recently shown (letter to K. Dennis, December 2, 1985) that if the rank of the unit group is at least 4, then every element of $SL_2 R$ is the product of six elementary matrices. In [5] it is shown that every element of $SL_2 R$ is the product of a bounded (but undetermined) number of elementary matrices provided the unit group is infinite. Related material can be found in [6, 7, 17, 18].

Example 5. Let $R$ be an associative ring with 1 such that $sr(R) \leq 1$ (many examples of such $R$ are collected in [25]). Theorem 2 says, in particular, that if $c(GL_3 R) < \infty$, then $c(E_n R) < \infty$ for all $n \geq 3$ and $c(E_n R) \leq 6$ for large $n$. For commutative $R$ this can be improved as follows.

Theorem 6. Let $R$ be a commutative associative ring with 1 such that $sr(R) \leq 1$. Then $c(GL_n R) \leq c(SL_n R) = c(E_n R) \leq 5$ for all $n \geq 3$.

2. Proof of Theorem 1

For any associative ring $R$ with 1 and any natural number $n$, let $e_n(R)$ denote the minimal number $e$ such that every matrix in $E_n R$ is the product of $e$ elementary matrices. Recall that a matrix is elementary if it differs from the identity matrix $I_n$ by at most one non-diagonal entry. We will use the notation $a^{ij}$, where $a \in R$, $1 \leq i \neq j \leq n$, for such a matrix. The group in which this element lies will be determined by context and should lead to no confusion. We denote by $E_n R$ the subgroup of $GL_n R$ generated by all elementary matrices.

The formula $a^{ij} = [a^{ik}, a^{kj}]$, where $i \neq k \neq j \neq i$, shows that every elementary matrix is a commutator if $n \geq 3$. In particular, $e_n(R) \geq c(E_n R)$ and the group $E_n R = [E_n R, E_n R]$ is perfect, if $n \geq 3$.

Let $t_n(R)$ (resp., $ut_n(R)$) denote the minimal number $t$ such that every matrix in $E_n R$ is the product of $t$ matrices such that each of them is triangular, i.e., either upper triangular with 1 along the main diagonal or lower triangular with 1 along the diagonal, and, in the case of $ut_n(R)$, the first matrix is upper triangular. It is clear that

$$t_n(R) \leq ut_n(R) \leq t_n(R) + 1 \leq e_n(R) + 1 \quad \text{and} \quad t_2(R) = e_2(R).$$
Since every triangular matrix in $E_n R$ is the product of $n(n-1)/2$ elementary matrices, $e_n(R) \leq t_n(R) n(n-1)/2$.

**Lemma 7.** For any associative ring $R$ with 1 and any $n \geq 2$ we have $ut_{n+1}(R) \leq ut_n(R)$.

**Proof.** We assume that the number $t = ut_n(R)$ is finite, and we have to prove that $ut_{n+1}(R) \leq t$, i.e., that every matrix in $E_{n+1} R$ is the product of $t$ triangular matrices, the first of them being upper triangular.

Consider the set $X$ of all matrices in $E_{n+1} R$ which can be written in this form. We want to prove that $X = E_{n+1} R$.

We set $Y = \{ g \in E_{n+1} R : gX = X \}$. Evidently, $Y$ is a subgroup of $E_{n+1} R$ and $Y$ contains the group $U$ of upper triangular matrices with 1 along the diagonal. Since the group $E_{n+1} R$ is generated by $U$ and $R^{i+1}$ ($i = 1, \ldots, n$), it suffices to prove that $Y \supseteq R^{i+1}$ (then $Y = E_{n+1} R$, hence $X = E_{n+1} R$, since $X$ is not empty).

Let $x = x_1 \cdots x_n \in X$, where $x_j$ are triangular and $x_1 \in U$. We want to prove that $y_i x \in X$ for any $y \in R$ and $1 \leq i \leq n$. Without loss of generality, we can assume that $x_j$ is upper triangular when $j$ is odd and $x_j$ is lower triangular when $j$ is even.

When $i < n$ (resp., $i = n$) we write $x_j = \begin{bmatrix} y_j & 0 \\ 0 & 1 \end{bmatrix} z_j$ (resp., $\begin{bmatrix} 0 & y_j \\ 1 & 0 \end{bmatrix} z_j$), where $y_j$ is a triangular matrix in $E_n R$, $z_j$ is triangular, $z_j$ may differ from $I_{n+1}$ only in the last column (resp., the first row) when $j$ is odd, and $z_j$ may differ from $I_{n+1}$ only in the last row (resp., the first column) when $j$ is even.

Now we collect all 

$$
\begin{array}{c}
E_n R \text{ parts}\end{array}
\begin{array}{c}
y_j \end{array}
\text{ together in front of } x \text{ and write }
\begin{array}{c}
x = \begin{bmatrix} y_1 \cdots y_n & 0 \\ 0 & 1 \end{bmatrix} z_1' \cdots z_r' \end{array}
\begin{array}{c}
(\text{resp., } \begin{bmatrix} 1 & 0 \\ 0 & y_1 \cdots y_r \end{bmatrix} z_1' \cdots z_r')
\end{array}
$$

where each $z_j'$ is the conjugate of $z_j$ by a matrix in $E_n R$ (resp., in $E_n R$), so $z_j'$ is of the same form as $z_j$.

By the definition of $t$, we can write $r^{i+1} y_1 \cdots y_i$ (resp., $r^{n+1} y_1 \cdots y_i$) as the product $y_i' \cdots y_i'$ with triangular $y_i'$ in $E_n R$ such that $y_j'$ is upper triangular when $j$ is odd and lower triangular when $j$ is even. Now we put new 

$$
\begin{array}{c}
E_n R \text{ parts}\end{array}
\begin{array}{c}
y_j' \end{array}
\text{ back in the places of } y_j. \text{ Each new } z_j' \text{ is of the same form as the previous } z_j \text{ or } z_j. \text{ So } r^{i+1} x \in X. \text{ The lemma is proved.}
$$

**Corollary 8.** If $e_n(R)$ is finite, then so is $e_{n+1}(R)$.

**Proof.** Use that
$$
e_{n+1}(R) \leq t_{n+1}(R) n(n+1)/2 \leq ut_{n+1}(R) n(n+1)/2
\leq ut_n(R) n(n+1)/2
\leq (t_n(R) + 1) n(n+1)/2 \leq (e_n(R) + 1) n(n+1)/2.$$

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In the next lemma we will assume that $R$ satisfies the $m$th stable range condition of Bass [1], which we write as $sr(R) \leq m$.

**Lemma 9.** Let $R$ be an associative ring with 1 with $sr(R) \leq m$. Then for any $n \geq m$ we have

$$GL_n R = ULUL \begin{bmatrix} GL_m R & 0 \\ 0 & 1_{n-m} \end{bmatrix} = UL \begin{bmatrix} GL_m R & 0 \\ 0 & 1_{n-m} \end{bmatrix} UL,$$

where $U$ (resp., $L$) is the group of all upper (resp., lower) triangular matrices in $E_m R$ with 1 along the main diagonal.

**Proof.** When $n - m = 1$, the first equality was proved in [1]. The general case follows easily by induction on $n - m$. The second equality follows by a similar argument (see the following remark) or can also be easily derived from the first.

**Remark 10.** By using the stable range condition directly in each step one can choose the triangular matrices to have entries in only one row or column. In fact, using [24] the innermost $L$ can be chosen to have at most $sr(R)$ entries yielding

$$e_{n+1}(R) \leq e_n(R) + sr(R) + 3n$$

and by induction for $n \geq m$

$$e_n(R) \leq e_m(R) + (n - m) sr(R) + 3(n(n - 1)/2 - m(m - 1)/2).$$

**Lemma 11.** If $A$ is the product of $c$ commutators in $GL_1 R$, then $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ is the product of $c + 2$ matrices from $R^{1,2} R^{2,1}$.

**Proof.** Set $U = R^{1,2} = U_2 R$ and $L = R^{2,1} = L_2 R$. Let $A = [B_1, D_1] \cdots [B_c, D_c]$ with $B_i, D_i$ in $GL_1 R$. Then $A = B_1 \cdots B_i B_{i+1} \cdots B_{2c}$, where $B_{c+i}$ is similar to $B_i^{-1}$ in $GL_1 R$ for $i = 1, \ldots, c$. We set

$$d_0 = \begin{bmatrix} B_1 \cdots B_c & 0 \\ 0 & (B_1 \cdots B_c)^{-1} \end{bmatrix}, \quad d_i = \begin{bmatrix} B_{c+i} & 0 \\ 0 & B_i \end{bmatrix}$$

for $i = 1, \ldots, c$. Then $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = d_0 d_1 \cdots d_c$. By the Whitehead Lemma [1], $d_i \in ULUL$ for $i \geq 0$. Since $d_i$ normalizes $U$, we have $d_0 d_1 \in d_0 ULUL = Ud_0 LUL < U(ULUL) LUL = ULULUL$ and similarly by an induction on $c$ that $d_0 d_1 \cdots d_c$ is the product of $c + 2$ matrices from $UL$.

**Corollary 12.** Let $R$ be an associative ring with 1, and $n$ a natural number. If $A$ is the product of $c$ commutators in $GL_n R$, then $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ is the product of $(2c + 4) n^2$ elementary matrices in $E_{2n} R$. 
Proof. Apply Lemma 11 to the ring $M_n R$ of $n \times n$ matrices over $R$, instead of $R$.

Now we are ready to prove Theorem 1. We gave a proof for Corollary 8 above, because we want to use a result of [22] which uses the corollary without proof. More precisely, our Corollary 8 is stated in as Lemma 1.1 (R. K. Dennis). But the "sketch of proof" in [22, p. 357] contains the statement that $t_n(R) \geq t_{n+1}(R)$ when $n \geq 3$, and we cannot prove this statement.

Assume now that $R = F[x]$ and $m \geq 2$ are as in Theorem 1. Since $R$ is euclidean, $SL_m R = E_m R$ for all $m$. By [22], $e_m(R) = \infty$ for all $m \geq 2$. By Lemma 9 with $n = 2m$ (using that $sr(R) \leq \dim(R) + 1 \leq 2$ by [1]) and Corollary 12:

$$e_{2m}(R) \leq (2c(SL_m R) + 4)m^2 + 4(2m(2m - 1)/2).$$

Since $e_{2m}(R) = \infty$, we obtain that $c(SL_m R) = \infty$ as well.

3. PROOF OF THEOREM 6

In [22] it is claimed without proof that every upper triangular matrix is the product of three commutators. The next lemma gives a strengthening of this result.

LEMMA 13. For any associative ring $R$ with 1 and any integer $n \geq 3$, every triangular matrix in $E_n R$ is the product of two commutators in $E_n R$.

Proof. The cases of upper and lower triangular matrices are similar, so let $A = (A_{i,j})$ be upper triangular, that is, $A_{i,j} = 0$ for $i > j$ and $A_{i,i} = 1$ for $i = 1, \ldots, n$.

If $V$ is upper triangular with $V_{i,i+1} = 1$ for $i = 1, \ldots, n - 1$, then we will show that $V$ is conjugate in $E_n R$ to the Jordan matrix $J$. This is the matrix which has 1 on the diagonal and the line above the diagonal (which we will refer to as the superdiagonal) and 0 elsewhere. If now $A_{i,i+1} = 0$ for $i = 1, \ldots, n - 1$, then $JA$ is similar to $J$, say $MJAM^{-1} = J$, and hence $A = J^{-1}M^{-1}JM$ is a commutator.

We show that $V$ is similar to $J$ by an upper triangular matrix $T$. The equation $TJ = VT$ yields the following equations in the entries for $l - k \geq 2$:

$$T_{k,l-1} = T_{k+1,l} + \sum_{2 \leq i} V_{k,k+i} T_{k-i,l}.$$

These can be solved by induction on $d = l - k \geq 2$, choosing $T_{1,1} = 0$ arbitrarily, with the other entries uniquely determined by the equations given above.
If $A_{2i,2i+1} = A_{2i,2i+2} = A_{2i-1,2i+1} = 0$ for all $i$, then the conjugate of $A$ by the permutation matrix with permutation $(2, 3)(6, 7)\cdots$ is an upper triangular matrix and its superdiagonal is zero. Thus, this conjugate of $A$ and hence $A$ itself is a commutator in $E_nR$.

In the general case, we write $A = BC$ with $B_{2i,2i+1} = A_{2i,2i+1}$, $B_{2i,2i+2} = A_{2i,2i+2} - A_{2i,2i+1}A_{2i+1,2i+2}$, $B_{2i-1,2i+1} = A_{2i-1,2i+1}$, $B_{2i,2i} = 1$, and $B_{2i+1,2i+2} = 0$, $B$ upper triangular. Then $C_{2i,2i+1} = C_{2i,2i+2} = C_{2i-1,2i+1} = 0$ and $C$ is upper triangular, so $C$ is a commutator, as observed above. It thus suffices to choose $B$ to be a commutator.

The matrix $B$ is upper triangular and can be written in block form with all blocks (except the last column in the case of odd $n$) of size 2 by 2:

$$B = \begin{bmatrix} I & b_1 \\ & I & b_2 \\ & & \ddots \\ & & & I & b_k \\ & & & & I \\ \end{bmatrix}, \quad \text{where} \quad b_i = \begin{bmatrix} B_{2i-1,2i+1} & 0 \\ B_{2i,2i+2} & B_{2i,2i+1} \end{bmatrix},$$

and $I = 1_2$ for $i = 1, \ldots, [(n-2)/2]$ with $k = [(n-1)/2]$ (when $n = 2k + 1$ is odd, the second column of $b_k$ is absent, so $b_k$ is a 2 by 1 matrix and the last $I$ on the diagonal of $B$ is 1). Let $d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $E_2R$. Then $d - 1_2$ and $d^{-1} - 1_2$ are both in $E_2R$. Let $D$ be a block diagonal matrix with alternating entries $1_2$ and $d$ along the diagonal chosen so that the $k$th block is $d$ (when $n$ is odd the last “block” of $D$ is 1).

For an appropriate matrix $B'$ of the same form as $B$, the matrix $B'' = [D, B']$ has the same entries on the diagonal and below as $B$, as well as the same superdiagonal $b_1, \ldots, b_k$. We have $A = BC = B''C''$, where $C''$ has zeros at the positions $(2i, 2i+1)$, $(2i, 2i+2)$, and $(2i-1, 2i+1)$. Thus, $A$ is the product of two commutators, $B''$ and $C''$ in $E_nR$.

**Corollary 14.** Under the conditions of Lemma 13, if a matrix $A$ is the product of $t > 1$ triangular matrices, then $A$ is the product of $3 + [t/2]$ commutators. In particular, $c(E_nR) \leq 3 + [t_n(R)/2]$.

**Proof.** First note that in any expression, the orders of two elements may be exchanged at the cost of introducing an extra commutator. Further, as conjugation of an entire expression does not change the number of commutators required to express it, we may assume that $t$ is even. We may as well assume $A = U_1L_1U_2L_2 \cdots$. Then $A = c_1U_2(U_1L_1)L_2 \cdots$ and by an induction we have $A = c_1c_2 \cdots c_scU'L'$, where $s = [t/2] - 1$. The result now follows from Lemma 13.
We can now complete the proof of Theorem 6. By $[1]$, $ut_2[R] \leq 4$. By Lemma 7, $ut_n(R) \leq 4$ for all $n$. Corollary 14 yields $c(E_n R) \leq 3 + [t_n(R)/2] \leq 3 + [ut_n(R)/2] \leq 3 + 2 = 5$. Theorem 6 is proved.

Remark 15. Let $R$ be any ring which is not 0. The matrix $[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}]$ can be expressed as the product of no fewer than three elementary matrices. If $u \neq 1$ is a unit with inverse $v$, then the matrix $[\begin{smallmatrix} v & 0 \\ 0 & v \end{smallmatrix}]$ can be expressed as the product of no fewer than four elementary matrices. Thus $ut_n(R) \geq 3$ for all non-trivial rings $R$. If $GL_1 R \neq 1$, then $ut_n(R) \geq 4$. If $R$ is the field with two elements or more generally any Boolean ring (i.e., $r^2 = r$ for all $r \in R$), then $ut_n(R) = 3$ for all $n \geq 2$. Conversely, if $ut_n(R) = 3$ and $R$ is commutative, it is easy to see that $R$ is Boolean. Boolean rings may well be the only rings with this property.

4. PROOF OF THEOREM 2

The following two statements are due to Harpe and Skandalis [14].

**Lemma 16.** Let $R$ be an associative ring with 1, $k$ a natural number, and $r_1, ..., r_k \in GL_1 R$. Assume that $r_1 \cdots r_k = 1$. Let $A$ be the diagonal matrix in $GL_k R$ with the diagonal entries $r_1, ..., r_k$. Then $A$ is a commutator in $GL_k R$.

**Proof.** Set $s_i = r_1 \cdots r_i$ for $i = 1, ..., k$. Let $B$ and $C$ be the diagonal matrices in $GL_k R$ with the diagonal entries $s_1, ..., s_k$ and 1, $s_1, ..., s_{k-1}$, respectively. Since $s_k = 1$, the matrices $B$ and $C$ are similar (in fact, conjugate by a permutation matrix). So $A = C^{-1}B$ is a commutator.

**Corollary 17.** Let $R$ and $k$ be as in Lemma 16, $a_i, b_i \in GL_1 R$. $r = [a_1, b_1] \cdots [a_k, b_k]$, and $B$ the diagonal matrix $[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}] \in GL_k R$. Then $B$ is the product of two commutators.

**Proof.** Let $C$ be the diagonal matrix with diagonal entries $[a_k, b_k]$, $[a_{k-1}, b_{k-1}]$, ..., $[a_1, b_1]$. Clearly $C$ is a commutator in $GL_k R$. On the other hand, $A = BC^{-1}$ is a diagonal matrix, and its diagonal entries are $r_1 = [a_1, b_1] \cdots [a_{k-1}, b_{k-1}]$, $r_2 = [a_k, b_k]^{-1}$, ..., $r_k = [a_1, b_1]^{-1}$. Since the product $r_1 \cdots r_k$ of these entries is 1, $A$ is a commutator by Lemma 16. Thus, $B = AC$ is the product of two commutators.

**Lemma 18.** Under the conditions of Lemma 16, $A \in (L_k R)(U_k R)(L_k R)$ ($U_k R$), where $L_k R$ (resp., $U_k R$) denotes the group of lower (resp., upper) triangular matrices.

**Proof.** Multiplying $A$ from the left by a lower triangular matrix $R$ we can replace each 0 on the line below the diagonal (subdiagonal)
by a 1. Next multiply $BA$ on the left by an upper triangular matrix $C$ to replace the diagonal entries $r_1, \ldots, r_k$ by $1, 1 + (1 - r_1) r_2, \ldots, 1 + (1 - r_1 \cdots r_{k-2}) r_{k-1}, \ldots, 1 + (1 - r_1 \cdots r_{k-1}) r_k = r_k$. The zeros on the superdiagonal have been replaced by $(1 - r_1) r_2, \ldots, (1 - r_1 \cdots r_{k-1}) r_k = r_k - 1$. Now it is clear that $CBA \in (L_k R)(U_k R)$, yielding the assertion.

**COROLLARY 19.** Let $R$ be an associative ring with 1, and $m$ and $t$ natural numbers. If $a_1, \ldots, a_t \in (U_m R)(L_m R)$, then

$$
\begin{pmatrix}
a_1 \cdots a_t & 0 \\
0 & 1_{mt-m_t}
\end{pmatrix} = B \in (L_{mt} R)(U_{mt} R)(L_m R)(U_m R).
$$

**Proof.** Let $A$ be the block diagonal matrix in $GL_{m_t} R$ with the diagonal entries $a_t, a_{t-1}, \ldots, a_1$. By Lemma 18, $BA^{-1} \in LULU$, where $L = L_m M_{m_t} R \subset L_{mt} R$, $U = U_m L_m R \subset U_{mt} R$. Thus $B \in LULUA = LUALU \subset LU(UL) LU = LULU$, since $A$ normalizes both $U$ and $L$ and $A \in UL$.

**THEOREM 20.** Let $R$ be an associative ring with 1 such that $t_m(R) < \infty$ for some $m \geq 2$. Then

(a) $t_n(R) \leq ut_n(R) < \infty$ for all $n \geq \text{sr}(R) + 1$;

(b) $ut_n(R) \leq 6$ for all $n \geq \min(m[(ut_m(R) + 1)/2])$, where the minimum is taken over all $m \geq \text{sr}(R) + 1$.

**Proof.** (a) In the case $n \geq m$, by Lemma 7, $t_n(R) \leq ut_n(R) \leq ut_m(R) \leq t_m(R) + 1 < \infty$. Now let $R^\infty$ be the direct product of countably many copies of $R$. Then $\text{sr}(R^\infty) = \text{sr}(R)$. For any $k \geq \text{sr}(R) + 1$, the condition $t_k(R) < \infty$ is equivalent to the homomorphism $K_k(R^\infty) \to (K_1 R)^\infty$ being an isomorphism. Since the condition holds for all sufficiently large $k$, it holds for all $k \geq \text{sr}(R) + 1$.

(b) Let $m \geq \text{sr}(R) + 1$. Set $t = [(ut_m(R) + 1)/2]$. We want to prove that $ut_n(R) \leq 6$ for all $n \geq mt$. By Lemma 7, we can assume that $n = mt$.

Let $A \in GL_n R$. By Lemma 9, $A \in UL[\begin{smallmatrix} B & 0 \\ 0 & 1_{n-m} \end{smallmatrix}] UL$, where $U = U_n R$ and $L = L_n R$. By Corollary 19 the middle term lies in $LULU$. Hence $A \in UL(ULU) UL = ULULU$.

Now we are prepared to prove Theorem 2. We will use that

$$
E_{n+1} R \cap \begin{pmatrix} GL_n R \\ 0 \\ 1 
\end{pmatrix} = \begin{pmatrix} E_n R \\ 0 \\ 1 
\end{pmatrix}
$$

when $n \geq \text{sr}(R) + 1$ and that $E_n R = [E_n R, E_n R]$ when $n \geq 3$ (see [1, 23]).

To prove (a) we take an arbitrary matrix $A$ in $GL_n R$, $n \geq m$. By Lemma 9, we write $A$ as the product of four triangular matrices and a matrix $B'$ of the form $[\begin{smallmatrix} B & 0 \\ 0 & 1_{n-m} \end{smallmatrix}]$ with $B$ in $GL_m R$. Clearly, $B'$ is the product
of \( c(GL_m \cdot R) \) commutators. By Corollary 14, the product of four triangular matrices is the product of five commutators. So \( A \) is the product of \( 5 + c(GL_m \cdot R) \) commutators. If \( A \in E_n \cdot R \), then \( B \in E_m \cdot R \), and we can write \( B \) as the product of \( c(E_m \cdot R) \) commutators in \( E_m \cdot R \). So \( A \) is the product of \( 5 + c(E_m \cdot R) \) commutators in \( E_n \cdot R \). Thus Theorem 2(a) is proved.

To prove (b), we proceed as above and write \( B = [C_1, D_1] \cdots [C_k, D_k] \) with \( k = c(GL_m \cdot R) \) and \( C_i, D_i \) in \( GL_m \cdot R \). We set

\[
P_i = \begin{bmatrix} C_i & 0 & 0 \\ 0 & C_i^{-1} & 0 \\ 0 & 0 & 1_{n-2m} \end{bmatrix} \in GL_n \cdot R, \quad Q_i = \begin{bmatrix} D_i & 0 & 0 \\ 0 & 1_{n-2m} & 0 \\ 0 & 0 & D_i^{-1} \end{bmatrix} \in GL_n \cdot R.
\]

By the Whitehead Lemma [1], \( P_i, Q_i \in E_n \cdot R \). So \( B' = [\begin{smallmatrix} B \ 
0 \ & 0 \\ \end{smallmatrix} \begin{smallmatrix} 0 \ & 0 \end{smallmatrix}] = [P_1, Q_1] \cdots [P_k, Q_k] \) is the product of \( k \) commutators in \( E_n \cdot R \). Thus, \( A \) is the product of \( 5 + k = 5 + c(GL_m \cdot R) \) commutators in \( E_n \cdot R \). Theorem 2(b) is proved.

To prove (c) note the condition \( c(GL_m \cdot R) < \infty \) for some \( m \geq sr(R) + 1 \) implies via Lemmas 9 and 11 that \( t_{2m}(R) < \infty \). Then by Theorem 20(a), \( t_m(R) < \infty \) for all \( m \geq sr(R) + 1 \). By Corollary 14, \( c(GL_m \cdot R) < \infty \) for all \( m \geq \max(3, sr(R) + 1) \).

To prove (d), we proceed as in the proof (b) above. When \( n \geq mk \), the matrix \( B' \) is the product of two commutators in \( GL_n \cdot R \) by Corollary 17. So \( A \) is the product of \( 5 + 2 = 7 \) commutators.

When \( n \geq 3mk \), we use that \( B'' = \begin{bmatrix} B \\ 0 & 0 \end{bmatrix} \in GL_{mk} \cdot R \) is the product of two commutators in \( GL_{mk} \cdot R \) to see that \( B' = \begin{bmatrix} B \\ 0 & 0 \end{bmatrix} \in GL_n \cdot R \) is the product of two commutators in \( E_n \cdot R \).

By Lemma 11, \( \begin{bmatrix} B \\ 0 & 0 \end{bmatrix} \) is the product of \( k + 2 \) matrices in \( (U_{2m} \cdot R)(L_{2m} \cdot R) \). By Corollary 19, \( \begin{bmatrix} C \ 
0 \ & 0 \end{bmatrix} \) is the product of two matrices in \( (L_{2m(k+2)} \cdot R)(U_{2m(k+2)} \cdot R) \). So our matrix is the product of two matrices in \( (L_n \cdot R)(U_n \cdot R) \) provided \( n \geq 2m(k+2) \). Therefore the original matrix \( A \in UL(ULUL')UL = ULULUL \) and by Corollary 14, \( A \) is the product of six commutators.

Theorem 2 is proved.

Remark 21. It follows from Corollary 17 that \( c(GLR) \leq 2 \) for any associative ring \( R \) with 1, where \( GLR = \lim GL_n \cdot R [1] \). In another paper [10] we show that in fact every matrix in \( ER = [GLR, GLR] \) is the product of a commutator in \( ER \) and a (finite) even permutation matrix (which is the commutator of even permutation matrices contained in \( ER \)).

Remark 22. By an analogue of Lemma 18 it follows that every element of \( ER \) is the product of four triangular matrices with the first being upper
triangular. Thus $ut_x(R) \leq 4$ for any ring $R$. A remark in [22, p. 358] also makes this assertion.

Remark 23. Wood [26] showed that $c(G) = \infty$ for the universal covering $G$ of the Lie group $SL_2 R$. A Borel (unpublished) observed that this can be generalized to any infinite covering of any simple Lie group.

Remark 24. There are several papers in which $c(GL_1 R)$ is investigated for $R$ rings of operators. The question which appears to be of most interest is determining whether or not this number is 1. For example, it is claimed in [2] that $c(GL_1 R) \leq 24$ when $R$ is the ring of compact operators on a complex Hilbert space. Our results are not directly applicable to rings $R$ without 1, but we hope that our methods can be extended to improve this result.

REFERENCES