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Generalized metric properties and kernels of set-valued maps ☆

Jiling Cao

Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland 1, New Zealand

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Dedicated to Professor Jun-iti Nagata for his 77th birthday

Abstract

We study some generalized metric properties near to stratifiability. It is shown that every upper semicontinuous set-valued map from a $\mathcal{G}$-space into a k-semistratifiable space has a compact kernel at every point of its domain.

MSC: 54E20; 54C60

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1. Preliminaries

The motivation of this paper is from a classical problem of Choquet on set-valued maps. To describe this problem, let $T : X \to 2^Y$ be a set-valued map, and $x \in X$ a point, where 2^Y is the power set of Y. Recall that T is *upper semicontinuous at* x if for each open set V containing $T(x)$, there exists an open neighborhood N of x such that $T(N) \subseteq V$. And T is said to be *upper semicontinuous on* X iff it is upper semicontinuous at every point of X. Moreover, a subset $K \subseteq X$ is said to be a *kernel of* T *at* x if for each open set V containing K, there is a neighborhood N of x with $T(N) \subseteq V \cup T(x)$. In [4, p. 70], Choquet stated (without a proof) that if T is upper semicontinuous at x, X and Y are metric spaces, then

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E-mail address: cao@math.auckland.ac.nz (J. Cao).

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T has a compact kernel at x. In [5,6], Dolecki introduced the notion of the *active boundary of T at x*, which is denoted by $\operatorname{Frac} T(x)$, and is defined by

$$\operatorname{Frac} T(x) = \bigcap_{N \in \mathcal{N}(x)} \overline{T(N) \setminus T(x)},$$

where $\mathcal{N}(x)$ is the set of all neighborhoods of x and $\overline{T(N) \setminus T(x)}$ denotes the closure of $T(N) \setminus T(x)$ in Y. Suppose that T is upper semicontinuous at a point $x \in X$ which has a countable local base: Dolecki [5] showed that $\operatorname{Frac} T(x)$ is compact when Y is a metric space; Dolecki and Rolewicz [7] showed that $\operatorname{Frac} T(x)$ is a subset of $T(x)$ when Y is first countable; and Dolecki and Lechicki [8] further showed that $\operatorname{Frac} T(x)$ is a kernel of T at x in the case when Y is metric. Thus, we conclude that T has a compact kernel at x if T is upper semicontinuous at x, X has a countable local base at $x \in X$ and Y is a metric space. This is the so-called Choquet–Dolecki theorem in the literature. Some of applications of this theorem can be found in [1]. A general problem which arises in this direction is: *For which classes of topological spaces containing X and Y, respectively, can we determine that an upper semicontinuous set-valued map $T : X \to 2^Y$ has compact kernels*? In fact, the metrizability of Y can be reduced to the agelicity, as Hansell et al. showed in [11], which implies that the range space Y can be the function space $C_{\mathrm{p}}(Z)$ for any compact Hausdorff Z. Moreover, Dolecki and Lechicki [8] also showed that $\operatorname{Frac} T(x)$ is compact when Y is Dieudonné complete.

In a recent paper, Cao et al. [3] studied a type of topological game, called the $\mathcal{G}(\mathcal{F})$-game (where $\mathcal{F}$ is a filterbase on a space), and the associated property $(**)$ defined by this game. It is shown in [3] that if T is upper semicontinuous from a $\mathcal{G}$-space X into a space Y with property $(**)$, then $\operatorname{Frac} T(x)$ is compact at every point $x \in X$. Furthermore, it is also shown in [3] that all Dieudonné complete spaces (thus all metrizable spaces) and all function spaces $C_{\mathrm{p}}(Z)$, where Z is compact Hausdorff, have property $(**)$. It is natural for us to consider the following question: *Can the metrizability of those spaces in the Choquet–Dolecki theorem be weakened to some generalized metric properties in sense of* [10]? The main goal of the present paper is to consider this question. It is discovered that the class of k-semistratifiable spaces introduced by Lutzer [12] is important for our purpose. In particular, the following theorem shall be proved.

Theorem 1.1. *Let $T : X \to 2^Y$ be an upper semicontinuous set-valued map from a $\mathcal{G}$-space X into a k-semistratifiable space Y. Then T has a compact kernel at every point of X.*

We need the following definition in the sequel.

Definition 1.2 [10,12]. A topological space X is said to be *semistratifiable* if for each closed set $F \subseteq X$, we can assign a decreasing sequence $(U(n, F))_{n \in \omega}$ of open subsets of X such that

(i) $F = \bigcap_{n \in \omega} U(n, F)$,
(ii) $U(n, F) \subseteq U(n, H)$ for all $n \in \omega$, whenever F and H are closed sets of X with $F \subseteq H$.

In addition, if also

(iii) if K is compact and H is closed with $K \cap H = \emptyset$, then there exists some $n \in \omega$ with $K \cap U(n, H) = \emptyset$,

then X is called *k-semistratifiable*. If (iii) is replaced by

(iii)′ $F = \bigcap_{n\in\omega} \overline{U(n, F)}$,

then X is called *stratifiable*.

Every stratifiable space is k-semistratifiable, and every k-semistratifiable space is semistratifiable. However, none of these implications is reversible in general. The proof of Theorem 1.1 will be given in Section 2. To do this, we shall need some special properties of k-semistratifiable spaces. In particular, we shall show that every k-semistratifiable space has property $(**)$. In the last section, we shall give some miscellaneous results on k-semistratifiable spaces, which are by-products of our investigation. Throughout the paper, all topological spaces are assumed to be regular and T_1.

2. Proof of Theorem 1.1

For a space X, let $o(X)$ be the family of all nonempty open subsets of X. In order to prove Theorem 1.1, we shall first give some useful properties of k-semistratifiable spaces.

Lemma 2.1 [9]. *A space X is k-semistratifiable if and only if there exists a map $g : \omega \times X \to o(X)$ such that* (i) $\bigcap_{n\in\omega} g(n, x) = \{x\}$ *for every $x \in X$, and* (ii) *for any two sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ in X with $x_n \to x$ and $x_n \in g(n, y_n)$ for all $n \in \omega$, $y_n \to x$.*

Property $(**)$ in [3] is defined by a topological game called the $\mathcal{G}(\mathcal{F})$-*game* played in a space X, where $\mathcal{F}$ is a (non-trivial) filterbase on X. Now, we briefly recall this game and its relevant concepts. The players of this game are α and β. Player α goes first (always) and chooses a point $x_0 \in X$, then β must respond by choosing a member $F_0 \in \mathcal{F}$. Following this, α must select another (possibly the same) point $x_1 \in F_0$ and in turn β must again respond to this by choosing a member (possibly the same) $F_1 \in \mathcal{F}$. When the players repeat this procedure infinitely many times, they produce a play

$$\mathbf{P} = \{(x_n, F_n):\ x_{n+1} \in F_n,\ n \in \omega\}$$

of the $\mathcal{G}(\mathcal{F})$-game. We shall say that β *wins* this play if the sequence $(x_n)_{n\in\omega}$ has an accumulation point in X. Otherwise, α is said to have *won* this play. A strategy for β is a map $\sigma : \mathbf{S}_{\mathrm{fin}}(X) \to \mathcal{F}$, where $\mathbf{S}_{\mathrm{fin}}(X)$ is the set of all finite sequences in X. We shall call a finite sequence $(x_0, x_1, \dots, x_k)$ or an infinite sequence $(x_n)_{n\in\omega}$ a σ-*sequence* if $x_{i+1} \in \sigma(x_0, \dots, x_i)$ for all $0 \leqslant i \leqslant k-1$ or all $i \in \omega$, respectively. A strategy σ for β is called *a winning strategy* if each infinite σ-sequence has an accumulation point in X. Finally, we shall call the pair $(\mathcal{F}, \sigma)$ a Σ-*filter* if $\mathcal{F}$ is a filter in X and σ is a winning strategy for β in the $\mathcal{G}(\mathcal{F})$-game. The space X is said to have *property* $(**)$ if for every Σ-filter $(\mathcal{F}, \sigma)$ in X, $\mathcal{F}$ has an accumulation point, i.e., $\bigcap\{\overline{F}:\ F \in \mathcal{F}\} \neq \emptyset$. It is proved in [3] that property $(**)$ is stable with respect to some basic topological operations: It is preserved by perfect images, it is arbitrarily productive and hereditary to closed subspaces.

Lemma 2.2. *Let X be a space in which each singleton is a G_δ-set, and let $(x_n)_{n\in\omega}$ be an infinite sequence in X. If every subsequence of $(x_n)_{n\in\omega}$ has an accumulation point in X, then $(x_n)_{n\in\omega}$ must have a convergent subsequence.*

Theorem 2.3. *Every k-semistratifiable space X has property $(**)$.*

Proof. Let $g:\omega\times X\to o(X)$ be a map as described in Lemma 2.1. We can further require that $g(n+1,x)\subseteq g(n,x)$ holds for every $(n,x)\in\omega\times X$. For each nonempty subset $A\subseteq X$ and $n\in\omega$, as usual, we define $\mathrm{st}(A,n)$ by

$$\mathrm{st}(A,n)=\bigcup\bigl\{g(n,y)\colon\ A\cap g(n,y)\neq\emptyset \text{ and } y\in X\bigr\}.$$

To show that X has property $(**)$, let $(\mathcal{F},\sigma)$ be a Σ-filter on X.

Claim 1. $\bigcap_{n\in\omega}\overline{F_n}\neq\emptyset$ *for each sequence $(F_n)_{n\in\omega}$ in $\mathcal{F}$.*

Proof. Fix a sequence $(F_n)_{n\in\omega}$ in $\mathcal{F}$. Since σ is a winning strategy for Player β, then there exists a σ-sequence $(x_n)_{n\in\omega}$ in X such that for every $n\in\omega$,

$$x_{n+1}\in\sigma(x_0,\dots,x_n)\cap\biggl(\bigcap_{0\leqslant i\leqslant n}F_i\biggr).$$

It follows that $(x_n)_{n\in\omega}$ has an accumulation point $x_*\in\bigcap_{n\in\omega}\overline{F_n}$. □

Claim 2. *For each pair $(H,n)\in\mathcal{F}\times\omega$, there exist an element $F(H,n)\in\mathcal{F}$ and a finite set $A(H,n)\subseteq H$ such that $\overline{F(H,n)}\subseteq\mathrm{st}(A(H,n),n)$.*

Proof. Suppose that Claim 2 is false. Then there is a pair $(H_*,n_*)\in\mathcal{F}\times\omega$ such that for each $F\in\mathcal{F}$ and each finite set $A\subseteq H_*$, $\overline{F}\not\subseteq\mathrm{st}(A,n_*)$. We start with any point $x_0(=y_0)\in H_*$. By assumption, $\overline{\sigma(x_0)\cap H_*}\not\subseteq\mathrm{st}(\{x_0\},n_*)$. Thus, there exist points $x_1,y_1\in X$ such that $y_1\in\overline{\sigma(x_0)\cap H_*}\setminus\mathrm{st}(\{x_0\},n_*)$ and $x_1\in\sigma(x_0)\cap H_*\cap g(n_*+1,y_1)$. Continuing this procedure inductively, we produce sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ in X such that

$$\text{(i)}\qquad y_{n+1}\in\overline{H_*\cap\biggl(\bigcap_{\substack{0\leqslant j\leqslant n,\\0\leqslant i_0\leqslant\cdots\leqslant i_j\leqslant n}}\sigma(x_{i_0},\dots,x_{i_j})\biggr)}\setminus\mathrm{st}\bigl(\{x_0,\dots,x_n\},n_*\bigr),$$

$$\text{(ii)}\qquad x_{n+1}\in H_*\cap g(n_*+n+1,y_{n+1})\cap\biggl(\bigcap_{\substack{0\leqslant j\leqslant n,\\0\leqslant i_0\leqslant\cdots\leqslant i_j\leqslant n}}\sigma(x_{i_0},\dots,x_{i_j})\biggr)$$

for every $n\in\omega$. Since σ is a winning strategy for Player β, each subsequence of $(x_n)_{n\in\omega}$ must have an accumulation point in X. By Lemma 2.2, $(x_n)_{n\in\omega}$ has a convergent subsequence $(x_{n_k})_{k\in\omega}$, which is convergent to a point $x\in X$. On the other hand, from (ii),

$x_n \in g(n, y_n)$ for all $n \in \omega$. Hence, $(y_{n_k})_{k\in\omega}$ is also convergent to x. Choose some elements $k_0 \in \omega$ such that $x_{n_k}, y_{n_k} \in g(n_*, x)$ for all $k \geqslant k_0$. It follows that $y_{n_{k+1}} \in \mathrm{st}(\{x_{n_k}\}, n_*)$ whenever $k \geqslant k_0$. This contradicts with (i). □

Now, fix any point $q \in X$ and put $A_{-1} = \{q\}$. Applying Claim 2 to the pair $(\sigma(q), 0)$, we obtain a finite set $A_0 \subseteq \sigma(q)$ and an element $F_0 \in \mathcal{F}$ such that $\overline{F_0} \subseteq \mathrm{st}(A_0, 0)$. Repeating this procedure infinitely many times inductively, one can construct a sequence $(F_n)_{n\in\omega}$ in $\mathcal{F}$ and a sequence $(A_n)_{n\in\omega}$ of finite sets of X such that for every $n \in \omega$,

(iii) $$A_n \subseteq \bigcap_{\substack{0\leqslant j\leqslant n,\\ -1\leqslant i_0\leqslant\cdots\leqslant i_j\leqslant n-1}} \bigl\{\sigma(x_{i_0}, \ldots, x_{i_j}) \colon (x_{i_0}, \ldots, x_{i_j}) \in A_{i_0} \times \cdots \times A_{i_j}\bigr\},$$

(iv) $\overline{F_n} \subseteq \mathrm{st}(A_n, n)$.

Then, by Claim 1, we have $\bigcap_{n\in\omega} \overline{F_n} \neq \emptyset$. Next, define the set K of X as

$$K = \bigcap_{n\in\omega} \overline{F_n} \setminus \bigcap_{F\in\mathcal{F}} \overline{F}.$$

Without loss of generality, we may assume that $K \neq \emptyset$. Otherwise, we obtain $\bigcap_{F\in\mathcal{F}} \overline{F} = \bigcap_{n\in\omega} \overline{F_n} \neq \emptyset$, and the proof is completed already.

Claim 3. *For each point $p \in K$, there exist $n(p) \in \omega$, $x(p) \in A_{n(p)}$ and $H(p) \in \mathcal{F}$ such that $p \in \mathrm{st}(\{x(p)\}, n(p))$ and $H(p) \cap \mathrm{st}(\{x(p)\}, n(p)) = \emptyset$.*

Proof. Suppose that Claim 3 is false. Then there exists a point $p \in K$ such that for each $n \in \omega$ and each $x \in A_n$ with $p \in \mathrm{st}(\{x\}, n)$ and each $H \in \mathcal{F}$, we have $\mathrm{st}(\{x\}, n) \cap H \neq \emptyset$. Since $p \notin \bigcap_{F\in\mathcal{F}} \overline{F}$, we can choose an element $F_* \in \mathcal{F}$ with $p \notin \overline{F_*}$. By (iv), there exists a sequence $(x_n)_{n\in\omega}$ in X such that $x_n \in A_n$ and $p \in \mathrm{st}(\{x_n\}, n)$ for every $n \in \omega$. Therefore, there exists a sequence $(z_n)_{n\in\omega}$ in X such that $p, x_n \in g(n, z_n)$ for each $n \in \omega$. It follows from Lemma 2.1 that $z_n \to p$. Moreover, by (iii), each subsequence of $(x_n)_{n\in\omega}$ is a σ-sequence, thus has an accumulation point in X. Since p is the only accumulation point of $(z_n)_{n\in\omega}$, we conclude that $x_n \to p$. Next, by applying our assumption inductively, with any first move y_0 of Player α, we can construct a σ-sequence $(y_n)_{n\in\omega}$ in X which satisfies the following condition

$$y_n \in \mathrm{st}\bigl(\{x_n\}, n\bigr) \cap F_* \cap \Biggl(\bigcap_{\substack{0\leqslant j\leqslant n-1,\\ 0\leqslant i_0\leqslant\cdots\leqslant i_j\leqslant n-1}} \sigma(y_{i_0}, \ldots, y_{i_j})\Biggr)$$

for all $n \geqslant 1$. Choose another sequence $(w_n)_{n\in\omega}$ in X such that $x_n, y_n \in g(n, w_n)$ for every $n \in \omega$. By k-semistratifiability of X, $w_n \to p$. By the construction of $(y_n)_{n\in\omega}$, each of its subsequences is a σ-sequence, and thus has an accumulation point in X. By Lemma 2.2, $(y_n)_{n\in\omega}$ has a convergent subsequence $(y_{n_k})_{k\in\omega}$. Suppose that $y_{n_k} \to y_*$. Since $(w_{n_k})_{k\in\omega}$ is convergent to p, then $y_* = p$. It follows that $p \in \overline{F_*}$, as $y_{n_k} \in F_*$ for every $k \in \omega$. But, this contradicts with the fact $p \notin \overline{F_*}$. □

Finally, as $\bigcup_{n\in\omega} A_n$ is a countable set, by applying Claim 3, we can choose a sequence $(H_n)_{n\in\omega}$ in $\mathcal{F}$ such that $(\bigcap_{n\in\omega}\overline{H_n})\cap K=\emptyset$. For every $n\in\omega$, define $L_n=\overline{F_n}\cap\overline{H_n}$. Then, by applying Claim 1 again, we obtain

$$\bigcap_{F\in\mathcal{F}}\overline{F}=\bigcap_{n\in\omega}L_n\neq\emptyset.$$

Therefore, we have shown that the space X has property $(**)$. □

Let X be a space. If Player β has a winning strategy in the $\mathcal{G}(\mathcal{N}(x))$-game for each point $x\in X$, then X is called a *$\mathcal{G}$-space* [2].

Theorem 2.4 [3, Theorem 3.3]. *Let $T:X\to 2^Y$ be an upper semicontinuous set-valued map from a $\mathcal{G}$-space X into a space Y with properties $(**)$. If each $T(x)$ is closed in the G_δ-topology on Y (which is generated by the family of all G_δ-sets of Y as a base), then* Frac $T(x)$ *is a compact kernel for T at each $x\in X$.*

Now, we are ready to prove our main result of this paper which is claimed in Section 1.

Proof of Theorem 1.1. Let $x\in X$ be an arbitrary point. Since every k-semistratifiable space has a G_δ-diagonal, the G_δ-topology on Y is the discrete one. This implies that $T(x)$ is closed in the G_δ-topology on Y. On the other hand, by Theorem 2.3, Y has property $(**)$. Therefore, by applying Theorem 2.4, we conclude that Frac $T(x)$ is a compact kernel of T at x. □

3. Miscellaneous results

In this section, we shall give some results on k-semistratifiable spaces and relevant properties.

Proposition 3.1. *Every k-semistratifiable $\mathcal{G}$-space is stratifiable.*

Proof. Let $g:\omega\times X\to o(X)$ be a map as described in Lemma 2.1. Suppose that there are a point $x\in X$ and a closed subset H in X with $x\notin H$, but $x\in\overline{\bigcup\{g(n,y):\ y\in H\}}$ for every $n\in\omega$. First, we choose some open neighborhood G of x such that $\overline{G}\cap H=\emptyset$. Since X is a $\mathcal{G}$-space, Player β has a winning strategy σ for the $\mathcal{G}(\mathcal{N}(x))$-game. Let Player α's first move be x_0. By our assumption, there must exist some point $y_0\in H$ such that $\sigma(x_0)\cap G\cap g(0,y_0)\neq\emptyset$. Inductively, we can obtain sequences $(x_n)_{n\in\omega}$, $(y_n)_{n\in\omega}$ in X such that for each $n\in\omega$, $y_n\in H$ and

$$x_{n+1}\in G\cap g(n+1,y_{n+1})\cap\left(\bigcap_{\substack{0\leqslant j\leqslant n,\\ 0\leqslant i_0\leqslant\cdots\leqslant i_j\leqslant n}}\sigma(x_{i_0},\ldots,x_{i_j})\right)$$

for every $n\in\omega$. It follows that each subsequence of $(x_n)_{n\in\omega}$ is a σ-sequence, and thus has an accumulation point in X. By Lemma 2.2, $(x_n)_{n\in\omega}$ has a convergent subsequence, say $(x_{n_k})_{k\in\omega}$. Suppose that x_{n_k} is convergent to some point $x_* \in \overline{G}$. Then, by k-semi-stratifiability of X and the construction of $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ in the above, $(y_{n_k})_{k\in\omega}$ is also convergent to x_*, and $x_* \in H$. It follows that $x_* \in \overline{G} \cap H$. We have derived a contradiction. Hence, $x \notin \overline{\bigcup\{g(n,y)\colon\ y \in H\}}$ for some $n \in \omega$. By [10, Theorem 5.8], X is stratifiable. □

Corollary 3.2 [12]. *Every first countable k-semistratifiable space is stratifiable.*

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