Abstract

We have proved that each 3-connected multigraph $G$ without loops and trivial 2-cycles embedded into a compact 2-dimensional manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) \leq 0$ contains a $k$-path, a path on $k$ vertices, such that each vertex of this path has, in $G$, degree at most $6k(1 - \chi(\mathbb{M})/3)$ for even $k \geq 2$, and at most $\lfloor (6k - 2)(1 - \chi(\mathbb{M})/3) \rfloor$ for odd $k \geq 3$, or does not contain any $k$-path. These bounds are shown to be the best possible. Moreover, we have proved that for any graph other than a path no similar bound exists. We also have proved that such multigraphs $G$ of order at least $k \geq 2$ contain subgraphs of order $k$ such that every vertex of these subgraphs has, in $G$, the degree at most $\lfloor (4k + 4)(1 - \chi(\mathbb{M})/3) \rfloor$. This bound is also sharp. For large multigraphs of this type the bounds hold with $\chi(\mathbb{M})$ replaced by 0; these bounds are again sharp. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 05C10; 05C38; 52B10

Keywords: Graphs; Path; Polyhedral map; Embeddings

1. Introduction

This paper continues the investigations of [1,2,8,10]. Some of the definitions of [3,8] are repeated.

All graphs are simple; multigraphs can have loops and multiple edges.

In this paper all manifolds are compact 2-dimensional manifolds. If a graph $G$ is embedded in a manifold $\mathbb{M}$ then the closures of the connected components of $\mathbb{M} - G$ are called the faces of $G$. If each face is a closed 2-cell and each vertex has valence at least three then $G$ is called a map in $\mathbb{M}$, otherwise $G$ is called an embedding into $\mathbb{M}$ or a graph on $\mathbb{M}$. If, in addition, no two faces of a map have a multiply connected
union then \( G \) is called a polyhedral map in \( \mathbb{M} \). This condition on the union of two faces is equivalent to say that any two faces that meet, meet on a single vertex or a single edge. When two faces in a map meet in one of these two ways we say that they meet properly.

Let in the sequel \( S_g \ (\mathbb{N}_q) \) be an orientable (a non-orientable) surface of genus \( g \) (resp. \( q \)).

The degree of a face \( x \) of a graph on \( \mathbb{M} \) is the number of edges incident to \( x \); if \( x \) bounds an edge at both sides then this edge is counted twice. Vertices and faces of degree \( i \) are called \( i \)-vertices and \( i \)-faces, respectively. Let \( v_i(G) \) and \( p_j(G) \) denote the number of \( i \)-vertices and \( j \)-faces, respectively. For a graph \( G \) on \( \mathbb{M} \) let \( V(G), E(G) \) and \( F(G) \) be the vertex set, the edge set and the face set of \( G \), respectively. The degree of a vertex \( A \) in \( G \) is denoted by \( \deg_G(A) \) or \( \deg(A) \) if \( G \) is known from the context. A path and a cycle on \( k \) vertices is defined to be the \( k \)-path and the \( k \)-cycle, respectively. A \( k \)-path passing through vertices \( A_1, A_2, \ldots, A_k \) is denoted by \([A_1, A_2, \ldots, A_k]\) provided that \( A_iA_{i+1} \in E(G) \) for any \( i = 1, 2, \ldots, k - 1 \).

1.1. Light paths of order \( k \)

It is an old classical consequence of the famous Euler’s formula that each planar graph contains a vertex of degree at most 5. A beautiful theorem of Kotzig \[11,12\] states that every 3-connected planar graph contains an edge with degree-sum of its endvertices being at most 13. This result was further developed in various directions and served as a starting point for discovering many structural properties of embeddings of graphs. For example, Ivančo \[5\] has proved that every graph on \( S_g \) of minimum degree at least three contains an edge with degree sum of its end vertices being at most \( 2g + 13 \) if \( 0 \leq g \leq 3 \) and at most \( 4g + 7 \), if \( g \geq 4 \). For results in this topic see e.g. \[1,4,8,13\].

Fabrici and Jendrol’ \[1\] have proved that every 3-connected planar graph \( G \) with a path on \( k \) vertices contains a path \( P_k \) on \( k \) vertices such that each vertex of this path has, in \( G \), degree \( \leq 5k \); the bound \( 5k \) being the best possible. In the same paper \[1\], they have asked if an analogous result can be found for closed 2-manifolds other than the sphere. More precisely, they have asked the following

**Problem 1.** For a given connected graph \( H \) let \( \mathcal{G}(H, \mathbb{M}) \) be the family of all polyhedral maps on a closed 2-manifold \( \mathbb{M} \) with Euler characteristic \( \chi(\mathbb{M}) \) having a subgraph isomorphic with \( H \). What is the minimum integer \( \phi(H, \mathbb{M}) \) such that every polyhedral map \( G \in \mathcal{G}(H, \mathbb{M}) \) contains a subgraph \( K \) isomorphic with \( H \) for which \( \deg_G(A) \leq \phi(H, \mathbb{M}) \) for every vertex \( A \in V(K) \)?

(If such minimum does not exist we write \( \psi(H, \mathbb{M}) = \infty \). If \( \phi(H, \mathbb{M}) < +\infty \) then the graph \( H \) is called light in \( \mathcal{G}(H, \mathbb{M}) \).)
The answer to this question for $S_0$ is contained in

**Theorem 1** (Fabrici and Jendrol’ [1]). Let $k$ be an integer, $k \geq 1$. Then

$$\varphi(P_k, S_0) = 5k,$$

$$\varphi(H, S_0) = \infty \text{ for any } H \neq P_k.$$  

A slight modification of the method used in [1] provides

**Theorem 2** (Fabrici and Jendrol’ [2]). Let $k$ be an integer, $k \geq 1$. Then

$$\varphi(P_k, N_{-1}) = 5k$$

and

$$\varphi(H, N_{-1}) = \infty \text{ for any } H \neq P_k.$$  

For compact 2-manifolds of higher genera we obtained

**Theorem 3** (Jendrol’ and Voss [8]). Let $k$ be an integer, $k \geq 1$, and $M$ be a closed 2-manifold with Euler characteristic $\chi(M) \leq 0$. Then

\begin{itemize}
  \item[(i)]
  $$2 \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{5 + \sqrt{49 - 24\chi(M)}}{2} \right\rfloor \leq \varphi(P_k, M) \leq k \left\lfloor \frac{5 + \sqrt{49 - 24\chi(M)}}{2} \right\rfloor, \quad k \geq 1,$$
  
  and
  \item[(ii)]
  $$\varphi(H, M) = \infty \text{ for any } H \neq P_k.$$
\end{itemize}

In Theorem 3 the upper bound is sharp for even $k \geq 2$. For odd $k \geq 3$ we studied the upper bound in [7,9].

The precise bounds for the torus $S_1$ and the Klein bottle $N_2$ are known.

**Theorem 4** (Jendrol’ and Voss [7,6]). Let $k$ be an integer, $k \geq 1$. Then

$$\varphi(P_k, S_1) = \varphi(P_k, N_2) = \begin{cases} 6k & \text{if } k = 1 \text{ or } k \text{ is even} \\ 6k - 2 & \text{if } k \text{ is odd, } k \geq 3. \end{cases}$$

In this paper, we drop the condition that the considered graphs are simple and have a polyhedral embedding into a compact 2-manifold $M$. These graphs are replaced by 3-connected multigraphs having an embedding into $M$ so that no null-homotopic 1- or 2-cycles occur. Such cycles are also called *trivial*. 
Problem 2. For a given connected graph $H$ let $\mathcal{G}(H, \mathbb{M})$ be the family of all 3-connected graphs which can be embedded on the compact 2-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ having a subgraph isomorphic with $H$. What is the minimum integer $\psi(H, \mathbb{M})$ such that every graph $G \in \mathcal{G}(H, \mathbb{M})$ contains a subgraph $K$ isomorphic to $H$ for which

$$\deg_{G}(A) \leq \psi(H, \mathbb{M}) \quad \text{for every vertex } A \in V(K)?$$

(If such a minimum does not exist we write $\psi(H, \mathbb{M}) = \infty$. If such a minimum does exist $H$ is called light.)

Correspondingly, $\psi^m(H, \mathbb{M})$ is defined for the family $\mathcal{G}^m(H, \mathbb{M})$ of all 3-connected multigraphs without loops and trivial 2-cycles, and $\psi^l(H, \mathbb{M})$ is defined for the family $\mathcal{G}^l(H, \mathbb{M})$ of all 3-connected multigraphs without trivial 1- or 2-cycles; where all multigraphs of these classes are embedded in $\mathbb{M}$. Obviously, $\varphi(H, \mathbb{M}) \leq \psi(H, \mathbb{M}) \leq \psi^m(H, \mathbb{M}) \leq \psi^l(H, \mathbb{M}).$

If $G \in \mathcal{G}^m(H, \mathbb{S}_0)$ or $G \in \mathcal{G}^l(H, \mathbb{S}_0)$ then it has no loops or multiple edges. Since $G$ is 3-connected it is polyhedral. Hence for $\mathbb{M} = \mathbb{S}_0$ the assertion of Theorem 5 follows from Theorem 1. For $\mathbb{M} = \mathbb{N}_1$ we have to prove it.

**Theorem 5.** Let $k$ be an integer, $k \geq 1$. Then for $\mathbb{M} = \mathbb{S}_0$ and $\mathbb{M} = \mathbb{N}_1$ it holds

(i) $\varphi(P_k, \mathbb{M}) = \psi(P_k, \mathbb{M}) = \psi^m(P_k, \mathbb{M}) = \psi^l(P_k, \mathbb{M}) = 5k$, and
(ii) $\varphi(H, \mathbb{M}) = \psi(H, \mathbb{M}) = \psi^m(H, \mathbb{M}) = \psi^l(H, \mathbb{M}) = \infty$ for any $H \neq P_k$.

Zaks [13] proved for all triangulations of $\mathcal{G}^m(P_2, \mathbb{S}_g)$, $g \geq 1.$

$$\psi^m(P_2, \mathbb{S}_g) = 2 \times 6 \left(1 + \frac{|\chi(\mathbb{S}_g)|}{3}\right) = 8g + 4.$$  

This result will be generalized in Theorems 6 and 7.

**Theorem 6.** Let $k$ be an integer, $k \geq 1$, and $\mathbb{M}$ a compact 2-manifold with Euler characteristic $\chi(\mathbb{M}) \leq 0$. Then

(i) $\psi^m(P_1, \mathbb{M}) = \lfloor 6 + \frac{3}{2} |\chi(\mathbb{M})| \rfloor$.
(ii) $\psi^m(P_k, \mathbb{M}) = \begin{cases} 6k(1 + |\chi(\mathbb{M})|/3) & \text{for even } k \geq 2, \\ (6k - 2)(1 + |\chi(\mathbb{M})|/3) & \text{for odd } k \geq 3. \end{cases}$

(iii) $\psi^m(H, \mathbb{M}) = \infty$ for all $H \neq P_k$.

**Theorem 7.** Let $k$ be an integer, $k \geq 1$, and $\mathbb{M}$ a compact 2-manifold with Euler characteristic $\chi(\mathbb{M}) \leq 0$. Then

(i) $\psi^m(P_1, \mathbb{M}) = \psi^l(P_1, \mathbb{M}) = \lfloor 6 + \frac{3}{2} |\chi(\mathbb{M})| \rfloor$. 

(ii) $\psi^m(P_k, \mathcal{M}) = \psi^l(P_k, \mathcal{M})$

\[
\begin{cases}
6k \left( 1 + \frac{|\chi(\mathcal{M})|}{3} \right) & \text{for even } k \geq 2, \\
6k - 2 \left( 1 + \frac{|\chi(\mathcal{M})|}{3} \right) & \text{for odd } k \geq \max \left\{ 3, \frac{|\chi(\mathcal{M})|}{3} + 1 \right\}.
\end{cases}
\]

\[
\left( 6k - 2 \right) \left( 1 + \frac{|\chi(\mathcal{M})|}{3} \right) = \psi^m(P_k, \mathcal{M}) \leq \psi^l(P_k, \mathcal{M}) \leq 6k \left( 1 + \frac{|\chi(\mathcal{M})|}{3} \right)
\]

for odd $k$, $3 \leq k < \frac{|\chi(\mathcal{M})|}{3} + 1$.

(iii) $\psi^m(H, \mathcal{M}) = \psi^l(H, \mathcal{M}) = \infty$ for all $H \neq P_k$.

Bounds for $\psi(P_k, \mathcal{M})$ can be derived from $\varphi(P_k, \mathcal{M}) \leq \psi(P_k, \mathcal{M}) \leq \psi^m(P_k, \mathcal{M})$. But in general these bounds are not sharp. We will deal with better bounds for $\psi(P_k, \mathcal{M})$ in a forthcoming paper.

The result of Theorem 4 is also valid for polyhedral maps on 2-manifolds $\mathcal{M}$ of Euler characteristic $\chi(\mathcal{M}) < 0$, if these maps have enough vertices. Thus the following problem has been investigated.

**Problem 3.** Let $N \geq 1$ be an integer. For a given connected graph $H$ let $\mathcal{G}_N(H, \mathcal{M})$ be the family of all polyhedral maps of order $\geq N$ on a closed 2-manifold $\mathcal{M}$ with Euler characteristic $\chi(\mathcal{M})$ having a subgraph isomorphic with $H$. What is the minimum integer $\varphi_N(H, \mathcal{M})$ such that every polyhedral map $G \in \mathcal{G}_N(H, \mathcal{M})$ contains a subgraph $K$ isomorphic with $H$ for which

$$\deg_G(A) \leq \varphi_N(H, \mathcal{M}) \quad \text{for every vertex } A \in V(K)?$$

Obviously, $\varphi_1(H, \mathcal{M}) = \varphi(H, \mathcal{M})$.

Let $N_k$ denote the largest number of vertices in a connected graph with maximum degree $\leq 6k$ containing no path with $k$ vertices. Obviously, $N_k \leq (6k)^{k/2+2}$.

**Theorem 8** (Jendrol’ and Voss [6]). For any 2-manifold $\mathcal{M}$ with Euler characteristic $\chi(\mathcal{M}) < 0$, any integer $k \geq 1$ and any integer $N > 30000 \left( |\chi(\mathcal{M})| + 1 \right)^3 (N_k + 3(|\chi(\mathcal{M})| + 1))$,

$$\varphi_N(P_k, \mathcal{M}) = \begin{cases}
6k & \text{if } k = 1 \text{ or } k \text{ is even,} \\
6k - 2 & \text{if } k \geq 3 \text{ is odd.}
\end{cases}$$

Here we shall show that the bounds of Theorem 4 are also valid for large graphs of the classes $\mathcal{G}(H, \mathcal{M})$, $\mathcal{G}^m(H, \mathcal{M})$, and $\mathcal{G}^l(H, \mathcal{M})$.

**Problem 4.** Let $N \geq 1$ be an integer. For a given graph $H$, let $\mathcal{G}_N(H, \mathcal{M})$ be the family of all 3-connected graphs of order $\geq N$ which can be embedded on a compact 2-manifold $\mathcal{M}$ with Euler characteristic $\chi(\mathcal{M})$ having a subgraph isomorphic with $H$. 


What is the minimum integer $\psi_N(H, M)$ such that every graph $G \in \mathcal{G}_N(H, M)$ contains a subgraph $K$ isomorphic to $H$ for which
\[ \deg_G(A) \leq \psi_N(H, M) \quad \text{for every vertex } A \in V(K)? \]

Correspondingly, $\psi^m_N(H, M)$ is defined for the family $\mathcal{G}^m_N(H, M)$ of all 3-connected multigraphs on $M$ of order $\geq N$ without loops and trivial 2-cycles, and $\psi_N(H, M)$ is defined for the family $\mathcal{G}_N^l$ of all 3-connected multigraphs on $M$ of order $\geq N$ without trivial 1- and 2-cycles, where all multigraphs of these classes are embedded in $M$.

Obviously, $\varphi_N(H, M) \leq \psi_N(H, M) \leq \psi^m_N(H, M) \leq \psi^l_N(H, M)$.

**Theorem 9.** For any 2-manifold $M$ with Euler characteristic $\chi(M) \leq 0$, any integers $k \geq 1$ and $N > (6k + 1)(2N_k + 1)|\chi(M)|$,
\[ \psi_N(P_k, M) = \psi^m_N(P_k, M) = \psi^l_N(P_k, M) \]
\[ = \begin{cases} 6k & \text{if } k = 1 \text{ or } k \text{ is even,} \\ 6k - 2 & \text{if } k \geq 3 \text{ is odd}. \end{cases} \]

**1.2. Light subgraphs of order $k$**

Fabrici and Jendrol’ [2] have proved that every 3-connected planar graph $G$ of order at least $k$ contains a subgraph on $k$ vertices such that each vertex of this subgraph has, in $G$, a degree $\leq 4k + 3$, for $k \geq 3$. More precisely, for the sphere the following problem has been investigated, which is formulated here more generally.

**Problem 5.** Let $N, k$ be positive integers with $N \geq k$. Let $\mathcal{G}_N(k, M)$ be the family of all polyhedral maps of order $N \geq k$ on a compact 2-manifold $M$ of Euler characteristic $\chi(M)$. What is the minimum integer $\tau_N(k, M)$ such that every graph $G \in \mathcal{G}_N(k, M)$ contains a connected subgraph $H$ of order $k$ such that
\[ \deg_G(A) \leq \tau_N(k, M), \]
holds for every vertex $A \in V(H)$? Let $\tau(k, M) := \tau_N(k, M)$.

For the sphere Euler’s formula gives $\tau(1, S_0) = 5$. Kotzig’s result [11,12] yields $\tau(2, S_0) = 10$.

**Theorem 10 (Fabrici and Jendrol’ [2]).** Let $k$ be an integer, $k \geq 1$. Then
\begin{enumerate}
  
  \item $\tau(1, S_0) = 5$
  
  \item $\tau(2, S_0) = 10$
  
  \item $\tau(k, S_0) = 4k + 3$ for any $k \geq 3$.
\end{enumerate}

Since each connected graph with one, two or three vertices contains a path with one, two or three vertices, respectively, Theorem 2 [1] implies: $\tau(1, N_1) = 5$, $\tau(2, N_1) = 10$, and $\tau(3, N_1) = 15$.
Theorem 11 (Jendrol’ and Voss [10]). Let $k$ be an integer, $k \geq 1$. Then for the projective plane holds:

(i) $\tau(1, N_1) = 5$
(ii) $\tau(2, N_1) = 10$
(iii) $\tau(k, N_1) = 4k + 3$ for any $k \geq 3$.

Theorem 9 of [6] implies $\tau_N(1, M) = 6$ for $N > 6|\chi(M)|$; $\tau_N(2, M) = 12$ and $\tau_N(3, M) = 16$ for large $N$.

Theorem 12 (Jendrol’ and Voss [10]). For any closed 2-manifold $M$ with Euler characteristic $\chi(M) \leq 0$, any integer $k \geq 1$, and any integer $N > (8k^2 + 6k - 6)|\chi(M)|$ it holds:

(i) $\tau_N(1, M) = 6$, and
(ii) $\tau_N(k, M) = 4k + 4$ for any $k \geq 2$.

Moreover, $\tau_N(1, M) = 6$ for $N > 6|\chi(M)|$.

For the newly introduced classes we formulate the analogue problems.

**Problem 6.** Let $N, k$ be positive integers with $N \geq k$. Let $\mathcal{G}_N(k, M)$ be the family of all 3-connected graphs of order $N \geq k$ which can be embedded on the compact 2-manifold $M$ with Euler characteristic $\chi(M)$. What is the minimum integer $\Theta_N(k, M)$ such that every graph $G \in \mathcal{G}_N(k, M)$ contains a connected subgraph $H$ of order $k$ such that $\deg_G(A) \leq \Theta_N(k, M)$ holds for every vertex $A \in V(H)$?

Let $\Theta(k, M) = \Theta_N(k, M)$.

Correspondingly, $\Theta^m_N(k, M)$ is defined for the family $\mathcal{G}^m_N(k, M)$ of all 3-connected multigraphs without loops and trivial 2-cycles, and $\psi^l_N(k, M)$ is defined for the family $\mathcal{G}^l_N(k, M)$ of all 3-connected multigraphs without trivial 1- and 2-cycles, where all multigraphs of these classes are embedded in $M$ and have an order $\geq N$.

In this paper Theorems 13–15 shall be proved.

For the sphere $S_0$ and the projective plane the bounds of Theorems 10 and 11 are again valid.

Theorem 13. Let $k$ be an integer, $k \geq 1$. Then for $M = S_0$ and $M = N_1$ it holds

(i) $\tau(1, M) = \Theta(1, M) = \Theta^m(1, M) = \Theta^l(1, M) = 5$.
(ii) $\tau(2, M) = \Theta(2, M) = \Theta^m(2, M) = \Theta^l(2, M) = 10$.
(iii) $\tau(k, M) = \Theta(k, M) = \Theta^m(k, M) = \Theta^l(k, M) = 4k + 3$ for any $k \geq 3$.

For the classes $\mathcal{G}^m(k, M)$ and $\mathcal{G}^l(k, M)$ the results correspond to the results of Theorems 6 and 7 relating light paths.
Theorem 14. Let $k$ be an integer, $k \geq 1$, and $\mathbb{M}$ a compact 2-manifold with Euler characteristic $\chi(\mathbb{M}) \leq 0$. Then

(i) $\Theta^m(1, \mathbb{M}) = \Theta^l(1, \mathbb{M}) = \left[6 + \frac{3}{2} |\chi(\mathbb{M})|\right]$.

(ii) $\Theta^m(k, \mathbb{M}) = \Theta^l(k, \mathbb{M}) = \left[(4k + 4) \left(1 + \frac{|\chi(\mathbb{M})|}{3}\right)\right]$ for any $k \geq 2$.

Bounds for $\Theta(k, \mathbb{M})$ can be derived from $\Theta(k, \mathbb{M}) \leq \Theta^m(k, \mathbb{M})$. But in general, these bounds are not sharp. We deal with better bounds for $\Theta(k, \mathbb{M})$ in a forthcoming paper.

The following theorem generalizes the result of Theorem 12 for light subgraphs in large graph to light subgraphs in large multigraphs. They correspond to the results for light paths in large multigraphs of Theorem 9.

Theorem 15. For any 2-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \leq 0$, any integer $k \geq 1$ and any integer $N > (8k^2 + 6k - 6)|\chi(\mathbb{M})|$

(i) $\tau^N(1, \mathbb{M}) = \Theta^N(1, \mathbb{M}) = \Theta^m(1, \mathbb{M}) = 6$.

(ii) $\tau^N(k, \mathbb{M}) = \Theta^N(k, \mathbb{M}) = \Theta^m(k, \mathbb{M}) = 4k + 4$ for any $k \geq 2$.

2. Minimum degrees of graphs on $\mathbb{M}$

In the rest of this paper $\chi(\mathbb{M}) \leq 0$.

Let $G$ be a multigraph embedded in a compact 2-dimensional manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$. If $G$ is a map, i.e. each face is a 2-cell, then $G$ fulfils Euler’s formula

$$n - e + f = \chi(\mathbb{M}),$$

where

$$\chi(\mathbb{M}) = \begin{cases} 
2(1 - g) & \text{if } \mathbb{M} = \mathbb{S}_g, \\
2 - q & \text{if } \mathbb{M} = \mathbb{N}_q.
\end{cases}$$

If $G$ contains a face $F$ which is not a 2-cell then add an edge to its interior so that $F$ is not subdivided. Add edges in this way until a 2-cell embedding is obtained. Let $e^*$ denote the number of these edges then Euler’s formula is fulfilled with

$$n - (e + e^*) + f = \chi(\mathbb{M}),$$

where $n$, $e$ and $f$ denote the number of vertices, edges and faces of $G$, respectively. We summarise this in

Lemma 1. Let $G$ be the embedding of a connected multigraph in a compact 2-dimensional manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$. Then the Euler sum is

$$n - e + f \geq \chi(\mathbb{M}),$$

where $n$, $e$ and $f$ denote the number of vertices, edges and faces of $G$, respectively.
Lemma 2. Let $G$ be the embedding of a connected multigraph of order $n$ without 1- or 2-faces in a compact 2-dimensional manifold $M$ of Euler characteristic $\chi(M)$. Then, the number of edges of $G$ is $e \leq 3(n - \chi(M))$, and the minimum degree $\delta$ is $\delta \leq 6 - 6\chi(M)/n$.

Proof. If $G$ has at most three edges then Lemma 2 is true. Next, let $G$ have $\geq 4$ edges. By Lemma 1 we have

$$n - e + f \geq \chi(M). \tag{1}$$

By hypothesis, each face is bounded by at least three edges and

$$3f \leq 2e, \tag{2}$$

where the equality holds, if all faces of the embedding of $G$ are bounded by three edges. Formulae (1) and (2) imply

$$3\chi(M) \leq 3n - 3e + 3f \leq 3n - 3e + 2e,$$

and

$$e \leq 3(n - \chi(M)).$$

With $\delta n \leq 2e$ follows

$$\delta \leq 6 - \frac{6\chi(M)}{n}. \quad \square$$

3. Proofs of Theorems 6(ii) and 7(ii) — upper bounds

The proof follows the ideas of [1,10]. Suppose that there is a counterexample to one of our theorems having $n$ vertices. Let $G$ be a counterexample with the maximum number of edges among all counterexamples having $n$ vertices. A vertex $A$ of the graph $G$ is major (minor) if its degree exceeds (does not exceed) the bound of Theorem 6(ii) or Theorem 7(ii), respectively. There are three cases.

(A) If $k \geq 2$ is an even integer then a major vertex $A$ of $G$ has a degree $\deg_G(A) > k(6 + 2|\chi(M)|)$ (Theorems 6(ii) and 7(ii)).

(B) If $G$ has no loops then a major vertex $A$ of $G$ has a degree $\deg_G(A) > \lfloor (k - \frac{1}{2}) (6 + 2|\chi(M)|) \rfloor$ for all odd $k \geq 3$ (Theorem 6(ii)).

(C) If $G$ has loops then a major vertex $A$ of $G$ has a degree $\deg_G(A) > \lfloor (k - \frac{1}{2}) (6 + 2|\chi(M)|) \rfloor$ for all odd $k \geq \max \{3, |\chi(M)|/3 + 1 \}$ and $\deg_G(A) > k(6 + 2|\chi(M)|)$ for all odd $k$ with $3 \leq k < |\chi(M)|/3 + 1$ (Theorem 7(ii)).

Lemma 3. Each path on $k$ vertices contains a major vertex.

Lemma 4. Every $r$-face $z$, $r \geq 4$, of $G$ is incident only with minor vertices.
Proof. Suppose there is a major vertex $B$ incident with an $r$-face $z$, $r \geq 4$. Let $C$ be a vertex on $z$ diagonal with respect to $B$. Because $G$ is 3-connected we can insert an edge $BC$ into the $r$-face $z$ (Since $G$ is 3-connected no trivial 1- or 2-cycle can be generated by inserting $BC$). The resulting embedding is again a counterexample but with one edge more, a contradiction. \qed

Let $H = H(G)$ and $H' = H'(G)$ be the subgraphs of $G$ induced on all major or minor vertices of $G$, respectively. Our aim is to transform the embedding $G$ into a triangulation $G^*$ with the same set of major vertices so that the degrees of the major vertices do not decrease and the major vertices of $G^*$ induce a triangulation $H^*$ of $\mathcal{M}$. It may be that the triangulation $H^*$ has trivial 1- and 2-cycles, but no 1- or 2-faces.

Case I: The graph $H$ of major vertices has at least three vertices, i.e. $n(H) \geq 3$. Let $K$ be a component of $H'$ which is joined by edges with the vertices $A_1, A_2, \ldots, A_s$, $s \geq 3$, which are major vertices of $G$. (Note $s \geq 3$ because $G$ is 3-connected). Let $K'$ be obtained from $K \cup \{A_1, A_2, \ldots, A_s\}$ by adding all $A_i$, $K$-edges, $i = 1, 2, \ldots, s$.

Each small topological neighbourhood of any vertex is an open 2-cell. Thus, in a small topological neighbourhood of $A_i$ we can introduce a direction around $A_i$ and denote the $A_i$-edges by $k_1, k_2, \ldots$ so that $k_j$ is directly followed by $k_{j+1}$. Let $P_j$ be the endvertex of $k_j$ so that $P_j \neq A_i$ if $k_j$ is not a loop and $P_j = A_i$ if $k_j$ is a loop. Since by Lemma 4 each major vertex is incident only with triangular faces the edges $k_j, k_{j+1}$, and $P_jP_{j+1}$ bound a triangular face for all $j$. This implies

(1) If $k_\mu$ and $k_\nu$ are in a trivial cycle $C$ so that $k_{\mu+1}, k_{\mu+2}, \ldots, k_{\nu-1}$ are in its inner 2-cell part then the endvertices of these edges form a path $[P_\mu, P_{\mu+1}, \ldots, P_{\nu-1}, P_\nu]$.

Let $M_i$ denote the set of all $A_i$-edges joining $A_i$ with a vertex of $K$. Two edges $k_\mu, k_\nu \in M_i$ are said to be $i$-equivalent if $k_\mu, k_{\mu+1}, \ldots, k_{\nu-1}, k_\nu \in M_i$ or $k_\mu, k_{\nu}k_{\nu+1}, \ldots, k_{\mu-1}, k_\mu \in M_i$. The $i$-equivalence is an equivalence relation and $M_i$ is partitioned into equivalence classes. If $T_i = \{k_\mu, \ldots, k_\nu\}$, $\nu \neq \mu - 1$, is such a class then $P_{\mu-1}$ and $P_{\nu+1}$ are not in $K$, moreover, they are major vertices belonging to $A_1, \ldots, A_s$ (note possibly $P_{\mu-1} = A_i$ or $P_{\nu+1} = A_i$).

In the class $T_i = \{k_\mu, k_{\mu+1}, \ldots, k_\nu\}$ of $A_i$-edges we label the edges $k_\mu, k_{\mu+(k-1)}, k_{\mu+2(k-1)}, \ldots$ and their endvertices $P_{\mu}, P_{\mu+(k-1)}, P_{\mu+2(k-1)}, \ldots$. This will be accomplished in each $i$-equivalence class of $A_i$-edges. Let $W_i$ and $W'_i$ denote the sets of all labelled $A_i$-edges and labelled vertices of $K$, respectively. In $K$ the labelled vertices of $\bigcup_i W'_i$ are joined by some tree. From this tree the subgraph $D$ of $K'$ is obtained by adding the labelled $A_i$-edges of $\bigcup_i W_i$ and their endvertices.

Note, trivial 1-cycles are possible in $H$.

We form a (small) open 2-cell $F$ on $\mathcal{M}$ along $D$ completely containing $D'$, where $D'$ is obtained from $D$ by deleting the geometric points $A_1, \ldots, A_s$; so from each $A_i$-edge only the point $A_i$ is deleted. (Note: $D'$ is no subgraph because it contains edges with one missing endvertex). The major vertices $A_1, \ldots, A_s$ appear on the boundary of $F$ in the order $B_1, B_2, \ldots, B_s$; the vertex $A_i$ appears $\alpha_i$ times, where $\alpha_i$ denotes the number of labelled $A_i$-edges of $S_i$, i.e., $A_i = B_{j_1} = B_{j_2} = \cdots = B_{j_{\alpha_i}}$. We delete $K$ together with
all incident edges. Thus, from $K'$ only the (major) vertices $A_1, A_2, \ldots, A_s$ remain. We
form in $F$ a $t$-cycle $[B_1, B_2, \ldots, B_t]$ by introducing new edges $B_iB_{i+1}$, $i = 1, 2, \ldots, t$,
indices modulo $t$. If the old embedding already contains a $B_iB_{i+1}$-edge bounding a 2-cell together
with the new $B_iB_{i+1}$-edge, then delete the new $B_iB_{i+1}$-edge. So if a multiple edge occurs no two edges of them bound a 2-cell of the graph $H$ induced
by the major vertices. (Note: Since $G$ is 3-connected and by our construction each component of the graph induced by the minor vertices is joined to at least three major vertices.)

But trivial 2-cycles are possible in $H$. We have to show

(2) In the graph $H$ induced by the major vertices no 2-cell face is generated bounded
by a loop.

**Proof of (2).** Assume by introducing the cycle $[B_1B_2\ldots B_t]$ a loop $l$ is generated
forming a 2-cell face of $H$. In $G$, this loop $l$ corresponds to a cycle $C$ through a
major vertex $A_i$ containing two $A_i$-edges $k_x$ and $k_y$, where the interior of this cycle
is a 2-cell. By assertion (1) $k_x, k_{x+1}, \ldots, k_y$ are in the interior of $C$ and the vertices
$P_x, P_{x+1}, \ldots, P_y$ form a path $[P_{x}P_{x+1}\ldots P_{y}]$ of $G$.

If $P_{x}$ and $P_{y}$ are in the same equivalence class then $x = \mu + s(k - 1)$ and $y = \mu + t(k - 1)$. Hence $[P_{x}P_{x+1}\ldots P_{y}]$ is a path with at least $k$ minor vertices. A contradiction!

Consequently, $[P_{x}P_{x+1}\ldots P_{y}]$ contains a major vertex which is also in the interior
of the loop $l$. Hence, $l$ is not the boundary of a 2-cell face of $H$. Thus, the proof of
(2) is complete.

Next the interior of $F$ is triangulated by the $t - 3$ new edges $B_1B_i$, $i = 3, 4, t - 1$ (Fig. 1).

Into each triangle $[B_1B_iB_{i+1}]$, $i = 2, 3, \ldots, t$, indices modulo $t$, we insert a generalised
3-star $S_3$ consisting of a central vertex $Z$ and three paths, say $p_1$, $p_2$ and $p_3$, starting
in $Z$ and including $Z$; the path $p_3$ of length $[k/2]$ and the other two paths $p_1$ and
$p_2$ of length $[k/2]$. Then $B_1$ is joined to all $2[k/2] - 1$ vertices of $p_1 \cup p_2$ and $B_i$ and
$B_{i+1}$ are joined to all $[k/2] + [k/2] - 1 = k - 1$ vertices of $p_2 \cup p_3$ or $p_3 \cup p_1$,
respectively. In the resulting embedding the face $F$ is triangulated.
The vertices $B_j$, $j = 2, 3, \ldots, t$, are incident with $\geq \lceil k/2 \rceil + \lfloor k/2 \rfloor - 1 = k - 1$ minor vertices of the 3-stars. The vertex $B_j$ is adjacent with $(t - 2)(2\lfloor k/2 \rfloor - 1) \geq k - 1$ minor vertices of the 3-stars. If $k$ is even or $k \geq 3$ is odd and $t \geq 4$ then $B_i$ is incident with $(t - 2)(2\lfloor k/2 \rfloor - 1) \geq k - 1$ minor vertices of the 3-stars; otherwise $B_j$ is incident with $\geq k - 2$ minor vertices of the 3-star $S_j$.

**Case 1:** Let $k$ be even or $k \geq 3$ be odd and $t \geq 4$.

Each $B_j$ is incident with $\geq k - 1$ minor vertices of the inserted 3-star $S_j$. Since $A_i = B_{j1} = \ldots B_{jn}$, in the new embedding the vertex $A_i$ is adjacent with $\geq x_i(k - 1)$ edges joining $A_i$ with vertices of the 3-stars $S_j$. Recall $x_i$ is the number of labelled $A_i$-edges of $M_i$ in $K'$.

In $G$ each labelled $A_i$-edges of $K'$ is followed by at most $k - 2$ unlabelled $A_i$-edges of $K'$. Hence $K'$ has $\leq x_i(k - 1)$ $A_i$-edges. Thus, $\leq x_i(k - 1)$ $A_i$-edges are replaced by $\geq x_i(k - 1)$ $A_i$-edges. Since no edge joining major vertices is deleted the valencies of the major vertices are not decreased. $K$ is transformed into a set of triangles of $H$ with inserted 3-stars $S_j$.

**Case 2:** Let $k \geq 3$ be odd and $t = 3$. Then $F$ is a triangle-face $[B_1B_2B_3] = [A_1A_2A_3]$.

We consider two cases

**Case 2.1:** The component $K$ contains only trivial cycles. Hence, $K$ can be embedded into $F$. This can be done by adding successively paths of $K$ having with the already constructed embedding only the endvertices in common. Thus, $F$ is a triangle-face containing the component $K$ of major vertices. By Lemma 10 from our paper [6] we know that one of the vertices $B_1, B_2$ or $B_3$ is adjacent only with $\leq k - 2$ minor vertices of $K$. W.l.o.g. let $B_1$ be this vertex. Hence, replacing $K$ by $S_3$ we do not decrease the valencies of $B_1, B_2$ and $B_3$.

**Case 2.2:** $K$ contains a nontrivial cycle $C$. The face $F$ will be constructed in a different way. In $G[K \cup \{B_i\}]$ we join $B_i$ with $C$ by a path, in $G[K \cup \{B_j\}]$ we join $B_j$ with the already constructed subgraph by a path, and in $G[K \cup \{B_k\}]$ we join $B_k$ with the already constructed subgraph by a path. The obtained subgraph is again denoted by $D$ (an example of $D$ is depicted in Fig. 2).

We form again a (small) open part $P$ of $M$ along $D$ completely containing $D'$, where $D'$ is obtained from $D$ by deleting the geometric points $B_1, B_2$ and $B_3$; so from each $B_i$-edge only the point $B_i$ is deleted. (Note: $D'$ is no subgraph, because it contains “edges” with one missing endvertex.) Then we obtain a triangle (a 2-cell) with a deleted disc with bounding curve $c$ which corresponds to the former cycle $C$. On the boundary of $P$ the vertices $B_1, B_2$ and $B_3$ appear (Fig. 3).

Next, the curve $c$ is enlarged so that $c$ touches $A_3$, and $c \setminus \{A_3\}$ lies completely in the interior part of $P$. The new face is again denoted by $F$ (see Fig. 4).

The major vertices appear on the boundary of $F$ in the order $B_1, B_2, B_3, B_4$, where $B_3 = B_4$ (see Fig. 4). We delete $K$ together with all incident edges. Thus, from $K'$ only the major vertices $A_1, A_2, A_3$ remain. We form in $F$ a 4-cycle $[B_1B_2B_3B_4]$ by introducing new edges $B_iB_{i+1}$, $i = 1, 2, 3, 4$, indices modulo 4; where the new $B_iB_{i+1}$-edge is introduced if and only if it does not form a 2-face with an old $B_iB_{i+1}$-edge. We triangulate $F$ by introducing the edge $B_1B_3$ and proceed as in the case $t \geq 4$. 


Finally, $K$ is transformed into two triangles of $H$ each of them containing 3-star $S_3$ so that the number of edges joining minor vertices with the major vertices $A_i$ is not decreased. Since no edge joining two major vertices is deleted the valency of no major vertex is decreased. This procedure will be accomplished with all components $K$ of the subgraph $H'$ of $G$ induced by the minor vertices.

In all cases the obtained embedding is denoted by $\tilde{G}$, the subgraph of $\tilde{G}$ induced by the major vertices is $\tilde{H}$, where $V(\tilde{H}) = V(H)$. Thus, each component $K$ of $H'$ is replaced by some generalized 3-stars $S_3$ each lying in a triangle of $\tilde{H}$. By our construction the degrees of the vertices of $V(\tilde{H}) = V(H)$ did not decrease. Perhaps, $\tilde{G}$ is no longer a 2-cell embedding. We add successively a maximum number of edges so that the number of faces remains unchanged. Thus, a 2-cell embedding is obtained. Each face with more than three vertices is triangulated. In each free triangle a generalized 3-star is placed and joined with the vertices of the triangle as before. Thus, a triangulation $G^*$ of $M$ is obtained, where the subgraph $H^*$ induced by the major vertices is also a triangulation of $M$ its vertex set $V(H^*) = V(H)$ and $H$ is a subgraph of $H^*$. In $G^*$
trivial 1- and 2-cycles may occur; but there are no 1- or 2-faces. The triangulation $H^*$ satisfies the equation
\[ 2e(H^*) = 3f(H^*) \]
and Euler’s formula
\[ n(H^*) - e(H^*) + f(H^*) = \chi(M). \]
Hence,
\[ f(H^*) = 2(n(H^*) - \chi(M)) \tag{3} \]
and
\[ e(H^*) = 3(n(H^*) - \chi(M)) \tag{4} \]
The edges joining vertices of $H$ and the edges joining minor vertices with major vertices in $G$ contribute to the degree sum $\sum_{A \in V(H)} \deg_G(A)$. The number of the latter edges is not larger than the number of edges joining minor vertices with major vertices in $G^*$. Each generalized 3-star $S_3$ is joined by $3k - 3 - \varepsilon$ edges with major vertices, where $\varepsilon = 0$ if $k \geq 2$ is even and $\varepsilon = 1$ if $k \geq 3$ is odd. Consequently,
\[ \sum_{A \in V(H)} \deg_G(A) \leq \sum_{A \in V(H)} \deg_H(A) + f(H^*)(3k - 3 - \varepsilon) \]
and with (3)
\[ \sum_{A \in V(H)} \deg_G(A) \leq 2e(H) + (6k - 6 - 2\varepsilon)(n(H^*) + |\chi(M)|). \tag{5} \]
With $e(H) \leq e(H^*)$ and (4) we have
\[ \sum_{A \in V(H)} \deg_G(A) \leq (6 + 2(3k - 3 - \varepsilon))(n(H^*) + |\chi(M)|) \]
and
\[ \sum_{A \in V(H)} \deg_G(A) \leq (6k - 2\varepsilon)(n(H^*) + |\chi(M)|). \tag{6} \]
Inequality (6) implies the existence of a major vertex $B$ of degree
\[ \deg_G(B) \leq (6k - 2\varepsilon) \left(1 + \frac{|\chi(M)|}{n^*}\right) \leq (6k - 2\varepsilon) \left(1 + \frac{|\chi(M)|}{3}\right). \]
This contradiction completes the proof of the theorems in Case I.

Case II: The graph $H$ of major vertices has at most two vertices, i.e. $n(H) \leq 2$. If $n(H) = 0$ then the theorems are obviously true. If $n(H) \in \{1, 2\}$ the method of Case I cannot be applied. Since $G$ is 3-connected it has $n(G) \geq 4$ vertices and the subgraph $H'$ of minor vertices has precisely one component, say $K$. If $k = 2$ then the component $K$ has $n(K) = 1$ vertices. Hence, $4 \leq n(G) = n(H) + n(K) \leq 2 + 1$. This contradiction shows that $k \geq 3$.

Let $V(H) = \{A_1, \ldots, A_{n^*}\}$, where $n^* = n(H)$. We again construct a subgraph $D$ of $K'$ having the same properties as in Case I where $K'$ is obtained from $K \cup V(H)$ by
adding all $H,K$-edges. Again a 2-cell face $F$ is formed containing $D$ with bounding cycle $[B_1B_2\ldots B_t]$, $B_i \in \{A,B\}$, $t \geq 1$, if $t = 1$ then $D$ is a path joining the single major vertex of $K'$ with some inner vertex of $K'$. The new graph is denoted by $\tilde{G}$, the subgraph of $\tilde{G}$ induced by the major vertices is $\tilde{H}$, where $V(\tilde{H}) = V(H)$. Perhaps $\tilde{G}$ is no longer a 2-cell embedding. Then successively a maximum number of edges is added so that the number of faces remains unchanged. Thus, a 2-cell embedding is obtained. Next all faces $F'$, $F' \neq F$, with more than three edges are triangulated. The obtained graph is denoted by $G^*$, the subgraph of major vertices by $H^*$, where $V(H^*) = V(\tilde{H}) = V(H)$.

In $G$ each labelled $A_i$-edge of $K'$ is followed by at most $k - 2$ unlabelled $A_i$-edges of $K'$. Hence $H' = K$ is joined with the vertices of $H$ by $\leq t(k - 1)$ edges, and

$$\sum_{A \in V(H)} \deg_G(A) \leq 2e(H) + t(k - 1) \quad \text{where } t \geq 1 \text{ is an integer.} \quad (7)$$

$H^*$ is a 2-cell embedding with one face $F$ of valence $t$ and all other faces of valence 3 (Note it can be $t = 1$ or $t = 2$). Hence $t + (f^* - 1)3 = 2e^*$, and

$$3f^* = 2e^* - (t - 3), \quad (8)$$

where $n^*, e^*$ and $f^*$ denote the number of vertices, edges, and faces of $H^*$, respectively, $1 \leq n^* \leq 2$.

An upper bound for the degree $t$ of $F$ is obtained when $K_1$ or $K_2$ is embedded into $\mathbb{M}$ and a maximum number of edges is added so that no second face is generated. Thus, $|\chi(\mathbb{M})| + 2$ edges are added. Since each edge of this map bounds the only face from both sides, the valence of this face is $2(|\chi(\mathbb{M})| + 2 + (n(K_1) - 1))$. Hence,

$$1 \leq t \leq 2(|\chi(\mathbb{M})| + n^* + 1). \quad (9)$$

By Euler’s formula $n^* - e^* + f^* = \chi(\mathbb{M})$. This implies with (8)

$$e(H) \leq e(H^*) = e^* = 3|\chi(\mathbb{M})| + 3n^* - (t - 3). \quad (10)$$

With $k \geq 3$ the assertion (7), (10) and (9) imply

$$\sum_{A \in V(H)} \deg_G(A) \leq 2e(H) + t(k - 1) \leq 2k(|\chi(\mathbb{M})| + n^* + 1). \quad (11)$$

With $1 \leq n^* \leq 2$ there exists a major vertex $B \in V(H)$ with

$$\deg_G(B) \leq \frac{k}{n^*}(2|\chi(\mathbb{M})| + 2n^* + 2) \leq k(2|\chi(\mathbb{M})| + 4) \leq k(2|\chi(\mathbb{M})| + 6). \quad (12)$$

For $k \geq 1 + |\chi(\mathbb{M})|/3$ the following inequality holds:

$$\deg_G(B) \leq k(2|\chi(\mathbb{M})| + 4) \leq [(k - \frac{1}{3})2|\chi(\mathbb{M})| + 6)]. \quad (13)$$

Inequalities (12) and (13) imply that the major vertex $B$ has degree

$$\deg_G(A) \leq \begin{cases} \frac{(k - \frac{1}{3})(2|\chi(\mathbb{M})| + 6)}{k(2|\chi(\mathbb{M})| + 6)} & \text{for odd } k \geq 1 + \frac{|\chi(\mathbb{M})|}{3}, \\ \frac{(k - \frac{1}{3})(2|\chi(\mathbb{M})| + 6)}{k(2|\chi(\mathbb{M})| + 6)} & \text{otherwise}. \end{cases} \quad (14)$$
Eq. (14) contradicts conditions (A) and (C). These contradictions complete the proofs of Theorem 6(ii) for all even \( k \geq 2 \) and of the Theorem 7(ii) for all \( k \geq 2 \).

Next, let \( G \) be a multigraph without loops.

If \( n(H) = n^* = 1 \) then \( e(H) = 0 \) and with (7) the only major vertex \( B \) of \( H \) has a degree

\[
\deg_G(B) \leq t(k - 1) \leq (k - 1)2(|\mathcal{M}(\mathcal{M})| + 2) \leq \left( 6k - 2 \right) \left( 1 + \frac{|\mathcal{M}(\mathcal{M})|}{3} \right).
\]

If \( n(H) = n^* = 2 \) then (11) implies the existence of a vertex \( B \) with

\[
\deg_G(B) \leq k(|\mathcal{M}(\mathcal{M})| + 3) \leq \left( 6k - 2 \right) \left( 1 + \frac{|\mathcal{M}(\mathcal{M})|}{3} \right).
\]

These contradictions with condition (B) complete the proof of Theorem 6(ii) for all odd \( k \geq 3 \).

**Remark.** The triangulation \( G^* \) of Case I with \( n^* \geq 3 \) major vertices consists of the triangulation \( H^* \) of major vertices, and in each triangular face of \( H^* \) lies precisely on one component of the subgraph \( H^{*'} \) of minor vertices (a generalized 3-star \( S_3 \)) which contributes to the degree of each major vertex of its boundary at most \( k - 1 \). Hence,

\[
\delta(G^*) \leq \delta(H^*) + (k - 1)\delta(H^*) = k\delta(H^*).
\]

With Lemma 2 we have

\[
\delta(H^*) \leq \left[ 6 \left( 1 + \frac{|\mathcal{M}(\mathcal{M})|}{n^*} \right) \right] = 6 + \left[ \frac{6|\mathcal{M}(\mathcal{M})|}{n^*} \right]
\]

and \( G \) and \( G^* \) contain a vertex \( B \) of degree

\[
\deg_G(B) \leq \deg_{G^*}(B) \leq k \left( 6 + \left[ \frac{6|\mathcal{M}(\mathcal{M})|}{n^*} \right] \right).
\]

With \( n^* = 3 \) for even \( k \geq 2 \) we obtain the same bound as in Case I, and for odd \( k \geq 3 \) we have a bound worse than in Case I.

4. Proof of Theorem 9 — upper bounds

Suppose there is a counterexample to our theorem having \( n \geq N \) vertices. Let \( G \) be a counterexample with the maximum number of edges among all counterexamples having \( n \) vertices. A vertex \( A \) of \( G \) is major (minor) if \( \deg_G(A) > 6k - 2\varepsilon \) (\( \leq 6 - 2\varepsilon \), respectively), where \( \varepsilon = 0 \) for even \( k \geq 2 \) and \( \varepsilon = 1 \) for odd \( k \geq 3 \). If \( G \) is a 3-connected multigraph of \( \mathcal{G}_N^m(P_k, \mathcal{M}) \cup \mathcal{G}_N^l(P_k, \mathcal{M}) \) with one or two major vertices then the subgraph of minor vertices of \( G \) has precisely one component, say \( K \). The vertex number of \( G \) is \( n(G) = n(H) + n(K) \leq 2 + N_k < N \). This contradiction implies that the 3-connected multigraph \( G \in \mathcal{G}_N^m(P_k, \mathcal{M}) \cup \mathcal{G}_N^l(P_k, \mathcal{M}) \) has at least three major vertices. According to Case I of Section 3, we transform \( G \) into a triangulation \( G^* \) with subtriangulation
\(H^*\) generated by the major vertices of \(G^*\). Thus, (6) is valid again
\[
\sum_{A \in V(H)} \deg_G(A) \leq (6k - 2\varepsilon)(n(H^*) + |\chi(M)|).
\] (1)

Hence, there exists a vertex \(B \in V(H)\) of degree
\[
\deg_G(B) \leq (6k - 2\varepsilon) \left( 1 + \frac{|\chi(M)|}{n(H^*)} \right).
\] (2)

Next, a bound for \(n(H^*)\) will be derived. By the construction of Section 3 the number of vertices of \(H\) remain unchanged because \(V(H^*) = V(H)\). Further, each component of \(H'\) has been replaced by one or more components of the subgraph induced by the minor vertices. Hence, the number of faces did not decrease and \(f(H) \leq f(H^*)\).

By the definition of \(N_k\) each component of \(H'\) has \(N_k\) vertices. Hence, with \(f(H^*) \leq 2(n(H^*) + |\chi(M)|)\)
\[
n = n(G) \leq n(H) + N_k f(H) \leq n(H^*) + N_k f(H^*)
\]
\[
\leq n(H^*) + 2N_k(n(H^*) + |\chi(M)|).
\]

Consequently,
\[
n(H^*) > n(G) - \frac{n(G)}{2N_k + 1} - \frac{2N_k |\chi(M)|}{2N_k + 1}
\] (3)

and with \(n(G) \geq N > (6k + 1)(2N_k + 1)|\chi(M)|\) condition (2) implies that the degree of the major vertex \(B\) is
\[
\deg_G(B) \leq 6k - 2\varepsilon.
\]

This contradiction completes the proof of the theorem. \(\Box\)

5. Proof of Theorem 5(i)

Because lower bounds are known by Theorems 1 and 2 we shall only deal with the upper bounds. Suppose there is a counterexample to Theorem 5(i). Let \(G\) be a counterexample with the maximum number of edges among all counterexamples on \(n\) vertices. A vertex \(A\) of the graph \(G\) is major (minor) if its degree \(\deg_G(A) > 5k\) (\(\deg_G(A) \leq 5k\)). The notation of Section 3 is used here again.

Case I: Let \(1 \leq n(H) \leq 2\).

By assertions (9) and (11) of Case II of Section 3 we have (replace \(|\chi(M)|\) by \(-\chi(M)\)):
\[
\sum_{A \in V(H)} \deg_G(A) \leq 2e(H) + t(k - 1) \quad \text{where } 1 \leq t \leq 2(n(H) + 1 - \chi(M)).
\]

Hence,
\[
\sum_{A \in V(H)} \deg_G(A) \leq 2e(H) + 2(n(H) + 1 - \chi(M))(k - 1) \leq 10k.
\]

Consequently, \(G\) contains a major vertex \(B\) of degree \(\deg_G(B) \leq 5k\). This contradiction completes the proof of theorem.
Case II: Let \( n(H) \geq 3 \).

The proof will be completed as in Case I of Section 3. So the multigraphs \( \tilde{G}, \tilde{H}, \tilde{H}' \) and \( G^*, H^*, H'^* \) are constructed as before. The multigraphs \( G^* \) and \( H^* \) are again (not necessarily polyhedral) triangulations.

The subgraph \( H^* \) of major vertices of \( G^* \) has \( n^* := n(\tilde{H}) \geq 3 \) vertices. By Lemma 2 there is, in \( H^* \), a major vertex \( A \) such that \( \text{deg}_{H^*}(A) \leq 6 - 6\chi(\mathcal{M})/n(H^*) \) and \( \text{deg}_{\tilde{H}^*}(A) \leq 5 \).

Let \( e_1, e_2, \ldots, e_\beta \) be the \( A \)-edges of \( H^* \) in this cyclic order around \( A \), \( \beta = \text{deg}_{H^*}(A) \), where each loop occurs twice in this sequence. The edges \( e_i \) and \( e_{i+1} \) belong to the same triangular-face, say \( A_i \), \( i = 1, 2, \ldots, \beta \), indices modulo \( \beta \). By construction in the cyclic order of the edges around \( A \) in \( G^* \) between \( e_i \) and \( e_{i+1} \) at most \( k - 1 \) edges occur which join \( A \) with the vertices of the tree \( S_3 \) inserted in \( A_i \), \( i = 1, 2, \ldots, \beta \). Thus,

\[
\text{deg}_{G^*}(A) \leq \text{deg}_{H^*}(A) + (k - 1) \text{deg}_{H^*}(A) = k \text{deg}_{H^*}(A) \leq 5k.
\]

But \( A \) is a major vertex and has degree \( \text{deg}_{G^*}(A) \geq \text{deg}_{G}(A) > 5k \). This contradiction completes the proof of Theorem 5(i).

6. Proofs of Theorems 6(ii) and 7(ii) — lower bounds

In order to prove the lower bound, we use a construction of J. Zaks [13] for the case of the path \( P_2 \) on orientable compact 2-manifolds and generalize it to all \( P_k \), \( k \geq 2 \), on an arbitrary 2-manifold \( \mathcal{M} \) of Euler characteristic \( \chi(\mathcal{M}) \leq 0 \). The construction starts with the triangulation of the quadrangle \( R \) presented in Fig. 5. Identifying opposite sides results in a triangulation \( T_1 \) of the torus \( \mathbb{S}_1 \) by a multigraph on three vertices in which each pair of vertices is joined by three edges.
Reversing one side of the quadrangle $R$ of Fig. 5 and then identifying opposite sides of $R$ results in a triangulation $Q_2$ of the Klein bottle $N_2$ by a multigraph on three vertices in which each pair of vertices is joined by three edges. Introducing a crosscap into $T_1$ and adding a 3-cycle (i.e. adding three further edges forming a cycle) via a crosscap as indicated in Fig. 6 results in a triangulation of $N_3$ by a multigraph on three vertices each pair of vertices in joined by four edges. We define $T_g$ and $Q_g$ inductively. Suppose $T_{g-1}$, $g \geq 2$, or $Q_{g-2}$, $q \geq 4$, have already been constructed. Delete a triangular face in each of the two triangulations $T_{g-1}$ and $T_1$ or $Q_{g-2}$ and $T_1$, respectively, and identify the boundaries properly along the three edges. The obtained triangulation of $S_g$ or $N_g$ is denoted by $T_g$ and $Q_g$, respectively.

$T_g$ and $Q_g$ are triangular embeddings of the multigraph on three vertices into $M = S_g$ or $M = N_q$ respectively, each pair of vertices is joined by $|\chi(M)| + 3$ edges, and consequently $T_g$ and $Q_g$ have 3 vertices, $3(|\chi(M)| + 3)$ edges and $2(|\chi(M)| + 3)$ triangular faces. Each of the three vertices has degree $2|\chi(M)| + 6$.

Let $A_1, A_2, A_3$ be the vertices of $H$. Next let $H = T_g$ if $M = S_g$ and $H = Q_g$ if $M = N_q$. Into each triangle we insert a generalized 3-star $S_3$ consisting of a central vertex $Z$ and three paths, say $p_1, p_2$ and $p_3$, starting in $Z$ and including $Z$; the path $p_3$ has length $\lceil k/2 \rceil$ and the other two paths $p_1$ and $p_2$ have length $\lfloor k/2 \rfloor$.

We partition the set of the triangle faces of $H$ into three classes $C_1, C_2$, and $C_3$ so that $|C_1| \geq |C_2| \geq |C_3|$ and $|C_i| - |C_j| \leq 1$ for all $1 \leq i, j \leq 3$. In $C_i$ we join $A_i$ with all $2\lfloor k/2 \rfloor - 1$ vertices of $p_1 \cup p_2$, and $A_{i+1}$ and $A_{i+2}$ with all $\lfloor k/2 \rfloor + \lfloor k/2 \rfloor - 1 = k - 1$ vertices of $p_2 \cup p_3$ or $p_3 \cup p_1$, respectively (indices modulo 3). The obtained graph is $G$. The degree

$$\deg_H(A_i) = 2(|\chi(M)| + 3).$$

If $k \geq 2$ is an even integer then with $f = 2(|\chi(M)| + 3)$

$$\deg_G(A_i) = \deg_H(A_i) + (k - 1)(|C_1| + |C_2| + |C_3|)$$

$$= k(6 + 2|\chi(M)|).$$

If $k \geq 3$ is an odd integer then

$$\deg_G(A_3) = \deg_G(A_2) \geq \deg_G(A_1)$$

$$= \deg_H(A_1) + |C_1|(k - 2) + |C_2|(k - 1) + |C_3|(k - 1)$$

$$= 6 + 2|\chi(M)| + (k - 1)(|C_1| + |C_2| + |C_3|) - |C_1|$$

$$= k(6 + 2|\chi(M)|) - \left[\frac{6 + 2|\chi(M)|}{3}\right].$$

If $k \geq 3$ is odd then each $P_k$ of $G$ contains a vertex of degree $k(6 + 2|\chi(M)|)$. If $k \geq 3$ is odd then each $P_k$ of $G$ contains a vertex of degree $k(6 + 2|\chi(M)|) - \lceil 6 + 2|\chi(M)|/3 \rceil = \lceil (k - 1/3)(6 + 2|\chi(M)|) \rceil$. If $k = 1$ the multigraph $G$ is not 3-connected because it has only 3 vertices. In the next section a 3-connected multigraph on 4 vertices will be constructed.
7. Proof of Theorems 6(i) and 7(ii)

The method is similar to that of Section 6. The construction starts with the triangulation of the quadrangle presented in Fig. 7. Identifying opposite sides results in a triangulation $T'_1$ of the torus $S_1$ by a multigraph $L_1$ on 4 vertices each pair of vertices is joined by two edges.

Reversing one side of the quadrangle $R'$ of Fig. 7 and then identifying opposite sides of $R'$ results in a triangulation $Q'_2$ of the Klein bottle $N_2$ by the multigraph $L_1$. Introducing a crosscap into $Q'_2$ and adding a 3-star (i.e. adding three further edges forming a 3-star) via a crosscap as indicated in Fig. 8 results in a triangulation $Q'_3$ of $N_3$ by a multigraph obtained from $L_1$ by adding a 3-star. We define $T'_g$ and $Q'_q$ inductively. Choose two triangular faces having a common edge in each of the two triangulations $T'_{g-1}$ and $T'_1$ or $Q'_{q-2}$ and $T'_1$, respectively. Delete these two faces together with their common edge. Both triangulations have a hole bounded by a 4-cycle. Identify these two 4-cycles so that always a vertex incident with the deleted edge of one triangulation is identified with a vertex being not incident with the deleted edge in the other triangulation. The new triangulation of $S_g$ or $N_q$ is denoted by $T'_g$ and $Q'_q$, respectively.

$T'_g$, $g \geq 1$, and $Q'_2p$, $p \geq 1$, are triangular embeddings of the multigraph $L_g$ or $L_2p$ on 4 vertices each pair of vertices is joined by $2 + \lfloor \chi(M) \rfloor /2$ edges, and consequently $T'_g$ and $Q'_2p$ have 4 vertices, $6(2 + \lfloor \chi(M) \rfloor /2) = 12 + 3\lfloor \chi(M) \rfloor$ edges and $8 + 2\lfloor \chi(M) \rfloor$ triangular faces. Each of the four vertices has degree $6 + \lfloor \chi(M) \rfloor$.

$Q'_{2p+1}$, $p \geq 1$, is a triangular embedding of a multigraph obtained from $L_{2p}$ by adding a 3-star. $Q'_{2p+1}$ has 4 vertices, $6(2 + \lfloor \chi(M) \rfloor /2) = 12 + 3\lfloor \chi(M) \rfloor$ edges and $8 + 2\lfloor \chi(M) \rfloor$ triangular faces. (Note that $\chi(Q'_{2p+1}) = \chi(Q'_{2p}) + 1$). Each of three vertices have degree $[6 + \frac{3}{2}\lfloor \chi(M) \rfloor]$, the fourth has degree $[6 + \frac{3}{2}\lfloor \chi(M) \rfloor] + 2$. This construction completes the proof of the lower bound for the minimum degree $\psi''(P_1, M)$, $\psi'(P_1, M) \geq [6 + \frac{3}{2}\lfloor \chi(M) \rfloor]$. 

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Fig. 7.

Fig. 8.
8. Subgraphs other than paths

It is enough to prove

**Lemma 5.** Let $H$, $H \neq P_k$, be a connected graph embedable in a compact 2-manifold $M$ of Euler characteristic $\chi(M) \leq 0$, and $m \geq 4$ an integer. Then there is a 3-connected triangulation $G$ of $M$ without trivial 1- and 2-cycles such that every subgraph $K$ of $G$, isomorphic to $H$, contains a vertex $A$ of degree
\[
\deg_G(A) \geq m.
\]

**Proof.** First we pack the graph $H$ into a triangulation $T$ of $M$ without trivial 1- and 2-cycles. Then into each triangular face $\alpha = [A_1A_2A_3]$ of $T$ we insert an $2m$-path and join all vertices with $A_1$, $m + 1$ vertices with $A_2$ and $m$ vertices with $A_3$. The resulting triangulation $G$ has the required properties with respect to $H$, namely for every vertex $A$ of $T$ we have $\deg_G(A) \geq m$ and every subgraph of $G$ which is not a path that has at least one vertex in $T$.

9. Sketch of the proofs for Theorems 13–15

For $k \leq 3$ the assertions of the theorems follow from the corresponding theorems for paths. Therefore, let be $k \geq 4$ in the following.

The proofs are the same as in Sections 3 and 4 with slight changes. The multigraph $G$ is now a counterexample to Theorems 14 or 15. A careful reading of the proofs of Sections 3 and 4 show that it is sufficient to change the generalized 3-stars $S_3$ used there by generalized 3-stars $S'_3$ of order $k - 1$. This can be done so that the degree of no major vertex is decreased (see [10]). The difference between the proofs is the fact that in each face $\alpha$ of the transformed graph $H^*$ the minor vertices are joined with the major vertices of $\alpha$ by at most $2k - 1$ edges and not by at most $3k - 3 - \epsilon$ edges as in Sections 3 and 4. The effect of this is that the essential formulas (5) and (6) of Section 3 are changed to
\[
\sum_{A \in V(H)} \deg_G(A) \leq 2e(H) + (4k - 2)(n(H) + |\chi(M)|)
\]
\[
\text{with } e(H) \leq 3(n(H) + |\chi(M)|).
\]

Hence, $H$ contains a vertex of $B$ of degree
\[
\deg_G(B) \leq (4k + 4) \left( 1 + \frac{|\chi(M)|}{n^*} \right).
\]

By formula (3) of Section 5 we know that
\[
n(H^*) > \frac{n(G)}{2N_k + 1} - \frac{2N_k}{2N_k + 1} |\chi(M)|,
\]
where $N_k$ is an upper bound for the components of the subgraph $H'$ of $G$ generated by the minor vertices. But in the cases considered here all components of $H'$ have an order $\leq k - 1$. Hence, $N_k$ can be replaced by $k - 1$ and we obtained:

$$n(H^*) > \frac{n(G)}{2k - 1} - \frac{2k - 2}{2k - 1} |\chi(M)|.$$  \hfill (3)

With $n(G) \geq N > (8k^2 + 6k - 6)|\chi(M)|$ the condition (2) implies that the degree of the major vertex $B$ is

$$\deg_G(B) \leq 4k + 4.$$  

This proves the upper bounds of Theorem 15(ii). The lower bounds are obtained from the example of [10]. Thus, the proof of Theorem 15(ii) is complete.

If $G \in \mathcal{D}^l(k, M)$ or $G \in \mathcal{D}^m(k, M)$ then we consider two cases.

Case I: Let $n^* \geq 3$.

Then the vertex $B$ of (2) has a degree

$$\deg_G(B) \leq (4k + 4) \left(1 + \frac{|\chi(M)|}{3}\right).$$

Case II: Let $1 \leq n^* \leq 2$.

Since, $G$ is 3-connected the subgraph $H'$ induced by the minor vertices of $G$ has precisely one component of order $\leq k - 1$. Then, the number of edges of $G$ is not greater then in a triangulation on $M$ with $n^* + k - 1$ vertices. Hence, $G$ has $3(|\chi(M)| + n^* + k - 1)$ edges, at least $k - 2$ of them join vertices of $K$. Hence, at most $3(|\chi(M)| + n^* + k - 1) - (k - 2)$ edges are incident with a vertex of $H$. At most $3(|\chi(M)| + n^*)$ of them join vertices of $H$. Therefore, we count $3(|\chi(M)| + n^*)$ edges twice in the degree sum of the vertices of $H$. The remaining $3(|\chi(M)| + n^* + k - 1) - (k - 2) - 3(|\chi(M)| + n^*) = 2k - 1$ edges will be counted once. Consequently,

$$\sum_{A \in V(H)} \deg_G(A) \leq 2(3|\chi(M)| + 3n^*) + 2k - 1 \leq 6|\chi(M)| + 6n^* + 2k - 1.$$  

Then $H$ contains a vertex $B$ of degree

$$\deg_G(B) \leq 6 + \frac{6|\chi(M)| + 2k - 1}{n^*}.$$  

If $k \geq 4$ then

$$\deg_G(B) \leq 6|\chi(M)| + 6 + (2k - 1) \leq (4k + 4) \left(1 + \frac{|\chi(M)|}{3}\right).$$

If $k = 3$ and $n^* = 2$ then

$$\deg_G(B) \leq 3|\chi(M)| + 6 + k - \frac{1}{2} \leq (4k + 4) \left(1 + \frac{|\chi(M)|}{3}\right).$$

If $k = 3$ and $n^* = 1$ or $k = 2$ and $1 \leq n^* \leq 2$ then $G$ is not 3-connected. This proved the upper bounds of Theorem 14(ii).
The lower bound is obtained by replacing the generalized 3-stars $S_3$ by the 3-stars $S'_3$ of order $k - 1$ in triangulations constructed in Section 7. Thus, the proof of Theorem 14(ii) is complete. Theorem 13 can be proved analogously to Theorem 5(i) in Section 6.

References