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Comparison of higher-order accurate schemes for solving the two-dimensional unsteady Burgers' equation

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Abstract

This paper is devoted to the testing and comparison of numerical solutions obtained from higher-order accurate finite difference schemes for the two-dimensional Burgers' equation having moderate to severe internal gradients. The fourth-order accurate two-point compact scheme, and the fourth-order accurate Du Fort Frankel scheme are derived. The numerical stability and convergence are presented. The cases of shock waves of severe gradient are solved and checked with the fourth-order accurate Du Fort Frankel scheme solutions. The present study shows that the fourth-order two-point compact scheme is highly stable and efficient in comparison with the fourth-order accurate Du Fort Frankel scheme.

Keywords: Two-dimensional Burgers' equation; Higher-order accurate schemes; Compact scheme; Du Fort Frankel scheme

1. Introduction

The basic fluid flow equations describing unsteady transport problems form a parabolic–hyperbolic system of partial differential equations. The interaction of the nonlinear terms and the dissipative viscous terms in these equations can result in relatively severe gradients in the solution. In addition, it is well known that the accuracy of the numerical computations and the computational efficiency are highly dependent on the numerical schemes used to solve the fluid flow equations. Central finite difference methods may work well for smooth solutions but they fail when severe gradients or discontinuities are present. This is common in the shock wave problems [3,7,19,24]. Therefore, they become less suitable.

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Upwinding type finite differences can be a remedy for the numerical oscillations and dispersions, but they have a large amount of "numerical viscosity" that smoothes the solution in much the same way that physical viscosity would, but to an extent that is unrealistic [32]. Furthermore, when grid nonuniformity becomes important, the lower-order upwinding methods can create additional numerical smearing effects [22]. Standard four-point finite difference methods such as Leonard's method [20], Quick ($\frac{1}{8}$) or / SPUDS ($\frac{1}{6}$) schemes, are good in their higher-order accuracies and in reducing numerical smearing effects. However, it is found that Quick ($\frac{1}{8}$) and SPUDS ($\frac{1}{6}$) schemes generate spurious oscillations or overshoots in the neighborhood of discontinuities and lack accuracy. The same conclusion is also reported in [22]. TVD finite difference schemes [12–14,34,37] guarantee oscillation-free solutions, but they are limited to second-order accuracy. Third-order accurate TVD schemes are reported in [11,22]. The latter conducted truncation error analysis for a set of schemes including Leonard's scheme, one-point upstream scheme and a third-order accurate TVD scheme, for the solution of one-dimensional linear convection diffusion equation and two-dimensional Burgers' equation. However, in their comparisons, nonlinearity was either not present or it played a minor role especially in the two-dimensional test case. Therefore, the perfect numerical methods should possess both higher-order accuracy and sharp resolution of discontinuities without excessive smearing. Moreover, higher-order accurate numerical methods are attractive for problems with long computational time or with higher accuracy solutions.

As mentioned in [23], at the cost of slight additional computational complexity, the fourth-order schemes achieve results in the 5% accuracy range with approximately half the spatial resolution in each space direction compared with the second-order schemes (i.e., a factor 8 fewer grid points in three dimensions). But, the objection to the standard higher-order schemes comes from the additional nodes necessary to achieve the higher-order accuracy. This precludes the use of implicit methods as the resulting matrix is not of tridiagonal form, and it is necessary to use fictitious nodes for the boundary conditions. Also, the standard higher-order schemes do not allow easily for nonuniform grids, unless at the expense of the order of accuracy. On the other hand, the compact schemes that treat the function and its necessary derivatives as unknowns at the grid nodes, like the Pade scheme [21], are fourth-order accurate, and compact in the sense that they are reduced to the tridiagonal form. The compact schemes generally consist of finite difference schemes which involve two or three grid points. The three-point schemes consist of methods, which are fourth-order accurate for uniform grids, such as Kreiss scheme [2,5,17,25], and of methods that allow variable grids such as the cubic spline methods of Rubin and Graves [1,30,31]. The disadvantage of the higher-order compact schemes involving three nodal points is that the boundary conditions are no longer sufficient and they do not allow easily for nonuniform grids, unless at the expense of the order of accuracy. Also, the complexity of the resulting nonlinear finite difference equations and the associated difficulty in solving them efficiently make these schemes difficult to use.

On the other hand, the compact scheme with two nodal points, like a second-diagonal Pade scheme, is fourth-order accurate even for nonuniform spatial grids, and no fictitious points or extra formulae are needed for Dirichlet boundary conditions; see [16,21]. Liniger et al. studied the numerical solutions of stiff systems of ordinary differential equations using compact two-point implicit methods. They introduced three main compact schemes with different orders of accuracy, and with some very favorable properties. In particular, their schemes have A-stability and they account for the exponential character of the rapidly decaying solutions directly, which are referred to exponential fitting methods. Inspite of the fact that many articles have appeared in the literature concerning the applications of the higher-order accurate schemes including the compact schemes to fluid dynamics problems, see, for example [4,9,10,15,26,33,36]. However, not much work has been done in the area of application of two-point compact schemes, like the fourth-order accurate second-diagonal Pade approximation, to multi-dimensional cases. As such, the present study aims at studying the feasibility of extending the two-point compact scheme to solve the unsteady two-dimensional Burgers' equation.

Burgers' equation retains the nonlinear aspects of the governing equations in many practical transport problems such as vorticity transport, hydrodynamic turbulence, shock wave theory, wave processes in thermoelastic medium, transport and dispersion of pollutants in rivers and sediment transport [7,8,24,28,35]. Burgers' equations have the same convective and diffusion form as the incompressible Navier–Stokes equations. The unsteady two-dimensional Burgers' equation in one unknown variable takes the following form:

$$\frac{\partial \Phi}{\partial t} = L(\Phi),$$
$$L(\Phi) = -\Phi\,\frac{\partial \Phi}{\partial x} - \Phi\,\frac{\partial \Phi}{\partial y} + v\,\frac{\partial^2 \Phi}{\partial x^2} + v\,\frac{\partial^2 \Phi}{\partial y^2},$$
$$x_0 \leqslant x \leqslant x_N, \quad y_0 \leqslant y \leqslant y_N, \quad t > 0 \tag{1.1}$$

with the initial conditions;

$$\Phi(x, y, 0) = \Phi_0(x, y) \tag{1.2}$$

and the Dirichlet boundary conditions;

$$\Phi(x_0, y, t) = \Phi_1(y), \quad \Phi(x_N, y, t) = \Phi_2(y), \tag{1.3}$$

$$\Phi(x, y_0, t) = \Phi_3(x), \quad \Phi(x, y_N, t) = \Phi_4(x), \tag{1.4}$$

where v is equal to $1/Re > 0$, and Re is the Reynolds number. For a small value of v, Burgers' equation behaves merely as hyperbolic partial differential equation, and the problem becomes very difficult to solve as a steep shock-like wave fronts developed, as reported in [18].

In the present study, higher-order accurate two-point compact alternating direction implicit algorithm is introduced to solve the two-dimensional unsteady Burgers' equation, for flow problems with moderate to severe internal and boundary gradients. The method is the natural extension of A-stable fourth-order accurate second-diagonal Pade approximation to solve multi-dimensional flow problems. Comparison of the present scheme with the fourth-order Du Fort Frankel scheme is made in terms of accuracy and computational efficiency.

2. The numerical schemes

In this section, the present numerical schemes; namely the fourth-order accurate compact scheme and the fourth-order Du Fort Frankel scheme, are derived for the two-dimensional unsteady Burgers' equation (1).

2.1. Fourth-order accurate two-point compact alternating direction implicit scheme

Liniger et al. [21] have introduced the following linear one-step formula that is containing real free parameters (a and b). It takes the following form:

$$[\psi_{i+1} - \psi_i] - \frac{\Delta x}{2}[(1+a)\psi_{x_{i+1}} + (1-a)\psi_{x_i}] + \frac{\Delta x^2}{4}[(b+a)\psi_{xx_{i+1}} - (b-a)\psi_{xx_i}] = e_{\mathrm{T}}, \tag{2.1}$$

$$e_{\mathrm{T}} = \frac{\Delta x^3}{4}\int_0^1 [2\xi^2 - 2(1-a)\xi + (b-a)]\frac{\partial^3}{\partial x^3}\psi(x+\xi\Delta x)\,\mathrm{d}\xi. \tag{2.2}$$

For the case of $a = 0$ and $b = \frac{1}{3}$, the formula has a fourth-order accuracy, which is known as the two-point second-diagonal Pade approximation:

$$[\psi_{i+1} - \psi_i] - \frac{\Delta x}{2}[\psi_{x_{i+1}} + \psi_{x_i}] + \frac{\Delta x^2}{12}[\psi_{xx_{i+1}} - \psi_{xx_i}] = e_{\mathrm{T}}, \tag{2.3}$$

$$e_{\mathrm{T}} = \frac{\Delta x^5}{24}\int_0^1 \xi^2(\xi-1)^2\frac{\partial^5}{\partial x^5}\psi(x+\xi\Delta x)\,\mathrm{d}\xi, \tag{2.4}$$

where ψ_x, ψ_{xx} are the first and the second derivatives of the variable $\psi(x)$. Using the above-mentioned scheme, Eq. (2.3) and an ADI-type time marching procedure for the temporal derivative, the fourth-order accurate two-point compact alternating direction implicit algorithm for the Burgers' equation, can be obtained by first rewriting Eq. (1.1) as follows:

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial x}\left[v\frac{\partial \Phi}{\partial x} - 0.5\Phi^2\right] + \frac{\partial}{\partial y}\left[v\frac{\partial \Phi}{\partial y} - 0.5\Phi^2\right]. \tag{2.5}$$

Then, the alternating direction implicit-type time marching procedure requires, in one full time step, the solution of

x-sweep:

$$[\Phi_t]^{n+1/2} = [(v\Phi_x - 0.5\Phi^2)_x]^{n+1/2} + [g]^n, \tag{2.6a}$$

$$[\Phi_x]^{n+1/2} = [F]^{n+1/2}. \tag{2.6b}$$

y-sweep:

$$[\Phi_t]^{n+1} = [(v\Phi_y - 0.5\Phi^2)_y]^{n+1} + [f]^{n+1/2}, \tag{2.7a}$$

$$[\Phi_y]^{n+1} = [G]^{n+1}, \tag{2.7b}$$

where f, g are the first and the second term on the right-hand side of Eq. (2.5). The solution procedure consists of solving, first, Eq. (2.6) in the solution vector $[\Phi, F]^{\mathrm{T}}$ at time level $n + \frac{1}{2}$, (x-sweep), then solving Eq. (2.7) in the solution vector $[\Phi, G]^{\mathrm{T}}$ at time level $n+1$, (y-sweep). Noting that $\alpha = 1/\Delta t$, and in order to apply the compact scheme to the solution in the x-sweep, a vector $\vec{Q}$ and its derivatives with respect to x, for Eq. (2.6), have been defined as follows:

$$\vec{Q}_{ij}^{n+1/2}=\begin{bmatrix} v\Phi-0.5\Phi^2 \\ v\Phi \end{bmatrix}_{ij}^{n+1/2}=\begin{bmatrix} vF-0.5\Phi^2 \\ v\Phi \end{bmatrix}_{ij}^{n+1/2}, \tag{2.8}$$

$$\vec{Q}_{xij}^{n+1/2}=\begin{bmatrix} \Phi_t \\ vF \end{bmatrix}_{ij}^{n+1/2}-\begin{bmatrix} g \\ 0 \end{bmatrix}_{ij}^{n}=\begin{bmatrix} \alpha\Phi \\ vF \end{bmatrix}_{ij}^{n+1/2}-\begin{bmatrix} g+\alpha\Phi \\ 0 \end{bmatrix}_{ij}^{n}, \tag{2.9}$$

$$\begin{aligned}\vec{Q}_{xxij}^{n+1/2}&=\begin{bmatrix} F_t \\ \Phi_t+\Phi\Phi_x \end{bmatrix}_{ij}^{n+1/2}-\begin{bmatrix} g_x \\ g \end{bmatrix}_{ij}^{n}\\ &=\begin{bmatrix} \alpha F \\ \alpha\Phi+\Phi F \end{bmatrix}_{ij}^{n+1/2}-\begin{bmatrix} \alpha F+g_x \\ \alpha\Phi+g \end{bmatrix}_{ij}^{n},\end{aligned} \tag{2.10}$$

where g and g_x are approximated by a fourth-order accurate finite differences. Having substituted the vector $\vec{Q}$ and its derivatives into the above two-point second diagonal Pade approximation, Eq. (2.3), by replacing ψ_i by the vector $\vec{Q}$, we have two nonlinear coupled finite difference equations in the solution vector $[\Phi, F]^{\mathrm{T}}$. Newton's method is used to linearize the equations, and the numerical solution is obtained by iteration. The resulting linearized equations form a block tridiagonal matrix system of order N, as in the following form:

$$a_i\vec{\delta}_{i-1}+b_i\vec{\delta}_i+c_i\vec{\delta}_{i+1}=\vec{r}_i, \quad i=1,2,\ldots,N, \tag{2.11}$$

where a_i, b_i, and c_i are block matrices of order two, $\vec{\delta}=[\delta\Phi, \delta F]^{\mathrm{T}}$ is the change in the solution vector, and r is the right-hand side vector, each of order two. At each iteration, the LU-factorization algorithm is used to obtain the solution of system (2.11). Similarly, the solution procedure of the Burgers' equation in the y-sweep, using Eq. (2.7).

2.2. Fourth-order accurate Du Fort Frankel scheme

Let the interval $[x_0, x_N]$ be discretized into N grid steps of size Δx where $\Delta x=(x_i-x_{i-1})$, i is an index of any grid-point in x direction. Similarly, the interval $[y_0, y_M]$ is discretized into M grid steps of size Δy, where $\Delta y=(y_j-y_{j-1})$, j is an index of any grid point in y-direction, and n is an index for the temporal grid point. The explicit form of the Du Fort Frankel scheme for the two-dimensional unsteady Burgers' equation, Eq. (1.1), using Kreiss fourth-order accurate approximations [23] for the spatial derivatives, takes the following form:

$$\begin{aligned}\left[\frac{\Phi_{ij}^{n+1}-\Phi_{ij}^{n-1}}{2\Delta t}\right]=&-\Phi_{ij}^{n}D_x\left[1-\frac{\Delta x^2}{6}\delta_x^2\right]\Phi_{ij}^{n}+v\delta_x^2\left[1-\frac{\Delta x^2}{12}\delta_x^2\right]\Phi_{ij}^{n}\\ &-\Phi_{ij}^{n}D_y\left[1-\frac{\Delta y^2}{6}\delta_y^2\right]\Phi_{ij}^{n}+v\delta_y^2\left[1-\frac{\Delta y^2}{12}\delta_y^2\right]\Phi_{ij}^{n}\\ &+\mathrm{O}(\Delta t^2+\Delta x^4+\Delta y^4),\end{aligned} \tag{2.12}$$

where

$$D_x \Phi_{ij} = \frac{1}{2\Delta x}(\Phi_{i+1j} - \Phi_{i-1j}), \tag{2.13}$$

$$\delta_x^2 \Phi_{ij} = \frac{1}{\Delta x^2}(\Phi_{i+1j} - 2\Phi_{ij} + \Phi_{i-1j}). \tag{2.14}$$

Define $c_x^n = \Phi_{ij}^n \, \Delta t/\Delta x$, $c_y^n = \Phi_{ij}^n \, \Delta t/\Delta y$ to be the local courant numbers in x and y directions, $d_x = v\,\Delta t/\Delta x^2$, and $d_y = v\,\Delta t/\Delta y^2$. The above-finite difference equation is the fourth-order accurate leap-frog scheme for Eq. (1.1), and in order to obtain the final form of the fourth-order accurate explicit Du Fort Frankel scheme for the two-dimensional unsteady Burgers' equation, the center node value (Φ_i) in the diffusion terms in Eq. (2.12) are replaced by their average value at time levels $(n-1)$ and $(n+1)$. The final form of the fourth-order accurate explicit Du Fort Frankel scheme for the two-dimensional unsteady Burgers' equation, Eq. (1.1), becomes

$$\begin{aligned}\Phi_{ij}^{n+1} = {} & A\Phi_{ij}^{n-1} + B\Phi_{i+2j}^n + C\Phi_{i+1j}^n + D\Phi_{i-1j}^n + E\Phi_{i-2j}^n \\ & + F\Phi_{ij+2}^n + G\Phi_{ij+1}^n + H\Phi_{ij-1}^n + L\Phi_{ij-2}^n,\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}&A = (1 - 2.5d_x - 2.5d_y)/Q, \quad B = (c_x^n - d_x)/6Q,\\ &C = (-8c_x^n + 16d_x)/6Q, \quad D = (8c_x^n + 16d_x)/6Q,\\ &E = -(c_x^n + d_x)/6Q, \quad F = (c_x^n - d_y)/6Q,\\ &G = (-8c_y^n + 16d_y)/6Q, \quad H = (8c_y^n + 16d_y)/6Q,\\ &L = -(c_x^n + d_x)/Q, \quad Q = (1 + 2.5d_x + 2.5d_y).\end{aligned} \tag{2.16}$$

2.3. Numerical stability limits and convergence

The implicit formulation of the two-point compact scheme to the Burgers' equations is always unconditionally stable. In this case, the accuracy of the numerical solution depends on the size of the discretizations, and higher accuracy can be obtained by finer discretization. Moreover, the present higher-order scheme allows us to use large discretization in comparison with the second-order schemes. In addition, it is well known that, for the convection–diffusion equation, the leap-frog scheme is unconditionally unstable, while the Du Fort Frankel scheme has a stability limit [27]. Therefore, it is necessary to use Von Neumann stability analysis to define the stability limit. Let the numerical solution $\Phi(x, y, t)$ be represented by a finite Fourier series, and for linear stability, the behavior of a single term of the series can be examined, as follows:

$$\Phi(i\Delta x, j\Delta y, n\Delta t) = G(t)\mathrm{e}^{\mathrm{I}[k_x i\Delta x + k_y j\Delta y]}, \tag{2.17}$$

where $G(t)$ is the amplitude function at time-level n of this term whose wave numbers in the x and y directions are k_x and k_y, and $\mathrm{I} = \sqrt{-1}$. Defining the x and y phase angles as $\theta_x = k_x\Delta x$ and $\theta_y = k_y\Delta y$, then, Eq. (2.17) becomes

$$\Phi_{ij}^n = G^n \mathrm{e}^{\mathrm{I}[i\theta_x + j\theta_y]}. \tag{2.18}$$

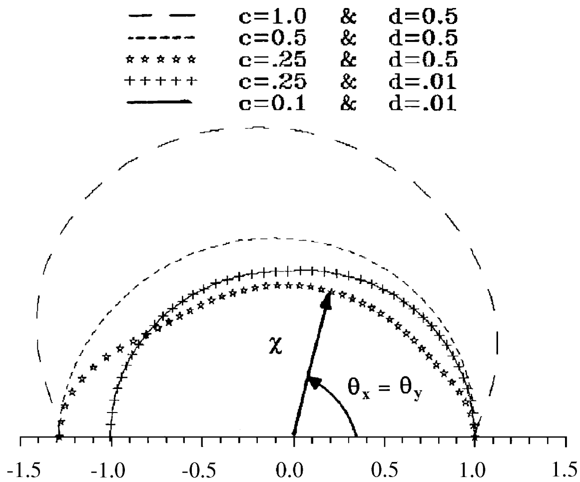


Fig. 1. The computed amplification factors of the numerical solution of two-dimensional unsteady Burgers' equation (1), using the explicit fourth-order Du Fort Frankel scheme for different values of $|c|$ and d.

Substituting Eq. (2.18) into Eq. (2.15), we obtain a quadratic equation for the amplification factor ζ, as follows:

$$\varsigma^2 - \lambda\varsigma - A = 0. \tag{2.19}$$

For the special case of $d_x = d_y = d$ and $c_x = c_y = c$, the modulus of the amplification factor χ, is defined by the following equation:

$$\chi(c, d, \theta_x, \theta_y) = \max\left(\left|\frac{1}{2}\left[\lambda + \sqrt{\lambda^2 + 4A}\right]\right|, \left|\frac{1}{2}\left[\lambda - \sqrt{\lambda^2 + 4A}\right]\right|\right), \tag{2.20}$$

where, $\lambda = \lambda(c, d, Q, \theta)$, and Q are defined by Eq. (2.16). The modulus of the amplification factor has been computed for different values of $|c|$ and d and plotted, as shown in Fig. 1. This shows that the fourth-order accurate Du Fort Frankel scheme is unstable for the range: $(0.35 \leqslant |c| \leqslant 1.0)$ e.g., $\chi(1, 0.5, \pi/2, \pi/2) = 1.77$. For small values of $|c|$ and d $(|c| < 0.35)$, the instability only occurs for phase angles close to π. Moreover, for smaller values of d $(d < 0.1)$, the scheme has a neutral stability e.g., $\chi(0.25, 0.01, \pi, \pi) = 1.0$.

Concerning the consistency of the present schemes, the finite difference equation using the fourth-order accurate two-point compact alternating direction implicit algorithm is consistent in the sense that the local truncation error, $e_{\mathrm{T}} = \mathrm{O}[\Delta x^5, \Delta t.\Delta x^2, \Delta t.\Delta x]$ tends to zero as Δt and Δx tend to zero. For Du Fort Frankel scheme equation, Eq. (2.15), whose truncation error, $\mathrm{e}_T = \mathrm{O}[\Delta t^2, (\Delta t/\Delta x)^2, \Delta x^4]$, the consistency condition requires the truncation error tends to zero upon $(\Delta t/\Delta x)^2$ approach zero as Δt and Δx approach zero. For this reason, and a much smaller time step than allowed by the above stability limit is implied. This concludes that each of the finite difference approximations to the two-dimensional unsteady Burgers' equation, the fourth-order explicit Du Fort Frankel scheme and the compact scheme, satisfies the consistency condition. Then, the stability of the scheme will be the necessary and sufficient conditions for convergence, which is true for linear PDEs. But, for the present nonlinear PDEs (1), the results of the test cases will verify the convergence, but with higher restricted stability limit.

3. Numerical experiments

3.1. Problem case-1

In the first problem, the one-dimensional unsteady Burgers' equation with imposed initial and boundary conditions provided by the exact solution [35] has been considered, as follows:

$$\Phi_e(x,t) = 1.0 - 0.9\left[\frac{r_1}{r_1+r_2+r_3}\right] - 0.5\left[\frac{r_2}{r_1+r_2+r_3}\right],$$

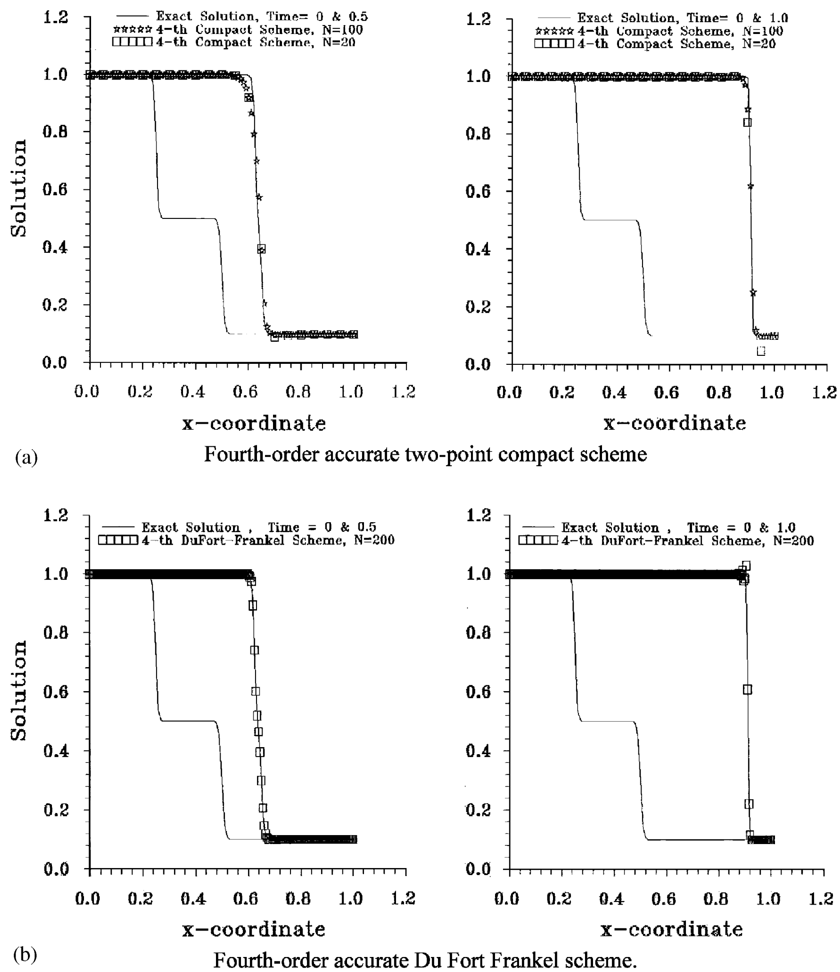


Fig. 2. The computed solutions of one-dimensional unsteady Burgers' equation, for $\nu = 0.001$ and at times $= 0.5$ and 1.0, using compact scheme with two-grid sizes $N = 20$ and 100, and Du Fort Frankel scheme with $N = 200$: (a) fourth-order accurate two-point compact scheme and (b) fourth-order accurate Du Fort Frankel scheme.

Table 1
Comparison of execution times for computed solutions of one-dimensional unsteady Burgers' equation at time = 1.0 and v=0.001

No. of points	Fourth compact scheme	Fourth Du Fort Frankel scheme
$N = 50$	$\Delta t = 0.1$ Ex. time = 1.2 s	$\Delta t = 0.00001$ Ex. time = 65 s
$N = 100$	$\Delta t = 0.025$ Ex. time = 5.0 s	$\Delta t = 0.00001$ Ex. time = 130 s
$N = 200$	$\Delta t = 0.0025$ Ex. time = 60 s	$\Delta t = 0.00002$ Ex. time = 130 s

where

$$
\begin{aligned}
r_1 &= \exp[-0.05(x - 0.5 + 4.95t)/v],\\
r_2 &= \exp[-0.25(x - 0.5 + 0.75t)/v],\\
r_3 &= \exp[-0.5(x - 0.375)/v].
\end{aligned}
\tag{3.1}
$$

The computed solutions for the case of $v = 0.001$ and their comparison with the exact solutions are shown in Fig. 2. It can be seen that the solution contains two shocks, one of which is overtaken by the other. The fourth-order two-point compact scheme is capable of capturing this fact without any numerical instability or oscillations, even with a few grid points (20 points), compared with the computed solutions using the fourth-order accurate Du Fort Frankel scheme, which requires 200 grid points to have reasonable accuracy. Moreover, the computed results of Du Fort Frankel scheme exhibit numerical overshoots at the shock. This indicates that the fourth-order Du Fort Frankel scheme is unstable at the shocks, and a much smaller time step and grid step sizes than allowed by the linear stability limit are required. To compare the computational efficiency of the two schemes, the execution times, using a personal computer, necessary to obtain stable solutions at time = 1.0, $v = 0.001$, and with comparable accuracy for the present two schemes are listed in Table 1. The two-point compact scheme computations with $\Delta t_{\max} = 0.025$ required about 5 s in comparison with 130 s needed by the fourth-order Du Fort Frankel scheme computations with $\Delta t_{\max} = 0.00001$ for the same grid and accuracy. This indicates that the fourth-order two-point compact scheme is efficient and economical for solving the one-dimensional unsteady Burgers' equation with severe gradients.

3.2. Problem case-2

The second test case is the solution of two-dimensional unsteady Burgers' equation that is dominated by moderate gradients [22]. Eq. (1.1) with the following Dirichlet boundary conditions is solved by the present two schemes:

$$
\begin{aligned}
&\Phi(0, y, t) = 1/[1 + e^{y/(2v)}], \quad \Phi(1, y, t) = 1/[1 + e^{(1+y)/(2v)}],\\
&\Phi(x, 0, t) = 1/[1 + e^{x/(2v)}], \quad \Phi(x, 1, t) = 1/[1 + e^{(1+x)/(2v)}]
\end{aligned}
$$

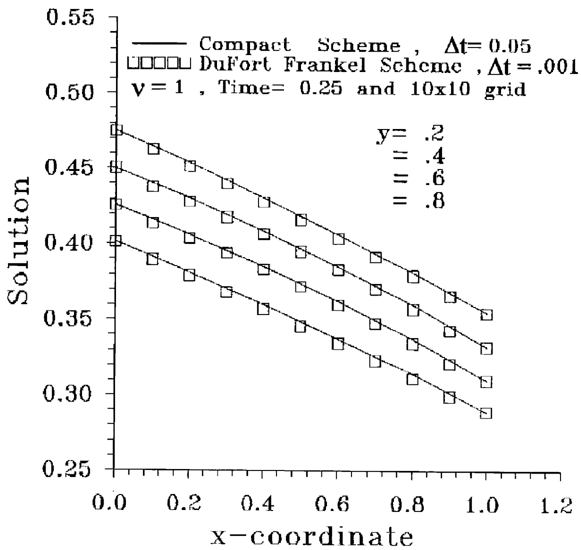


Fig. 3. The computed steady state solutions of two-dimensional unsteady Burgers' equation for test case 2, using compact scheme with $\Delta t = 0.05$ and their comparison with Du Fort Frankel scheme with $\Delta t = 0.001$.

and the initial condition

$$\Phi(x, y, 0) = 1/[1 + e^{(x+y)/2v}], \quad 0 \leqslant x \leqslant 1, \ 0 \leqslant y \leqslant 1. \tag{3.2}$$

The computed steady state solutions of Eq. (1.1), with the above initial and boundary conditions have been obtained using the two higher-order schemes for different grid and time step sizes. Comparison of the computed steady state solutions is shown in Fig. 3, for a grid (10×10), $v = 1.0$, and at time $= 0.25$. Each of the present scheme reaches the same steady state solution with comparable accuracy. However the fourth-order compact solution requires a large time step size ($\Delta t = 0.05$), compared with the fourth-order Du Fort Frankel scheme that required smaller time step size ($\Delta t = 0.001$). Concerning the computational efficiency, the two schemes are comparable in this case. The computed steady solutions using compact scheme at time $= 0.25$, with different grid sizes and their comparison with the exact steady solutions are presented in Fig. 4. It is sufficient to obtain convergent steady solutions with only 5 grid points in each spatial direction.

3.3. Problem case-3

In this case, the solution of the two-dimensional unsteady Burgers' equation with a steep oblique shock in the domain: $-0.1 \leqslant x \leqslant 0.1$, and $-0.05 \leqslant y \leqslant 0.05$, is considered. The two-dimensional unsteady Burgers' equation, Eq. (1.1), is solved using the following Dirichlet boundary conditions that are set to form an

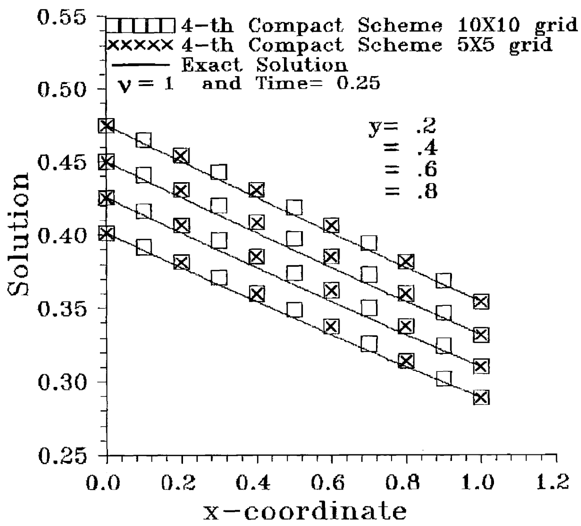
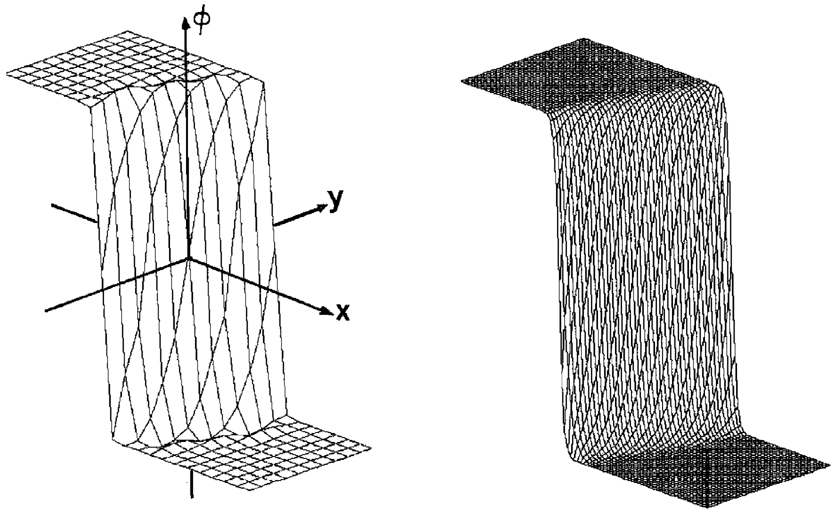


Fig. 4. The computed steady state solutions of two-dimensional unsteady Burgers' equation for test case 2, using compact scheme with different grid sizes, and their comparison with the exact solutions.

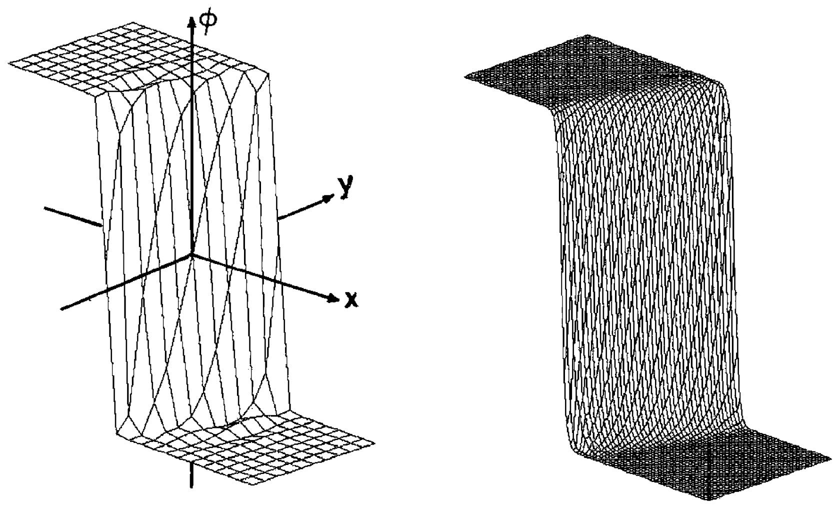
"oblique" shock in the domain [29]:

$$\Phi(x,-0.05,t) = -\tanh\left[\frac{x+0.02}{2v}\right], \quad \Phi(x,-0.05,t) = -\tanh\left[\frac{x-0.02}{2v}\right],$$
$$\Phi(-0.1,y,t) = -\tanh\left[\frac{-0.1-0.4y}{2v}\right], \quad \Phi(0.1,y,t) = -\tanh\left[\frac{0.1+0.4y}{2v}\right],$$
$$-0.1 \leqslant x \leqslant 0.1, \quad -0.05 \leqslant y \leqslant 0.05. \tag{3.3}$$

The computed steady state solutions using the two schemes, with $v = 0.002$, and on two different grids (80×40), (20×10), are shown in Fig. 5. Again, the fourth-order two-point compact scheme is capable of producing convergent and stable solution with steep oblique shock on relatively coarse grid (20×10), compared with the fourth-order Du Fort Frankel solution that required finer grid (80×40), and small time step size to be stable. Furthermore, the computational efficiency of the two schemes has been tested. The execution times, using a personal computer, necessary to obtain stable steady state solutions at time $= 0.1$ and $v = 0.002$, with comparable accuracy for the present two schemes, are listed in Table 2. The compact scheme computations with $\Delta t_{\max} = 0.05$ required about 16 s, compared with 61 s needed by the fourth-order Du Fort Frankel scheme with $\Delta t_{\max} = 0.00025$, for the same grid (80×40) and accuracy. This indicates that the fourth-order Du Fort Frankel scheme is about four times less economical than the fourth-order two-point compact scheme. Moreover, the compact scheme efficiently solves the same problem with steeper oblique shock cases of $v = 0.001$–0.0002, without any oscillations; see Fig. 6. This concludes that the fourth-order two-point compact scheme is stable and efficient for solving the two-dimensional unsteady Burgers' equation, especially with severe gradients.



(a) Fourth-order accurate two-point compact scheme



(b) Fourth-order accurate Du Fort Frankel scheme.

Fig. 5. The computed steady state solutions of two-dimensional unsteady Burgers' equation for an oblique shock case, with $v = 0.002$, for two different grid sizes; (80×40) & (20×10), using compact scheme and Du Fort Frankel scheme: (a) fourth-order accurate two-point compact scheme and (b) fourth-order accurate Du Fort Frankel scheme.

4. Conclusion

In conclusion, the fourth-order two-point compact scheme and the fourth-order accurate Du Fort Frankel scheme are used to solve the two-dimensional unsteady Burgers' equation having moderate to severe internal gradients. The compact scheme is found to be efficient and stable when compared with

Table 2
Comparison of execution times for computed solutions of two-dimensional unsteady Burgers' equation at time = 0.1 and v=0.002, for the oblique shock problem

Grid points	Fourth compact scheme	Fourth Du Fort Frankel scheme
20×10	$\Delta t = 0.05$ Ex. time = 1 s	$\Delta t = 0.00025$ Ex. time = 4 s
40×20	$\Delta t = 0.05$ Ex. time = 4 s	$\Delta t = 0.00025$ Ex. time = 15 s
80×40	$\Delta t = 0.05$ Ex. time = 16 s	$\Delta t = 0.00025$ Ex. time = 61 s

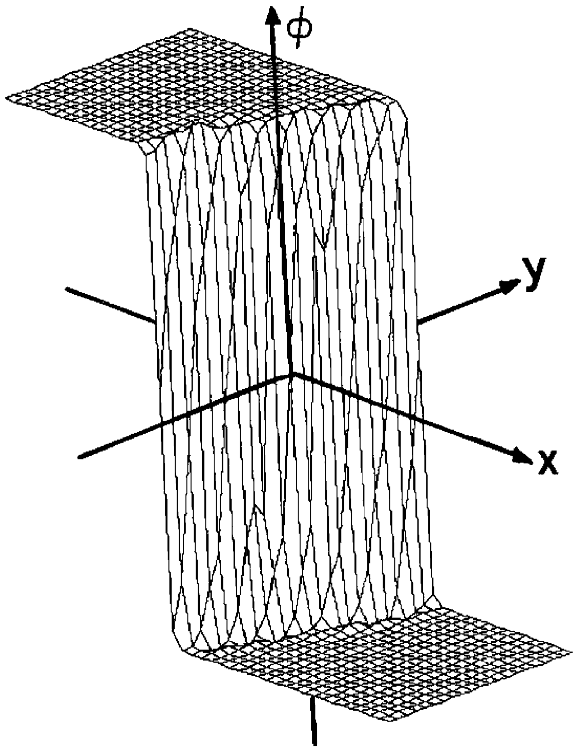


Fig. 6. The computed steady state solutions of two-dimensional unsteady Burgers' equation for an oblique shock case, with $v = 0.0005$ and grid size (40×20), using the fourth-order accurate compact scheme.

the other scheme; it has the following features:

(1) It results in finite difference equations that involve only two-nodal points and, therefore, it is formally fourth-order accurate on all grid points, even for nonuniform grids.
(2) It has A-stability in the sense of Dahlquist, and accounts for the exponential character of rapidly varying solutions [6,21].

(3) It utilizes Newton's method for linearization with a quadratic convergence.
(4) It requires only the given Dirichlet boundary conditions.

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