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On a conjecture about enumerating (2 + 2)-free posets

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ABSTRACT

Recently, Kitaev and Remmel posed a conjecture concerning the generating function for the number of unlabeled (2+2)-free posets with respect to the number of elements and the number of minimal elements. In this paper, we present a combinatorial proof of this conjecture.

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1. Introduction

A poset is said to be (2 + 2)-free if it does not contain an induced subposet that is isomorphic to 2 + 2, the union of two disjoint 2-element chains. In a poset, let $D(x)$ be the set of *predecessors* of an element x (the strict down-set of x). Formally, $D(x) = \{y : y < x\}$. A poset P is (2 + 2)-free if and only if its sets of predecessors, $D(P) = \{D(x) : x \in P\}$ can be written as

$$D(P) = \{D_0, D_1, \ldots, D_k\}$$

where $\emptyset = D_0 \subset D_1 \subset \cdots \subset D_k$; see [1,2]. In such a context, we say that $x \in P$ has *level i* if $D(x) = D_i$. An element x is said to be a *minimal* element if x has level 0.

Let p_n be the number of unlabeled (2 + 2)-free posets on n elements. El-Zahar [4] and Khamis [5] used a recursive description of (2 + 2)-free posets to derive a pair of functional equations that define the generating function for the number p_n. But they did not solve these equations. Recently, using functional equations and the Kernel method, Bousquet-Mélou et al. [2] showed that the generating function for the number p_n of unlabeled (2 + 2)-free posets on n elements is given by

$$P(t) = \sum_{n\geq 0} p_n t^n = \sum_{n\geq 0} \prod_{i=1}^{n} (1 - (1 - t)^i). \tag{1.1}$$

Note that throughout this paper, the empty product as usual is taken to be 1. In fact, they studied a more general function of unlabeled (2 + 2)-free posets according to number of elements, number

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of levels and level of minimum maximal elements. Zagier [8] proved that Formula (1.1) is also the generating function for certain involutions introduced by Stoimenow [7].

Given a sequence of integers $x = (x_1, x_2, \ldots, x_n)$, we say that the sequence x has an ascent at position i if $x_i < x_{i+1}$. The number of ascents of x is denoted by $\mathrm{asc}(x)$. A sequence $x = (x_1, x_2, \ldots, x_n)$ is said to be an *ascent sequence of length n* if it satisfies $x_1 = 0$ and $0 \le x_i \le \mathrm{asc}(x_1, x_2, \ldots, x_{i-1}) + 1$ for all $2 \le i \le n$. Ascent sequences were introduced by Bousquet-Mélou et al. [2] to unify three combinatorial structures. Bousquet-Mélou et al. [2] constructed bijections between unlabeled (2 + 2)-free posets and ascent sequences, between ascent sequences and permutations avoiding a certain pattern, between unlabeled (2 + 2)-free posets and a class of involutions introduced by Stoimenow [7].

Recently, Kitaev and Remmel [6] extended the work of Bousquet-Mélou et al. [2]. They found the generating function for unlabeled (2 + 2)-free posets when four statistics are taken into account, one of which is the number of minimal elements in a poset. The key strategy used by Bousquet-Mélou et al. [2] and Kitaev and Remmel [6] is to translate statistics on (2 + 2)-free posets to statistics on ascent sequences using the bijection between unlabeled (2 + 2)-free posets and ascent sequences given by Bousquet-Mélou et al. [2]. Let $p_{n,k}$ be the number of (2 + 2)-free posets on n elements with k minimal elements, with the assumption $p_{0,0} = 1$. Under the bijection between unlabeled (2 + 2)-free posets and ascent sequences, the number of unlabeled (2 + 2)-free posets on n elements with k minimal elements is equal to that of ascent sequences of length n with k zeros. Kitaev and Remmel [6] showed that the generating function for the number $p_{n,k}$ is given by

$$P(t,z) = \sum_{n\ge 0, k\ge 0} p_{n,k} z^k t^n = 1 + \sum_{n\ge 0} \frac{zt}{(1-tz)^{n+1}} \prod_{i=1}^{n} (1-(1-t)^i),$$

by counting ascent sequences with respect to the length and the number of zeros. Moreover, they conjectured that the function $P(t, z)$ can be written in a simpler form.

Conjecture 1.1.

$$P(t,z) = \sum_{n\ge 0, k\ge 0} p_{n,k} z^k t^n = \sum_{n\ge 0} \prod_{i=1}^{n} (1-(1-t)^{i-1}(1-zt)). \tag{1.2}$$

The objective of this paper is to give a combinatorial proof of Conjecture 1.1. In order to prove the conjecture, we need two more combinatorial structures: upper triangular matrices with non-negative integer entries such that all rows and columns contain at least one non-zero entry, which was introduced by Dukes and Parviainen [3], and upper triangular (0, 1)-matrices in which all columns contain at least one non-zero entry.

Let $\mathcal{A}_n$ be the collection of upper triangular matrices with non-negative integer entries which sum to n. A (0, 1)-*matrix* is a matrix in which each entry is either 0 or 1. Let $\mathcal{M}_n$ be the set of (0, 1)-matrices in $\mathcal{A}_n$ in which all columns contain at least one non-zero entry. Denote by $\mathcal{I}_n$ the set of matrices in $\mathcal{A}_n$ in which all rows and columns contain at least one non-zero entry. Given a matrix A, denote by $A_{i,j}$ the entry in row i and column j. Let $\dim(A)$ be the number of rows in matrix A. The sum of all entries in row i is called the *row sum* of row i, denoted by $\mathrm{rsum}_i(A)$. The *column sum* of column i, denoted by $\mathrm{csum}_i(A)$, can be defined similarly. A row is said to be *zero* if its row sum is zero.

Let A be a matrix in $\mathcal{M}_n$, define $\min_i(A)$ to be the least value j such that $A_{j,i}$ is non-zero. A column i of A is said to be *improper* if it satisfies one of the following two cases: (1) $\mathrm{csum}_i(A) \ge 2$; (2) for $1 < i \le \dim(A)$, we have $\mathrm{csum}_i(A) = 1$, $\mathrm{rsum}_i(A) = 0$, and $\min_i(A) < \min_{i-1}(A)$. Otherwise, column i is said to be *proper*. Matrix A is said to be *improper* if there is at least one improper column in A; otherwise, matrix A is said to be *proper*. Since each column of a proper matrix must contain exactly one 1, all proper matrices in $\mathcal{M}_n$ have dimension n. Given an improper matrix $A \in \mathcal{M}_n$, define $\mathrm{index}(A)$ to be the largest value i such that column i is improper. Note that the $\mathrm{index}(A)$ of a matrix A defined here is different from that introduced by Dukes and Parviainen [3]. Denote by $\mathcal{PM}_n$ the set of proper matrices in $\mathcal{M}_n$.

Example 1.2. Consider the following matrix $A \in \mathcal{M}_8$:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have $\dim(A) = 6, \min_1(A) = 1, \min_2(A) = 2, \min_3(A) = 1, \min_4(A) = 2, \min_5(A) = 2, \min_6(A) = 1$. There are two improper columns, columns 3 and 6. Hence, we have $\text{index}(A) = 6$.

Denote by $\mathcal{PM}_{n,k}$ the set of matrices $A \in \mathcal{PM}_n$ with $\text{rsum}_1(A) = k$ and $\mathcal{I}_{n,k}$ the set of matrices $A \in \mathcal{I}_n$ with $\text{rsum}_1(A) = k$. Dukes and Parviainen [3] constructed a recursive bijection between the set $\mathcal{I}_n$ and the set of ascent sequences of length n. Under their bijection, they showed that the number of upper triangular matrices $A \in \mathcal{I}_n$ with $\text{rsum}_1(A) = k$ is equal to the number of ascent sequences of length n with k zeros, which implies that the cardinality of $\mathcal{I}_{n,k}$ is also given by $p_{n,k}$. In this paper, we will prove Conjecture 1.1 by showing that the generating function for the number of matrices in $\mathcal{I}_{n,k}$ is given by the right-hand side of Formula (1.2).

In Section 2, we present a parity reversing and weight preserving involution on the set $\mathcal{M}_n \setminus \mathcal{PM}_n$. In Section 3, we prove that the right-hand side of Formula (1.2) is the generating function for the number of matrices in $\mathcal{PM}_{n,k}$. Moreover, we show that there is a bijection between the set $\mathcal{PM}_{n,k}$ and the set $\mathcal{I}_{n,k}$ in answer to Conjecture 1.1.

2. A parity reversing and weight preserving involution

In this section, we will construct a parity reversing and weight preserving involution on the set $\mathcal{M}_n \setminus \mathcal{PM}_n$. Before constructing the involution, we need some definitions.

Given a matrix $A \in \mathcal{M}_n$, the *weight* of matrix A is assigned by $z^{\text{rsum}_1(A)}$. Given a subset S of the set $\mathcal{M}_n$, the *weight* of S, denoted by $W(S)$, is the sum of the weights of all matrices in S. We define the *parity* of matrix A to be the parity of the number $n - \dim(A)$. Denote by $\mathcal{EM}_n$ (resp. $\mathcal{OM}_n$) the set of matrices in $\mathcal{M}_n$ whose parity is even (resp. odd).

Theorem 2.1. *There is a parity reversing and weight preserving involution Φ on the set $\mathcal{M}_n \setminus \mathcal{PM}_n$. Furthermore, we have*

$$W(\mathcal{EM}_n) - W(\mathcal{OM}_n) = W(\mathcal{PM}_n).$$

Proof. Given a matrix $A \in \mathcal{M}_n \setminus \mathcal{PM}_n$, suppose that $\text{index}(A) = i$. We now have two cases. (1) We have $\text{csum}_i(A) \geq 2$. (2) We have $1 < i \leq \dim(A)$, $\text{csum}_i(A) = 1$, $\text{rsum}_i(A) = 0$, and $\min_i(A) < \min_{i-1}(A)$.

For Case (1), we obtain a new matrix $\Phi(A)$ from matrix A in the following way. In A, replace the entry in row $\min_i(A)$ of column i with zero. Then, insert a new zero row between row i and row $i+1$ and insert a new column between column i and $i+1$. Let the new column be filled with all zeros except that the entry in row $\min_i(A)$ is filled with 1. In this case, we have $\Phi(A) \in \mathcal{M}_n \setminus \mathcal{PM}_n$ with $\text{index}(\Phi(A)) = i+1$, $\dim(\Phi(A)) = \dim(A) + 1$ and $\text{rsum}_1(\Phi(A)) = \text{rsum}_1(A)$.

For Case (2), we may obtain a new matrix $\Phi(A)$ by reversing the construction for Case (1) as follows. In A, replace the entry in row $\min_i(A)$ of column $i-1$ with 1. Then remove column i and row i. In this case, we have $\Phi(A) \in \mathcal{M}_n \setminus \mathcal{PM}_n$ with $\text{index}(\Phi(A)) = i-1$, $\dim(\Phi(A)) = \dim(A) - 1$ and $\text{rsum}_1(\Phi(A)) = \text{rsum}_1(A)$.

In both cases, the map Φ reverses the parities and preserves the weights of the matrices. Hence, we obtain the desired parity reversing and weight preserving involution on the set $\mathcal{M}_n \setminus \mathcal{PM}_n$. Note that if a matrix $A \in \mathcal{M}_n$ is proper, then there is exactly one 1 in each column. Hence for each $A \in \mathcal{PM}_n$, the parity of A is even. By applying the involution, we can deduce that

$$W(\mathcal{EM}_n) - W(\mathcal{OM}_n) = W(\mathcal{PM}_n). \quad \square$$

Example 2.2. Consider the following two matrices in $\mathcal{M}_7$:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For matrix A, we have $\text{index}(A) = 3$. Thus we have

$$\Phi(A) = \begin{bmatrix} 1 & 1 & 0 & \mathbf{1} & 0 \\ 0 & 1 & 1 & \mathbf{0} & 0 \\ 0 & 0 & 1 & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} & 1 \end{bmatrix},$$

where the new inserted row and column are illustrated in bold.

For matrix B, we have $\text{index}(B) = 4$. Thus we have

$$\Phi(B) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In fact, we have $\Phi(A) = B$ and $\Phi(B) = A$.

3. Proof of Conjecture 1.1

In this section, we will show that the right-hand side of Formula (1.2) is the generating function for the number of matrices in $\mathcal{PM}_{n,k}$. Furthermore, we prove that there is a bijection between the set $\mathcal{PM}_{n,k}$ and the set $\mathcal{I}_{n,k}$, which implies Conjecture 1.1.

Let

$$A(t,z) = \sum_{n\geq 0} \prod_{i=1}^{n} (1 - (1-t)^{i-1}(1-zt)).$$

With the assumption that the empty product is as usual taken to be 1, we have

$$A(t,z) = 1 + \sum_{n\geq 1} \prod_{i=1}^{n} \sum_{j=1}^{i-1} \left(\binom{i-1}{j} + z\binom{i-1}{j-1} \right) (-1)^{j-1} t^j.$$

Define $A_n(z)$ to be the coefficient of t^n in $A(t,z)$ for $n \geq 1$, that is

$$A(t,z) = 1 + \sum_{n\geq 1} A_n(z) t^n. \tag{3.1}$$

Thus we have

$$A_n(z) = \sum_{d=1}^{n} \sum_{n_1+n_2+\cdots+n_d=n} (-1)^{n-d} \prod_{j=1}^{d} \left(\binom{j-1}{n_j} + z\binom{j-1}{n_j-1} \right),$$

where the second summation is over all compositions $n_1 + n_2 + \cdots + n_d = n$ such that $n_j \geq 1$ for $j = 1, 2, \ldots, d$.

Lemma 3.1. *For $n \geq 1$, we have*

$$A_n(z) = W(\mathcal{EM}_n) - W(\mathcal{OM}_n).$$

Proof. Let $\mathcal{M}(n_1, n_2, \ldots, n_d)$ be the set of matrices in $\mathcal{M}_n$ with d columns in which the column sum of column j is equal to n_j for all $1 \leq j \leq d$. In order to get a matrix $A \in \mathcal{M}(n_1, n_2, \ldots, n_d)$, we should choose n_j places in column j to arrange 1's for all $1 \leq j \leq d$. We have two cases. (1) If $A_{1,j} = 0$, then we have $\binom{j-1}{n_j}$ ways to arrange 1's in column j. (2) If $A_{1,j} = 1$, then we have $\binom{j-1}{n_j-1}$ ways to arrange the remaining 1's in column j. In the former case, column j contributes 1 to the weight of A. While in the latter case, column j contributes z to the weight of A. Altogether, column j contributes $\binom{j-1}{n_j} + z\binom{j-1}{n_j-1}$ to the weight of $\mathcal{M}(n_1, n_2, \ldots, n_d)$, which implies that

$$W(\mathcal{M}(n_1, n_2, \ldots, n_d)) = \prod_{j=1}^{d}\left(\binom{j-1}{n_j} + z\binom{j-1}{n_j-1}\right).$$

It is clear that the parity of each matrix in $\mathcal{M}(n_1, n_2, \ldots, n_d)$ is the parity of the number $n-d$. When d ranges from 1 to n and $n_1, n_2, \ldots, n_d$ range over all compositions $n_1 + n_2 + \cdots + n_d = n$ such that $n_j \geq 1$ for all $1 \leq j \leq d$, we get the desired result. □

Denote by $a_{n,k}$ the cardinality of the set $\mathcal{PM}_{n,k}$. Assume that $a_{0,0} = 1$.

Theorem 3.2. *We have*

$$A(t, z) = \sum_{n\geq 0, k\geq 0} a_{n,k} z^k t^n = \sum_{n\geq 0}\prod_{i=1}^{n}(1 - (1-t)^{i-1}(1-zt)).$$

Proof. Combining Theorem 2.1 and Lemma 3.1, we deduce that $A_n(z) = W(\mathcal{PM}_n)$ for $n \geq 1$. Note that $W(\mathcal{PM}_n) = \sum_{k=1}^{n} a_{n,k} z^k$ for $n \geq 1$. Hence we have

$$A(t, z) = 1 + \sum_{n\geq 1} A_n(z) t^n = \sum_{n\geq 0, k\geq 0} a_{n,k} z^k t^n,$$

which implies the desired result. □

From Theorem 3.2, in order to prove Conjecture 1.1, it suffices to prove that $a_{n,k} = p_{n,k}$. In a matrix A, the operation of adding column i to column j is defined by increasing $A_{k,j}$ by $A_{k,i}$ for each $k = 1, 2, \ldots, \dim(A)$. Note that a matrix $A \in \mathcal{M}_n$ is proper if and only if it satisfies

- each column has exactly one 1;
- if $\mathrm{rsum}_i(A) = 0$, then we have $\min_i(A) \geq \min_{i-1}(A)$ for $2 \leq i \leq \dim(A)$.

This observation will be essential in the construction of the bijection between the set $\mathcal{PM}_{n,k}$ and the set $\mathcal{I}_{n,k}$.

Theorem 3.3. *There is a bijection between the set $\mathcal{PM}_{n,k}$ and the set $\mathcal{I}_{n,k}$.*

Proof. Let A be a matrix in $\mathcal{PM}_{n,k}$, we now construct a matrix A' in $\mathcal{I}_{n,k}$. If there is no zero row in A, then we do nothing for A and let $A' = A$. In this case, the resulting matrix A' is contained in $\mathcal{I}_{n,k}$. Otherwise, we can construct a new upper triangular matrix A' by the following *removal* algorithm.

- Find the least value i such that row i is a zero row. Then we obtain a new upper triangular matrix by adding column i to column $i-1$ and removing column i and row i.
- Repeat the above procedure for the resulting matrix until there is no zero row in the resulting matrix.

Clearly, the obtained matrix A' is a matrix in $\mathcal{I}_n$. Since the algorithm preserves the sums of entries in each non-zero row of A, we have $\mathrm{rsum}_1(A') = \mathrm{rsum}_1(A)$. Hence, the resulting matrix A' is in $\mathcal{I}_{n,k}$.

Conversely, we can construct a matrix in $\mathcal{PM}_{n,k}$ from a matrix in $\mathcal{I}_{n,k}$. Let B be a matrix in $\mathcal{I}_{n,k}$. If the sum of entries in each column is equal to 1, then we do nothing for B and let $B' = B$. Otherwise, we can construct a new upper triangular matrix B' by the following *addition* algorithm.

- Find the largest value i such that $\mathrm{csum}_i(B) \geq 2$. Then we obtain a new upper triangular matrix by decreasing the entry in row $\max_i(B)$ of column i by 1, where $\max_i(B)$ is defined to be the largest value j such that $B_{j,i}$ is non-zero. Since B is upper triangular, we have $\max_i(B) \leq i$.

- Insert one column between column i and column $i+1$ and one zero row between row i and row $i+1$ such that the new inserted column is filled with all zeros except that the entry in row $\max_i(B)$ is filled with 1.
- Repeat the above procedure for the resulting matrix until there is no column whose column sum is larger than 1.

Clearly, the obtained matrix B' is a matrix in $\mathcal{M}_n$. From the construction of the above algorithm we know that the column sum of each column in B' is equal to 1. Furthermore, if row j is a zero row, then we must have $\min_j(B') \geq \min_{j-1}(B')$. Thus, the resulting matrix B' is proper. Since the algorithm preserves the sums of entries in each non-zero row of B, we have $\mathrm{rsum}_1(B') = \mathrm{rsum}_1(B)$. Hence, the resulting matrix B' is in $\mathcal{PM}_{n,k}$. This completes the proof. $\square$

Example 3.4. Consider a matrix $A \in \mathcal{PM}_{6,3}$. By applying the removal algorithm, we get

$$A = \begin{bmatrix} 1 & 1 & \mathbf{0} & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & \mathbf{0} & 0 & 1 & 1 \\ 0 & 0 & \mathbf{0} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 1 & 1 & \mathbf{0} & 0 \\ 0 & 1 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 1 & 1 & \mathbf{0} \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \leftrightarrow A' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

where the removed rows and columns are illustrated in bold at each step of the removal algorithm. Conversely, given $A' \in \mathcal{I}_{6,3}$, by applying the addition algorithm, we can get $A \in \mathcal{PM}_{6,3}$, where the inserted new rows and columns are illustrated in bold at each step of the addition algorithm.

Combining Theorems 2.1, 3.2 and 3.3, we obtain a combinatorial proof of Conjecture 1.1. Note that specializing $z = 1$ implies a combinatorial proof of Formula (1.1), which was proved by Bousquet-Mélou et al. [2] by using functional equations and the Kernel method.

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