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# Graph of triangulations of a convex polygon and tree of triangulations ☆

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**Abstract**

Define a graph $G_T(n)$ with one node for each triangulation of a convex $n$-gon. Place an edge between each pair of nodes that differ by a single flip: two triangles forming a quadrilateral are exchanged for the other pair of triangles forming the same quadrilateral. In this paper we introduce a tree of all triangulations of polygons with any number of vertices which gives a unified framework in which several results on $G_T(n)$ admit new and simple proofs. 

*Keywords:* Simple polygon; Triangulation; Flipping; Binary tree; Rotation; Outerplanar graph; Hamiltonian graph

## 1. Introduction

Triangulating a polygon plays a central role in Computational Geometry, and is a basic step in many algorithms. A related structure is the triangulation of a set $S$ of $n$ points of the plane. When two adjacent triangles form a convex quadrilateral then the shared diagonal can be *flipped* and a new triangulation of $S$ is obtained. This is a well-known process, that allows the construction of the Delaunay Triangulation by successive flips selected with a local criterion [6], and that is also useful for enumerative purposes [1].

For a given polygon or point set, its graph of triangulations is defined as the graph having as nodes its triangulations, that are considered adjacent when they differ by a flip. These graphs are widely studied in [11]. In this paper we focus on the special and relevant case of convex polygons: all convex $n$-gons have the same graph of triangulations, which we denote by $G_T(n)$. This graph is isomorphic to the *rotation graph* of binary trees of with $n-2$ internal nodes, denoted $RG(n-2)$. The graph $RG(n-2)$ has one vertex for each binary tree with $n-2$ internal nodes, and an edge between nodes $T$ and $T'$ if there is a rotation that changes $T$ into $T'$. By taking a fixed edge $e$ of a convex polygon as a root, any triangle with base $e$ has two additional sides that can each be recursively considered as roots for subtrees;

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in this way we obtain a one to one correspondence between binary trees with $n-2$ internal nodes and triangulations of an $n$-gon in which diagonal flips correspond to rotations, so $RG(n-2)$ is isomorphic to $G_T(n)$. In [16] Sleator et al. considered and solved the problem of determining the diameter of $RG(n)$ by using 3-dimensional hyperbolic geometry. In [13] Lucas proved that the rotation graph has a Hamiltonian cycle, with a long and intricate proof, using a particular way of encoding binary trees. Some of the results in [13] were revisited and some others added in [14], where the authors mention the interest in obtaining a simpler proof that a Hamiltonian cycle exists. In [12] Lee proved that the graph of triangulations $G_T(n)$ can be realized as the skeleton of a convex $(n-3)$-polytope called the *associahedron*, a particular case of a more general construction known as *secondary polytopes* [2,17]. This fact and Balinski's theorem for polytopes [17] show that the vertex-connectivity of $G_T(n)$ is $n-3$. The realization is also used in [12] to prove that the automorphism group of $G_T(n)$ is the dihedral group $D_n$ of symmetries of a regular $n$-gon. To our knowledge, no general theorem on polytopes, in the spirit of [15], implies the existence of a Hamiltonian cycle in $G_T(n)$.

In this paper we introduce a hierarchy for all triangulations of polygons with any number of vertices, which are organized in an infinite tree; besides its intrinsic interest this gives us a unified framework in which several of the above results on $G_T(n)$ – Hamiltonicity, vertex-connectivity, center and group of automorphisms – admit new and simple proofs.

The paper is organized as follows. In Section 2 we give definitions and preliminary results, all triangulations of polygons are organized in a tree in Section 3, and in Section 4 we give some applications of such a structure.

## 2. Definitions and preliminaries

We use standard notations and terminology in graph theory as in [3]. In particular, the distance between two nodes $u$ and $v$, i.e., the length of the shortest path between $u$ and $v$, will be denoted $d(u,v)$, and the eccentricity of a node $u$ – the maximum distance from $u$ to any other vertex – will be denoted $e(u)$. The set of nodes with minimum eccentricity is the *center* of the graph.

A convex polygon $P$ with $n$ sides will be described by listing its vertices $v_1,\dots,v_n$ in counterclockwise order, the arithmetic of the indices being done mod $n$. The internal diagonal joining vertices $v_i$ and $v_j$ will be denoted $\delta_{i,j}$. For convenience sides of the polygon are considered as diagonals (but the adjective internal is not used), so in particular $\delta_{i,i+1}$ is the edge $v_iv_{i+1}$. Two diagonals are *noncrossing* when they share no interior points.

The partition of the interior of $P$ into triangles by means of a set of noncrossing diagonals is called a *triangulation* of the polygon. The partition uses always $n-3$ internal diagonals. The set of triangulations of a polygon $P$ will be denoted $\mathcal{T}(P)$. As a diagonal is described by the indices of its extreme points and a triangulation is given by the diagonals it uses, we can consider that all convex $n$-polygons, for $n$ fixed, have the same set of triangulations, that will be denoted simply $\mathcal{T}(n)$, and its cardinality by $t_n$. It is well known that the number $t_n$ agrees with the Catalan number $C_{n-2}=(1/(n-1))\binom{2n-4}{n-2}$ $(n\geqslant 3)$ [7]. Related counting problems for specific triangulations and for non-convex polygons have also been considered recently [5,8–10].

There is a geometric graph naturally associated with a triangulation $T\in\mathcal{T}(P)$, whose nodes are the vertices of $P$, and whose arcs are the edges of the polygon and the diagonals of the triangulation. When no confusion is possible, this graph and the triangulation itself will be essentially identified. A vertex of

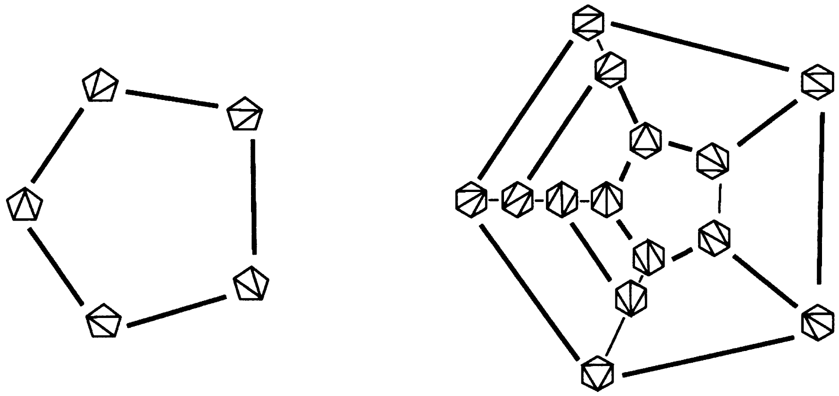

Fig. 1. The graphs $G_T(5)$ and $G_T(6)$.

degree 2 is called an *ear* of $T$; every triangulation has at least two ears. We define the *labeled degree sequence* of $T$ as the sequence $d_1 d_2 \ldots d_n$, where $d_i$ is the degree of $v_i$ in the graph associated with $T$. As mentioned above, the vertices of $P$ are taken in counterclockwise order.

As all triangulation have ears, a triangulation can be easily reconstructed from its labeled degree sequence: find an ear, remove it, and decrease by one its neighboring degrees; then apply the procedure recursively. This bijection between $\mathcal{T}(P)$ and the labeled degree sequences of the $T \in \mathcal{T}(P)$ will be used later.

As $P$ is convex, for every two adjacent triangles in a triangulation $T_1 \in \mathcal{T}(P)$ the diagonal of the quadrilateral they form can be *flipped*, resulting in a new triangulation $T_2$ nearly equal to the former one: we will say that $T_1$ and $T_2$ are *adjacent* and we write $T_1 \sim T_2$. More formally: two triangulations $T_1, T_2 \in \mathcal{T}(P)$ are called adjacent when there are indices $i < j < k < l$ (circularly) such the quadrilateral $v_i v_j v_k v_l$ is present in both $T_1$ and $T_2$, and $T_2 = T_1 - \delta_{i,k} + \delta_{j,l}$.

The *graph of triangulations* $G_T(P)$ of the polygon $P$ has one node for each triangulation of $P$ and an edge between each pair of nodes that correspond to adjacent triangulations, this is, they differ by a single flip.

All convex polygons with $n$ vertices have the same graph of triangulations, denoted simply by $G_T(n)$. For small $n$ we have $G_T(3) = K_1$, $G_T(4) = K_2$, $G_T(5) = C_5$, where $K_m$ is the complete graph with $m$ nodes and $C_m$ is the cycle of length $m$. For $n \geqslant 6$ the situation becomes more intricate. Cases $n = 5$, $n = 6$ are shown in Fig. 1.

As all internal diagonals can be flipped, every triangulation will have exactly $n - 3$ adjacent triangulations. There are no triangles in $G_T(n)$, a result we need later which we prove next.

**Lemma 2.1.** *$G_T(P)$ is triangle-free, for every polygon $P$.*

**Proof.** We describe here the triangulations by the internal diagonals they use. Let $T \in \mathcal{T}(P)$ be $T = \{\delta_1, \ldots, \delta_{n-3}\}$ and let us denote $\delta_i'$ the diagonal obtained by the flip of $\delta_i$; then it is clear that $\delta_i' \notin T$ and that $\delta_i' = \delta_j'$ if and only if $i = j$. If $T \sim T_1$ and $T \sim T_2$ then we can assume without loss of generality that $T_1 = \{\delta_1', \delta_2, \ldots, \delta_{n-3}\}$, $T_2 = \{\delta_1, \delta_2', \ldots, \delta_{n-3}\}$. But then $\delta_1' \notin T_2$ and $\delta_2 \notin T_2$, so that $T_1 \not\sim T_2$. $\square$

There are certain triangulations specially simple in structure called *fans*: the fan $f_i$ is obtained by joining vertex $v_i$ to every other vertex. Note that for $n = 3, 4, 5$ all triangulations are fans. If $T \in \mathcal{T}(n)$ is any triangulation and $d_1 d_2 \ldots d_n$ its labeled degree sequence, then $d(T, f_i) = n - 1 - d_i$ (this is the number of diagonals one has to flip in order to go from $T$ to the fan $f_i$). As the labeled degree sequence determines $T$, so do the numbers $d(T, f_i)$, $i = 1, \ldots, n$.

## 3. A hierarchy for triangulations

In this section we organize all triangulations of polygons with any number of vertices – equivalently, all binary trees – as nodes in a certain (infinite) tree. This structure, of intrinsic interest, allows easy proofs of some properties of the graphs $G_T(n)$, as shown in Section 4.

The elements of $\mathcal{T}(n)$, the set of triangulations of the convex $n$-polygon, will lie on the level $n$ of our tree. To this end, we will accept as a convention the existence of convex polygons with 0, 1, 2 vertices, namely the empty set, a point and a segment. This is just a formality, and we will assume hereafter we deal with the case $n \geqslant 3$. Every $T \in \mathcal{T}(n)$ will have one father, belonging to $\mathcal{T}(n-1)$, and a number of sons, belonging to $\mathcal{T}(n+1)$. Formally: let $T \in \mathcal{T}(n)$ be such that $\delta_{i,n} \in T$; we construct its son $S^i(T)$ as the element in $\mathcal{T}(n+1)$ defined by

$$S^i(T) = \{\delta_{p,q} \mid p, q \neq n,\ \delta_{p,q} \in T\} \cup \{\delta_{p,n+1} \mid 1 \leqslant p \leqslant i,\ \delta_{p,n} \in T\}$$
$$\cup \{\delta_{p,n} \mid i \leqslant p \leqslant n,\ \delta_{p,n} \in T\} \cup \{\delta_{n,n+1}\}.$$

This operation can be quickly understood through a picture (refer to Fig. 2): the convex $n$-polygon is opened like an oyster through the diagonal $\delta_{i,n}$, having the vertex $v_i$ as a hinge. Old vertex $v_n$ splits into two vertices, $v_n$ neighboring $v_{n-1}$, and $v_{n+1}$ neighboring $v_1$; the same splitting occurs to the diagonal $\delta_{i,n}$. Diagonals in the shell containing $v_{n+1}$ are re-labeled if necessary. Finally, the edge $\delta_{n,n+1}$ is added.

The number of sons of a triangulation $T \in \mathcal{T}(n)$ is exactly the degree of $v_n$ in $T$; in particular, $T$ will have at least two sons, namely $S^1(T)$ and $S^{n-1}(T)$ (Fig. 3).

If $T \in \mathcal{T}(n)$ is a son of $\widetilde{T} \in \mathcal{T}(n-1)$ we also say that $\widetilde{T}$ is the father of $T$, and we write $\widetilde{T} = \mathit{father}(T)$ or simply $\widetilde{T} = f(T)$. The father $f(T)$ of $T$ is obtained from $T$ by contracting the edge $v_{n-1}v_n$ (and retaining the label $v_{n-1}$), a usual operation in graph theory. This also shows the uniqueness of the father: different triangulations cannot have a common son. We finally define a binary relation in $\mathcal{T}(n)$ by making $T_1$ related to $T_2$ if and only if they have the same father $f(T_1) = f(T_2)$ (we also say that $T_1$ and $T_2$ are *brothers*); this is clearly an equivalence relation, having $\mathcal{T}(n-1)$ as quotient set.

Now we have an (infinite) tree that has as nodes, by an obvious induction, all triangulations. Nodes at level $n$ are the elements of $\mathcal{T}(n)$, that were also the nodes of $G_T(n)$ (Fig. 4).

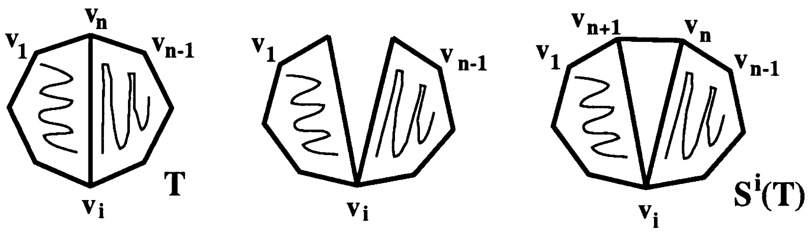


Fig. 2. Construction of the son $S^i(T)$ of $T$.

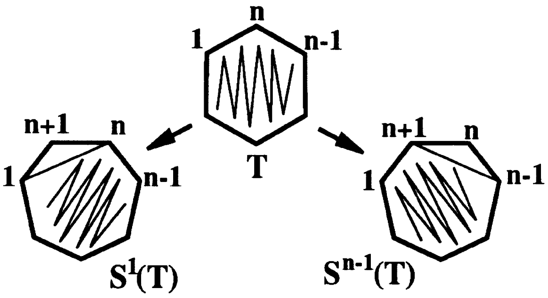


Fig. 3. The sons $S^1(T)$ and $S^{n-1}(T)$.

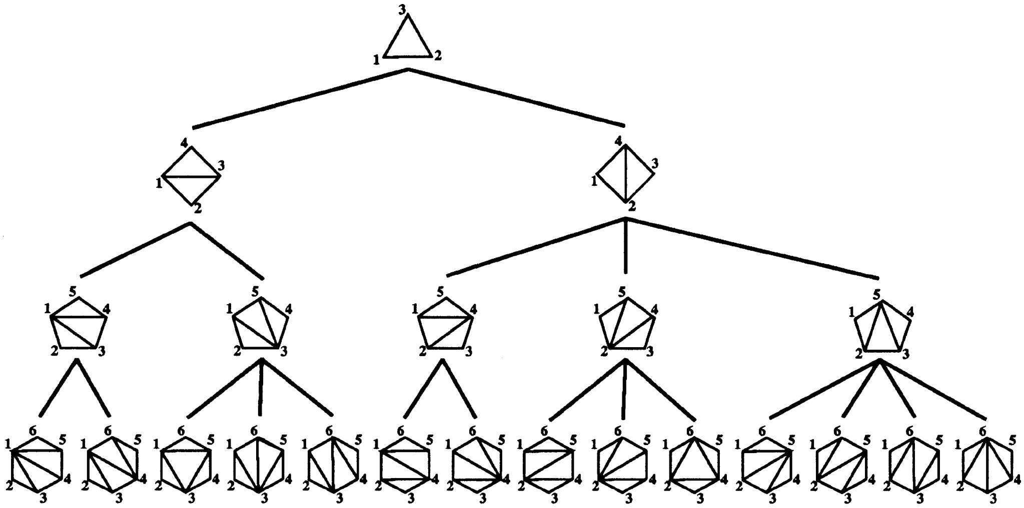

Fig. 4. Levels three to six of the tree of triangulations.

The two lemmas below relate the son/father operations in the tree with the adjacency "$\sim$" flip operation in $G_T(n)$. Proofs are straightforward and omitted.

**Lemma 3.1.** *Let $T$, $T_1$, $T_2 \in \mathcal{T}(n)$. The following properties hold*:

(a) $T_1 \sim T_2 \Rightarrow f(T_1) = f(T_2)$ *or* $f(T_1) \sim f(T_2)$.

(b1) $T_1 \sim T_2$ *and* $\delta_{i,n} \in T_1 \cap T_2 \Rightarrow S^i(T_1) \sim S^i(T_2)$.

(b2) $T_1 \sim T_2 \Rightarrow S^1(T_1) \sim S^1(T_2)$ *and* $S^{n-1}(T_1) \sim S^{n-1}(T_2)$.

(c) $T_1 \neq T_2$ *and* $S^i(T_1) \sim S^j(T_2) \Rightarrow i = j$.

(d) $T_1 \sim T_2 \Rightarrow |\#sons(T_1) - \#sons(T_2)| \leqslant 1$.

(e) *The sons of $T$ induce a subgraph on $G_T(n+1)$ that is a path having as extremes $S^1(T)$ and $S^{n-1}(T)$.*

(f1) *$S^1(T)$ has one neighbor which is a brother $S^j(T)$; the remaining $n-3$ neighbors are of the form $S^1(W_1), \ldots, S^1(W_{n-3})$. The analogous property holds for $S^{n-1}(T)$.*

(f2) *$S^i(T)$ $(i \neq 1, n-1)$ has two neighbors which are its brothers; the remaining $n-4$ neighbors are of the form $S^i(W_1), \ldots, S^i(W_{n-4})$, where the $W_i \neq T$ are distinct elements of $\mathcal{T}(n)$.*

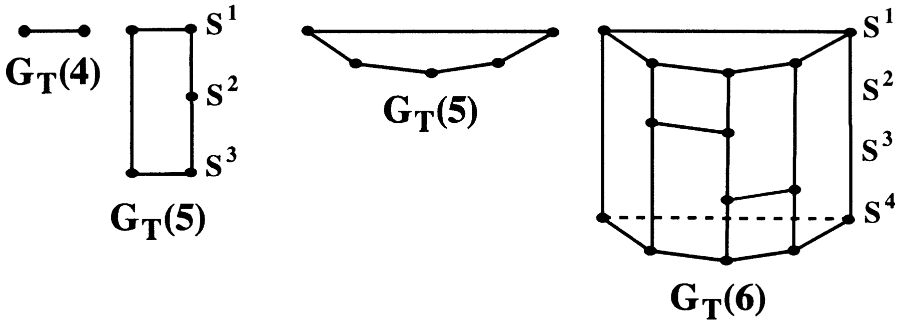


Fig. 5. Layers in $G_T(5)$ and in $G_T(6)$.

We need some suitable notations for two special triangulations of the convex $m$-gon: $F_m$ denotes the fan from $v_m$ (note the slight departure from previous notation, due to the fact that $m$ is fixed), and $E_{m,i}$, for $i = 2, \ldots, m-2$, is the triangulation in which all vertices except $v_i$ are joined with $v_m$, and $v_i$ is an ear.

**Lemma 3.2.**
(a) *The path formed by the sons of $F_n$ is exactly $S^1(F_n) \sim S^2(F_n) \sim \cdots \sim S^{n-2}(F_n) \sim S^{n-1}(F_n)$.*
(b) $S^{n-1}(F_n) = F_{n+1}$, $S^{n-2}(F_n) = E_{n+1,n-1}$.
(c) $S^i(E_{n,k}) \sim S^j(F_n) \Leftrightarrow i = j$ (*observe that $S^k(E_{n,k})$ does not exist*).

Lemma 3.1 tells us how to lift structures in $G_T(n)$ through the tree. Every substructure in $G_T(n)$ can be exactly lifted down to $G_T(n+1)$ via $S^1$ or via $S^{n-1}$ (that we will denote occasionally as the "layer" $S^1$ and the "layer" $S^{n-1}$). If we allow complete blow-up then every node of $G_T(n)$ has to be substituted by the path formed by its sons, and we have to deal with many "new" adjacencies. For lifting up we see that adjacencies are maintained or contracted at the father's level. Lemma 3.2 will be exploited in Section 4.

By thinking $G_T(n+1)$ as decomposed into layers $S^i$, where $S^1$ and $S^{n-1}$ are graphs both isomorphic to $G_T(n)$, one can imagine $G_T(n+1)$ as a kind of cylinder (see Fig. 5).

## 4. Applications of the hierarchy of triangulations

### *4.1. $G_T(n)$ is a Hamiltonian graph*

The tree of triangulations introduced above is a suitable tool that gives a reasonably simple constructive proof of the Hamiltonicity of $G_T(n)$.

**Theorem 4.1.** *$G_T(n)$ is a Hamiltonian graph for $n \geqslant 5$. More precisely, there is a Hamiltonian cycle in which $F_n$ and $E_{n,n-2}$ are neighbors.*

**Proof.** We proceed by induction on $n$. $G_T(5)$ is $C_5$ and $E_{5,3}$ is a neighbor of $F_5$ (Fig. 1). Let us assume now that $G_T(h)$ has a Hamiltonian cycle $C$ as in the statement. By Lemma 3.1, we obtain a copy of $C$ in $G_T(h+1)$ via $S^1$; and a second disjoint copy via $S^{h-1}$. For every node $x$ of $G_T(h)$ the nodes $S^1(x)$

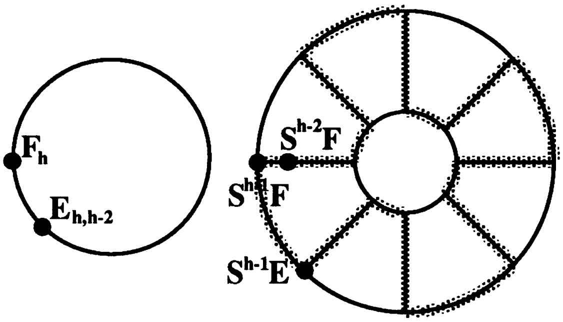


Fig. 6. Constructing a Hamiltonian cycle in $G_T(h+1)$ given such a cycle in $G_T(h)$.

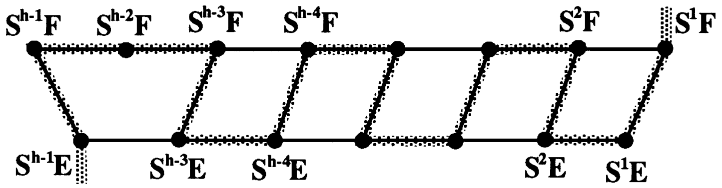


Fig. 7. Intertwining the sons of $F_h$ and $E_{h,h-2}$ when $t_h$ is odd.

and $S^{h-1}(x)$ are connected with the path formed by the sons of $x$. All the nodes of $G_T(h+1)$ belong to some of these paths. By Lemma 3.2, we have $S^{h-1}(F_h) = F_{h+1}$ and $S^{h-2}(F_h) = E_{h+1,h-1}$.

If the order $t_h$ of $G_T(h)$ is even, we simply travel through $G_T(h+1)$ as in a cogwheel (Fig. 6, center). If $t_h$ is odd, the construction of the cycle starts similarly (Fig. 6, right), but the sons of $F_h$ and $E_{h,h-2}$ have to be intertwined suitably.

Let us recall that $t_h$ (the Catalan number $C_{h-2}$) is odd if and only if $h = 2^k + 1$ for some $k$, so $h$ is odd too. Then $F_h$ has an even number of sons and $E_{h,h-2}$ has an odd number of sons (there is no $S^{h-2}(E_{h,h-2})$). The situation is depicted in Fig. 7, where we can also see the completion of the Hamiltonian cycle. □

### 4.2. Connectivity of $G_T(n)$

As a second example of application of the hierarchy introduced above, we compute here the connectivity of the graph $G_T(n)$ by inductively lifting down through the tree.

**Theorem 4.2.** *The vertex-connectivity of the graph $G_T(n)$ $(n \geqslant 5)$ is equal to $n-3$.*

**Proof.** As the degree is $n-3$ we only have to prove that the graph remains connected when any $n-4$ vertices are suppressed. This is clear for $n=5$. We assume that the property holds for $n=h$ and we proceed by induction: we prove that $G_T(h+1)$ is still connected after the removal of any set $W$ of $h-3$ nodes. There are two cases.

(i) $W \subset S^1$. Then we have a path between any two given nodes $x$, $y$ as follows: from $x$ to $S^{h-1}(f(x))$, then to $S^{h-1}(f(y))$, and finally to $y$. The same proof applies when all the removed nodes belong to the layer $S^{h-1}$.

(ii) $W \not\subset S^1$ and $W \not\subset S^{h-1}$, so every one of these two layers is a connected subgraph of $G = G_T(h+1) - W$. We can also be certain that we have some path from $S^1(w)$ to $S^{h-1}(w)$ in $G$ through brothers (a "family" path) because there are $t_h$ of such paths in $G_T(h+1)$, every two are disjoint, and $t_h > h-3$. So it is enough to prove that from any node we can reach either the layer $S^1$ or the layer $S^{h-1}$ through a "family" path in $G$. Let $x = S^i(y) \in G$, with $i \neq 1, h-1$. If both family paths from $x$ to $S^1$ and $S^{h-1}$ are broken then $x = S^i(y)$ will have at least one neighbor of the form $S^i(z)$ in $G$. The vertex $S^i(z)$ has $h-5$ neighboring vertices of the form $S^i(u)$ in $G_T(h+1)$ other than $S^i(y)$, and the situation is symmetric for $S^i(y)$; as $G_T(h+1)$ is triangle-free we get in such a way a total of $2h-10$ vertices in $G_T(h+1)$. Not all the family paths associated with these $2h-10$ vertices can get broken in $G$, because $2h-10 > h-5$, so in $G$ we can move from $x$, and inside the layer $S^i$, to a suitable vertex, then to an extreme layer. □

We see that $G_T(n)$ is a maximally connected graph, i.e., the vertex-connectivity is equal to the minimum degree.

### 4.3. Center and automorphism group of $G_T(n)$

Here we give a last example of application of the hierarchy of triangulations.

**Theorem 4.3.** *The center of $G_T(n)$ consists of the $n$ fans $f_1, f_2, \ldots, f_n$.*

**Proof.** As it is clear that the eccentricity of a fan is equal to $n-3$, it remains to show that if a triangulation $T$ is not a fan then $e(T) > n-3$ or, equivalently, that there exists $T'$ such that $d(T, T') \geqslant n-2$.

We prove this claim by induction on $n$ starting with $n = 6$ since for $n < 6$ all triangulations are fans. The case $n = 6$ is easily dealt with by inspection since there are only three different types of triangulations. If $T \in \mathcal{T}(n+1)$ is not a fan ($n \geqslant 6$), there is an ear vertex $v$ of $T$ such that its removal gives a triangulation $\widehat{T} \in \mathcal{T}(n)$ which is not a fan. By rotating the labels of $T$ if necessary we can assume that $v$ gets the label 1, so that $T = S^1(\widehat{T})$. By induction there is a triangulation $W \in \mathcal{T}(n)$ such that $d(\widehat{T}, W) \geqslant n-2$, which by isomorphism translates into $d(S^1(\widehat{T}), S^1(W)) \geqslant n-2$. So we get $d(T, S^{n-1}(W)) = d(S^1(\widehat{T}), S^{n-1}(W)) \geqslant n-1 = (n+1)-2$. □

As a corollary of the former theorem, we can now completely determine the automorphism group of $G_T(n)$. Since any two convex polygons are equivalent with respect to their triangulations, we are free to work with a regular polygon. It is clear that any symmetry of the regular polygon will induce a corresponding automorphism on the graph of triangulations, since adjacencies will be preserved. We next show that there are no more automorphisms.

**Corollary 4.4.** *The automorphism group $\Gamma(G_T(n))$ is isomorphic to the dihedral group $D_n$ of symmetries of a regular polygon with $n$ sides.*

**Proof.** It is straightforward to see that the distances between fans are

$$d(f_i, f_j) = \begin{cases} n-3, & \text{if } j = i \pm 1, \\ n-4, & \text{otherwise,} \end{cases}$$

where the indices are taken modulo $n$. Now let $\sigma$ be in $\Gamma(G_T(n))$ and consider the action of $\sigma$ on $C$, which being the center of the graph is an invariant set of vertices. Because of the above relations on the distances, if $\sigma f_1 = f_k$ then either $\sigma f_2 = f_{k+1}$ or $\sigma f_2 = f_{k-1}$. In the first case it follows that $\sigma f_3 = f_{k+2}$ and in the second case $\sigma f_3 = f_{k-2}$. Proceeding in this way we see that $\sigma$ is either a rotation or a reflection of the index set $[n]$. This shows that the restriction of $\Gamma(G_T(n))$ to the center is equivalent to the dihedral group $D_n$.

The second part of the proof is to show that an automorphism is completely determined by its action on the center or, in other words, that if $\sigma_{|C} = 1$ then $\sigma = 1$. Let $T$ be any triangulation and $d_1 d_2 \ldots d_n$ its (ordered) degree sequence. We know that $d(T, f_i) = n - 1 - d_i$, but $\sigma$ is trivial on the fans by hypothesis and an automorphism preserves distances, hence

$$d(\sigma T, f_i) = d(\sigma T, \sigma f_i) = d(T, f_i) = n - 1 - d_i.$$

As mentioned in the preliminaries, this implies that $T = \sigma T$, and we conclude that $\sigma = 1$. $\square$

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## Note added

After this paper was ready for publication we have learned of reference [4], where the authors give independent proofs of Theorems 4.2 and 4.3 similar to ours in spirit.

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