Abstract

Early number sense is built within child’s everyday life experience. A number is represented as a state (5 apples), address (5th floor), operator of comparison (Mike is 2 cm taller than Tom) and operator of change (I lost 5 Euros). This study presents didactical environments that enable young pupils to gain ample experience with a number as an operator that opens the doors to the understanding negative numbers. In the paper we describe how pupils gain experience with operators through the didactical environments of Stepping and Staircase. Such experience further enhances their effectiveness in solving of dynamic word problems.

1. Introduction

In the past six years, a research team of experts in the area of didactics of mathematics in Prague has studied and elaborated a teaching method called the schema-oriented education. The principal idea of this method is that a pupil is introduced to certain mathematical didactical environments where s/he solves problems that have already been grounded there. Through discussion and problem-solving, a pupil creates a schema of the particular environment in his consciousness. This schema consists of various and distinct mathematical ideas. The interlinking of several of such schemata leads to the construction of mathematical knowledge. For details see Hejný (2012). The study presents two closely related environments: Stepping and Staircase.

2. Methodology

In a longitudinal research, a series of didactical environments in the sense of Wittman (2001) were created and a large database of lesson protocols, videorecordings, written records of pupils’ work and teacher interviews. The database was continually reviewed, analyzed and classified. The focus of the investigation was on understanding the cognitive mechanisms underlying pupils’ problem-solving processes. The sets of tasks and problems aimed at pupils
between the ages 6 and 12 were repeatedly revised, extended and piloted. Long-term data was obtained from three classrooms.

3. Description of the Stepping and Staircase didactical environments

3.1. Stepping

The Stepping environment can be briefly introduced in several stages.

3.1.1. Introductory stage – Movement and practice of commands

The environment makes use of the natural rhythmical movement of stepping. To begin with, the teacher himself/herself demonstrates stepping to pupils, making for example four steps and counting one, two, three, four marking the pace. Pupils are then invited to join the teacher, first to count along with him/her and then also to make steps. At this stage of stepping the following is interconnected: numbers (their acoustic form), the acoustic rhythm (counting, clapping hands, telling a rhyme) and the rhythm of movement (stepping and clapping hands). This synchrony of rhythms plays a crucial role in mastering the algorithm of counting and is the foundation of arithmetic thinking.

While stepping the pupil becomes familiar with the semantic model of a number. The number is represented by the number of steps, i.e. it is a quantity. Stepping is movement, i.e. the model of the number is processual. Movement results in a change, i.e. the number has the role of an operator of change. As soon as the pupil has finished stepping according to the instruction, the perception (acoustic, visual, kinaesthetic) of the number of steps fades and disappears. Thus the model of the number is temporary.

3.1.2. First stage – multi-part commands

Two pupils, Adam and Eva, stand next to each other by the stepping stripe (Figure 1). The teacher says: Adam, take three steps forwards, then two steps forwards, start now! The class count, clap their hands while Adam is taking the steps. Now the teacher asks the class: Who can give a command to Eva so that she ends up standing next to Adam again? The pupil’s answer is: Eva, take five steps forwards, start now!

3.1.3. Second stage – stepping backwards and negative number

We enrich the stepping by steps backwards. This time the command for Adam is: Two steps forwards, three steps backwards, start now! The corresponding command for Eva is: one step backwards. In this way pupils are introduced to the semantic model of negative number. One step backwards is a representation of number -1, something the pupils will be learning about one or two years later.

In the Stepping environment the operators are not built on any state, or address - such connections usually lead to confusion when a pupil tries to solve certain word problems. When a second-graders are asked to consider the situation 6 people got off the bus and 7 got on, they may be taken aback and ask how many people were on the bus. The operator of change points at two virtual numbers: the state before the change and after the change. The Stepping environment should help pupils to understanding that in tackling the above problem there is no need for knowing the start- and end-states.
3.1.4. Third stage – recording

The number of instructions eventually becomes so large that they cannot be memorized. This is where the need of keeping a record naturally arises. A discussion with the class leads to the idea of using an arrow notation. Thus for example the two-part command *Three steps forwards, two steps backwards, start now!* is expressed as: \[
\begin{array}{c}
\rightarrow \\
\leftarrow
\end{array}
\]
\[
\begin{array}{c}
\rightarrow \\
\leftarrow
\end{array}
\]
The introduction of arrows weakens the temporary model of the used numbers. The stepping process is approached with the concept of recording.

3.1.5. Fourth stage – introduction to equations

Adam and Eva are standing next to each other. The teacher commands: *Eva, four steps forwards, start now!* Eva makes the steps. The teacher then gives the first part of the command: *Adam, three steps forwards,* ... and indicates with a gesture for the class to complete the second part of the instruction. This situation is expressed by arrows as follows: \[
\begin{array}{c}
\rightarrow \\
\leftarrow
\end{array}
\]
\[
\begin{array}{c}
\rightarrow \\
\leftarrow
\end{array}
\]

“Step equations” grow increasingly more complicated, for example: *Eva, two steps backwards, four steps forwards, three steps backwards and one step backwards, start now!* This is followed by another incomplete command: *Adam, three steps forwards, two backwards,...!* and pupils add: *three steps backwards.*

Later pupils solve problems expressed by arrows, e.g.: \[
\begin{array}{c}
\leftarrow \\
\rightarrow \\
\leftarrow
\end{array}
\]
\[
\begin{array}{c}
\leftarrow \\
\rightarrow \\
\leftarrow
\end{array}
\]

3.1.6. Fifth stage – arrow equations, equations with two unknowns

Pupils solve arrow equations such as: \[
\begin{array}{c}
\leftarrow \\
\rightarrow
\end{array}
\]
Using just three arrows find all the possible solutions. This is already a system of two linear equations with two unknowns. In standard mathematical symbols these arrow equations would be recorded as: \(-2 + x = y + 1\) and \(|x| + |y| = 3\). Thus a pupil gets first experience with the idea of absolute value.

3.1.7. Sixth stage – number notation

Translate the problem expressed by arrows to numbers and solve it. Verify by stepping.

\[
\begin{array}{c}
\rightarrow \\
\leftarrow
\end{array}
\]
\[
\begin{array}{c}
\rightarrow \\
\leftarrow
\end{array}
\]
\[
\begin{array}{c}
\rightarrow \\
\leftarrow
\end{array}
\]

Arrow notation is linked to numerical notation continuously. However, at this point it becomes essential for the following stage.

3.1.8. Seventh stage – turn about

Numerical record \(4 - (3 - 1)\) is translated into language of arrows as follows:

\[
\begin{array}{c}
\rightarrow \\
\leftarrow
\end{array}
\]
\[
\begin{array}{c}
\rightarrow \\
\leftarrow
\end{array}
\]

This corresponds to the verbal command: *Take four steps forwards, turn about, take three steps forwards, one backwards, turn about, start now!* The simplified command for the other pupil is: *Make two steps forwards, start now!* The command *turn about* stands for minus in front of a bracket.

This stage more or less concludes the environment Stepping at the primary school level. It provides the pupils with tools for the evaluating of expressions with minus in front of a bracket and with strategies for solving linear equations which they will encounter at the lower secondary school level. They have mastered the semantic model of a negative number – a step backwards and the model of absolute value of a number as the number of steps
regardless of their orientation. This environment also offers the possibility for a brief excursion into the worlds of combinatorics if we look for all the solutions of an equation, or of probability if we throw dice and of statistics if we keep record of repeated throws of dice.

We noticed in several schools that sixth and seventh-graders (ages 12 and 13) who had been instructed in a traditional way, adopted this stepping method from their younger schoolmates as a tool to understanding negative numbers and operations.

3.2. Staircase

In the stepping environment, a number was only representing an operator of change. Here the situation becomes richer because we add the concept of a number as an address. We find a similar situation in problems about buildings with multiple stories, the thermometer, elevators etc.

The transition from the Stepping environment to the Staircase environment is mirrored by a change of the setting: the stepping stripe is changed for the number line. At the same time, the symbolic record changes as well, for example the problem: “Adam is standing on step 2 and takes 3 steps forward. What step is he standing on now?” is recorded as $2 \rightarrow \rightarrow \rightarrow \ ?$.

4. Applying the Stepping and Staircase environments

Both of these environments have been successfully implemented and applied in other areas of mathematics such as combinatorics, probability, statistics, and mainly in solving word problems. Especially word problems that involve a dynamic element have evidently a significantly higher success rate for pupils who have experience with the Staircase environment.

A dynamic problem is a problem where time plays a decisive role. For instance problems involving the meeting of a biker and a pedestrian, filling up a swimming pool or problems about age. We will illustrate the idea on a particular problem-solving process.

**Problem 1.** Eva is three years old. When she is as old as Adam is today, Adam will be 15. How old is Adam today?

**Tom’s (13 yrs) solution:**

$\ x \ ... \ Adam \ today$

$\ y \ ... \ Adam \ when \ he^{’}s \ 15 \ (y = 15)$

$\ 3 \ ... \ Eva \ today$

$15 - 3 = 12$

Adam is 12 years old.

In Tom’s solution, we can notice the lack of understanding of the problem; he is unable to represent the problem in letters and relationships. The failure is caused by not being able to distinguish the ultimate invariant in dynamic problems: time passes at the same for each person. Had Tom discerned this invariant, he would clearly see that the time from “now” until “then” is the same for Eva and for Adam. Then $x - 3 = 15 - x$. This conceptual grasp of a dynamic problem is quite challenging for a large number of pupils at the lower secondary school level.

**Viky’s (9 yrs) solution:**

Viky, in the role of a stage director, enacts the problem in the classroom. She appoints a Commentator and four actors: Eva, Adam, Solution and Chronos, the god of Time. On the floor she places a stepping strip marked with numbers from 3 to 15 (one of the possible staircase models). She positions Eva on number 3 (Eva is 3 years old). Now Viky tries to guess how old Adam. She chooses number 7. She positions both Adam and Solution on number 7 on the stepping strip. Solution is to stay there for the duration of the entire sketch. Viky gives Chronos a sign to start. Chronos commands: “One year has passed, now.” Hearing this, Eva and Adam take one step forward moving onto the respective number marks. The commentator reports: “Eva is now 4, Adam is 8.” This process repeats three more times until the commentator reports that “Eva is now 7, we’re finished. But Adam is only 11, that solution didn’t work out.“ Already throughout the performance some pupils realized that the choice of number 7 was not the
right one and offer to try number 9. The second performance, now positioning Adam and Solution at number, proves to lead to the correct solution. It ends with the Commentator concluding: “Eva is 9, Adam is 15, we’re finished, the problem is solved.”

Soon after this performance, followed by several similarly staged problems, the teacher suggests to appoint also a Scribe who would write on the board everything the Commentator says. This way the successful solution of the problem above will be represented by Complete Table (Figure 2).

\[
\begin{array}{cccccccccc}
E & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
A & 9 & 10 & 11 & 12 & 13 & 14 & 15
\end{array}
\]

Figure 2. Complete Table

Filling out these tables, some pupils find out the importance of the first and last columns. In the next few problems these pupils leave the unnecessary columns out and create a short version of the initial table (Figure 3). They will also record the condition that Eva will be as old as Adam is today by inserting an equal sign as in Figure 4. If we now insert letter \(x\) into each of the empty boxes, and recall that from “now” to “then” means \((x – 3)\) for Eva and \((15 – x)\) for Adam, the desired equation, as indicated above, is discovered. We can find the same equation by making use of the fundamental fact that the age difference between Adam and Eva is time-invariant, i.e. the difference “today” \((x – 3)\) is the same as the difference “then” \((15 – x)\).

Let us take a closer look at the cognitive process taking place in the pupils’ minds: at the first stage, the dynamic situation is modelled by a dramatised act, implementing a trial-error strategy. As the movement of actors is repeated several times, a dynamic schema of the time passing for two different objects is being created in the pupils’ consciousness. Both primary invariants that will play a crucial role in the solution of dynamic problems are observed and re-observed by pupils: the distance between Eva and Adam stays the same and the number of steps that Eva takes from start to finish is the same as the number of steps Adam takes. When the process gets recorded by a table, conceptualizing begins. Table 1 does not change, pupils can even read it backwards, or skip around columns and rows, such process will help some pupils to make a key discovery. Only the first and last columns are essential in this problem. The reduced table is then a concept that leads to an easy conceptualization of the algebraic equation. To conclude, the main tool that leads pupils to the above described knowledge is the change from processual to conceptual understanding. Hejný (1999) has termed this phenomenon proceptual transfer, following the idea of procept as described in Gray and Tall (1994).

Both of the above environments have been elaborated by our team for pupils of ages 12 to 18.

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References