Linear Algebra and its Applications 430 (2009) 782–790

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

A note on noncommutative unique ergodicity and weighted means

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ARTICLE INFO

Article history:
Received 27 February 2008
Accepted 17 September 2008

Submitted by H. Schneider

AMS classification:
47A35
46L35
46L55

Keywords:
Uniquely ergodic
Markov operator
Riesz means

ABSTRACT

In this paper we study unique ergodicity of C^*-dynamical system $(\mathfrak{A},T)$, consisting of a unital C^*-algebra $\mathfrak{A}$ and a Markov operator $T:\mathfrak{A}\mapsto\mathfrak{A}$, relative to its fixed point subspace, in terms of Riesz summation which is weaker than Cesaro one. Namely, it is proven that $(\mathfrak{A},T)$ is uniquely ergodic relative to its fixed point subspace if and only if its Riesz means

$$\frac{1}{p_1+\cdots+p_n}\sum_{k=1}^{n}p_kT^kx$$

converge to $E_T(x)$ in $\mathfrak{A}$ for any $x\in\mathfrak{A}$, as $n\to\infty$, here E_T is an projection of $\mathfrak{A}$ to the fixed point subspace of T. It is also constructed a uniquely ergodic entangled Markov operator relative to its fixed point subspace, which is not ergodic.

1. Introduction

It is known [16,22] that one of the important notions in ergodic theory is unique ergodicity of a homeomorphism T of a compact Hausdorff space Ω. Recall that T is *uniquely ergodic* if there is a unique T-invariant Borel probability measure μ on Ω. The well known Krylov–Bogolyubov theorem [16] states that T is uniquely ergodic if and only if for every $f\in C(\Omega)$ the averages

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doi:10.1016/j.laa.2008.09.029

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)$$

converge uniformly to the constant $\int f\,\mathrm{d}\mu$, as $n \to \infty$.

The study of ergodic theorems in recent years showed that the ordinary Cesaro means have been replaced by weighted averages

$$\sum_{k=0}^{n-1} a_k f(T^k x). \tag{1.1}$$

Therefore, it is natural to ask: is there a weaker summation than Cesaro, ensuring the unique ergodicity. In [15] it has been established that unique ergodicity implies uniform convergence of (1.1), when $\{a_k\}$ is Riesz weight (see also [14] for similar results). In [4] similar problems were considered for transformations of Hilbert spaces.

On the other hand, since the theory of quantum dynamical systems provides a convenient mathematical description of irreversible dynamics of an open quantum system (see [1,5]) investigation of ergodic properties of such dynamical systems have had a considerable growth. In a quantum setting, the matter is more complicated than in the classical case. Some differences between classical and quantum situations are pointed out in [1,19]. This motivates an interest to study dynamics of quantum systems (see [8,9,12]). Therefore, it is then natural to address the study of the possible generalizations to quantum case of various ergodic properties known for classical dynamical systems. In [17,18] a non-commutative notion of unique ergodicity was defined, and certain properties were studied. Recently in [2] a general notion of unique ergodicity for automorphisms of a C^*-algebra with respect to its fixed point subalgebra has been introduced. The present paper is devoted to a generalization of such a notion for positive mappings of C^*-algebras, and its characterization in term of Riesz means.

The paper is organized as follows: Section 2 is devoted to preliminaries, where we recall some facts about C^*-dynamical systems and the Riesz summation of a sequence on C^*-algebras. Here we define a notion of unique ergodicity of C^*-dynamical system relative to its fixed point subspace. In Section 3 we prove that a C^*-dynamical system $(\mathfrak{A}, T)$ is uniquely ergodic relative to its fixed point subspace if and only if its Riesz means (see below)

$$\frac{1}{p_1+\cdots+p_n}\sum_{k=1}^{n} p_k T^k x$$

converge to $E_T(x)$ in $\mathfrak{A}$ for any $x \in \mathfrak{A}$, here E_T is a projection of $\mathfrak{A}$ onto the fixed point subspace of T. Note however that if T is completely positive then E_T is a conditional expectation (see [6,20]). On the other hand it is known [18] that unique ergodicity implies ergodicity. Therefore, one can ask: can a C^*-dynamical system which is uniquely ergodic relative to its fixed point subspace be ergodic? It turns out that this question has a negative answer. More precisely, in Section 4 we construct entangled Markov operator which is uniquely ergodic relative to its fixed point subspace, but which is not ergodic.

2. Preliminaries

In this section we recall some preliminaries concerning C^*-dynamical systems.

Let $\mathfrak{A}$ be a C^*-algebra with unit $\mathbb{1}$. An element $x \in \mathfrak{A}$ is called *positive* if there is an element $y \in \mathfrak{A}$ such that $x = y^*y$. The set of all positive elements will be denoted by $\mathfrak{A}_+$. By $\mathfrak{A}^*$ we denote the conjugate space to $\mathfrak{A}$. A linear functional $\varphi \in \mathfrak{A}^*$ is called *Hermitian* if $\varphi(x^*) = \overline{\varphi(x)}$ for every $x \in \mathfrak{A}$. A Hermitian functional φ is called *state* if $\varphi(x^*x) \geqslant 0$ for every $x \in \mathfrak{A}$ and $\varphi(\mathbb{1}) = 1$. By $S_{\mathfrak{A}}$ (resp. $\mathfrak{A}_h^*$) we denote the set of all states (resp. Hermitian functionals) on $\mathfrak{A}$. By $M_n(\mathfrak{A})$ we denote the set of all $n \times n$-matrices $a = (a_{ij})$ with entries a_{ij} in $\mathfrak{A}$.

Definition 2.1 A linear operator $T : \mathfrak{A} \mapsto \mathfrak{A}$ is called:

(i) *positive*, if $Tx \geqslant 0$ whenever $x \geqslant 0$;

(ii) *n-positive* if the linear mapping $T_n : M_n(\mathfrak{A}) \mapsto M_n(\mathfrak{A})$ given by $T_n(a_{ij}) = (T(a_{ij}))$ is positive;
(iii) *completely positive* if it is n-positive for all $n \in \mathbb{N}$.

A positive mapping T with $T\mathbb{1} = \mathbb{1}$ is called *Markov operator*. A pair $(\mathfrak{A}, T)$ consisting of a C^*-algebra $\mathfrak{A}$ and a Markov operator $T : \mathfrak{A} \mapsto \mathfrak{A}$ is called *a C^*-dynamical system*. The C^*-dynamical system $(\mathfrak{A}, \varphi, T)$ is called *uniquely ergodic* if there is a unique invariant state φ (i.e. $\varphi(Tx) = \varphi(x)$ for all $x \in \mathfrak{A}$) with respect to T. Denote

$$\mathfrak{A}^T = \{x \in \mathfrak{A} : Tx = x\}. \tag{2.1}$$

It is clear that $\mathfrak{A}^T$ is a closed linear subspace of $\mathfrak{A}$, but in general it is not a subalgebra of $\mathfrak{A}$ (see Section 3). We say that $(\mathfrak{A}, T)$ is *uniquely ergodic relative to* $\mathfrak{A}^T$ if every state of $\mathfrak{A}^T$ has a unique T-invariant state extension to $\mathfrak{A}$. In the case when $\mathfrak{A}^T$ consists only of scalar multiples of the identity element, this reduces to the usual notion of unique ergodicity. Note that for an automorphism such a notion has been introduced in [2].

Now suppose we are given a sequence of numbers $\{p_n\}$ such that $p_1 > 0, p_k \geqslant 0$ with $\sum_{k=1}^{\infty} p_k = \infty$. We say that a sequence $\{s_n\} \subset \mathfrak{A}$ is *Riesz convergent* to an element $s \in \mathfrak{A}$ if the sequence

$$\frac{1}{\sum_{k=1}^{n} p_k} \sum_{k=1}^{n} p_k s_k$$

converges to s in $\mathfrak{A}$, and it is denoted by $s_n \to s(R, p_n)$. The numbers p_n are called *weights*. If $s_n \to s$ implies $s_n \to s(R, p_n)$ then Riesz-converges is said to be *regular*. The regularity condition (see [13, Theorem 14]) is equivalent to

$$\frac{p_n}{p_1 + p_2 + \cdots + p_n} \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

Basics about (R, p_n) convergence can be found in [13].

Recall the following lemma which shows that Riesz convergence is weaker than Cesaro convergence (see [13,15]).

Lemma 2.2 [13, Theorem 16]. *Assume that $p_{n+1} \leqslant p_n$ and*

$$\frac{np_n}{p_1 + \cdots + p_n} \leqslant C \quad \forall n \in \mathbb{N} \tag{2.3}$$

for some constant $C > 0$. Then Cesaro convergence implies (R, p_n) convergence.

3. Unique ergodicity

In this section we are going to characterize unique ergodicity relative to $\mathfrak{A}^T$ of C^*-dynamical systems. To do it we need the following.

Lemma 3.1 (cf. [18,2]). *Let $(\mathfrak{A}, T)$ be uniquely ergodic relative to $\mathfrak{A}^T$. If $h \in \mathfrak{A}^*$ is invariant with respect to T and $h\lceil\mathfrak{A}^T = 0$, then $h = 0$.*

Proof. Let us first assume that h is Hermitian. Then there is a unique Jordan decomposition [21] of h such that

$$h = h_+ - h_-, \quad \|h\|_1 = \|h_+\|_1 + \|h_-\|_1, \tag{3.1}$$

where $\|\cdot\|_1$ is the dual norm on $\mathfrak{A}^*$. The invariance of h implies that

$$h \circ T = h_+ \circ T - h_- \circ T = h_+ - h_-.$$

Using $\|h_+ \circ T\|_1 = h_+(\mathbb{1}) = \|h_+\|_1$, similarly $\|h_+ \circ T\|_1 = \|h_+\|_1$, from uniqueness of the decomposition we find $h_+ \circ T = h_+$ and $h_- \circ T = h_-$. From $h\lceil\mathfrak{A}^T = 0$ one gets $h(\mathbb{1}) = 0$, which implies that $\|h_+\|_1 =$

$\|h_-\|_1$. On the other hand, we also have $\frac{h_+}{\|h_+\|_1} = \frac{h_-}{\|h_-\|_1}$ on $\mathfrak{A}^T$. So, according to the unique ergodicity relative to $\mathfrak{A}^T$ we obtain $h_+ = h_-$ on $\mathfrak{A}$. Consequently, $h = 0$. Now let h be an arbitrary bounded, linear functional. Then it can be written as $h = h_1 + ih_2$, where h_1 and h_2 are Hermitian. Again invariance of h implies that $h_i \circ T = h_i$, $i = 1, 2$. From $h \upharpoonright \mathfrak{A}^T = 0$ one gets $h_k \upharpoonright \mathfrak{A}^T = 0$, $k = 1, 2$. Consequently, according to the above argument, we obtain $h = 0$. □

Now we are ready to formulate a criterion for unique ergodicity of C^*-dynamical system in terms of (R, p_n) convergence. In the proof we will follow some ideas used in [2,15,18].

Theorem 3.2. *Let $(\mathfrak{A}, T)$ be a C^*-dynamical system. Assume that the weight $\{p_n\}$ satisfies*

$$P(n) := \frac{p_1 + |p_2 - p_1| + \cdots + |p_n - p_{n-1}| + p_n}{p_1 + p_2 + \cdots + p_n} \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

Then the following conditions are equivalent:

(i) *$(\mathfrak{A}, T)$ is uniquely ergodic relative to $\mathfrak{A}^T$.*

(ii) *The set $\mathfrak{A}^T + \{a - T(a) : a \in \mathfrak{A}\}$ is dense in $\mathfrak{A}$.*

(iii) *For all $x \in \mathfrak{A}$,*

$$T^n x \to E_T(x)\ (R, p_n),$$

where $E_T(x)$ is a positive norm one projection onto $\mathfrak{A}^T$ such that $E_T T = TE_T = E_T$. Moreover, the following estimation holds:

$$\left\| \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k T^k(x) - E_T(x) \right\| \leqslant P(n)\|x\|, \quad n \in \mathbb{N} \tag{3.3}$$

for every $x \in \mathfrak{A}$.

(iv) *For every $x \in \mathfrak{A}$ and $\psi \in S_{\mathfrak{A}}$*

$$\psi(T^k(x)) \to \psi(E_T(x))(R, p_n).$$

Proof. Consider the implication (i)⇒(ii). Assume that $\overline{\mathfrak{A}^T + \{a - T(a) : a \in \mathfrak{A}\}} \neq \mathfrak{A}$; then there is an element $x_0 \in \mathfrak{A}$ such that $x_0 \notin \overline{\mathfrak{A}^T + \{a - T(a) : a \in \mathfrak{A}\}}$. Then according to the Hahn–Banach theorem there is a functional $h \in \mathfrak{A}^*$ such that $h(x_0) = 1$ and $h \upharpoonright \overline{\mathfrak{A}^T + \{a - T(a) : a \in \mathfrak{A}\}} = 0$. The last condition implies that $h \upharpoonright \mathfrak{A}^T = 0$ and $h \circ T = h$. Hence, Lemma 3.1 yields that $h = 0$, which contradicts to $h(x_0) = 1$.

(ii)⇒(iii): It is clear that for every element of the form $y = x - T(x)$, $x \in \mathfrak{A}$ by (3.2) we have

$$\begin{aligned} \frac{1}{\sum_{k=1}^n p_k} \left\| \sum_{k=1}^n p_k T^k(y) \right\| &= \frac{1}{\sum_{k=1}^n p_k} \left\| \sum_{k=1}^n p_k (T^{k+1}(x) - T^k x) \right\| \\ &= \frac{1}{\sum_{k=1}^n p_k} \|p_1 Tx + (p_2 - p_1)T^2 x + \cdots \\ &\quad + (p_n - p_{n-1})T^n x - p_n T^{n+1} x\| \\ &\leqslant P(n)\|x\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.4}$$

Now let $x \in \mathfrak{A}^T$, then

$$\lim_{n\to\infty} \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k T^k(x) = x. \tag{3.5}$$

Hence, for every $x \in \mathfrak{A}^T + \{a - T(a) : a \in \mathfrak{A}\}$ the limit

$$\lim_{n\to\infty}\frac{1}{\sum_{k=1}^n p_k}\sum_{k=1}^n p_kT^kx$$

exists, which is denoted by $E_T(x)$. It is clear that E_T is a positive linear operator from $\mathfrak{A}^T+\{a-T(a):a\in\mathfrak{A}\}$ onto $\mathfrak{A}^T$. Positivity and $E_T\mathbb{1}=\mathbb{1}$ imply that E_T is bounded. From (3.4) one obviously gets that $E_TT=TE_T=E_T$. According to (ii) the operator E_T can be uniquely extended to $\mathfrak{A}$, this extension is denoted by the same symbol E_T. It is evident that E_T is a positive projection with $\|E_T\|=1$.

Now take an arbitrary $x\in\mathfrak{A}$. Then again using (ii), for any $\epsilon>0$ we can find $x_\epsilon\in\mathfrak{A}^T+\{a-T(a):a\in\mathfrak{A}\}$ such that $\|x-x_\epsilon\|\leqslant\epsilon$. By means of (3.4), (3.5) we conclude that

$$\left\|\frac{1}{\sum_{k=1}^n p_k}\sum_{k=1}^n p_kT^k(x_\epsilon)-E_T(x_\epsilon)\right\|\leqslant P(n)\|x_\epsilon\|.$$

Hence, one has

$$\begin{aligned}\left\|\frac{1}{\sum_{k=1}^n p_k}\sum_{k=1}^n p_kT^k(x)-E_T(x)\right\| &\leqslant \left\|\frac{1}{\sum_{k=1}^n p_k}\sum_{k=1}^n p_kT^k(x-x_\epsilon)\right\|\\ &\quad+\left\|\frac{1}{\sum_{k=1}^n p_k}\sum_{k=1}^n p_kT^k(x_\epsilon)-E_T(x_\epsilon)\right\|\\ &\quad+\|E_T(x-x_\epsilon)\|\\ &\leqslant 2\|x-x_\epsilon\|+P(n)\|x_\epsilon\|\\ &\leqslant P(n)\|x\|+(2+P(n))\epsilon,\end{aligned}$$

which with the arbitrariness of ϵ implies (3.3).

Consequently,

$$\lim_{n\to\infty}\frac{1}{\sum_{k=1}^n p_k}\sum_{k=1}^n p_kT^kx=E_T(x)$$

is valid for every $x\in\mathfrak{A}$.

The mapping E_T is a unique T-invariant positive projection. Indeed, if $\widetilde{E}:\mathfrak{A}\to\mathfrak{A}^T$ is any T-invariant positive projection onto $\mathfrak{A}^T$, then

$$\widetilde{E}(x)=\frac{1}{\sum_{k=1}^n p_k}\sum_{k=1}^n p_k\widetilde{E}(T^k(x))=\widetilde{E}\left(\frac{1}{\sum_{k=1}^n p_k}\sum_{k=1}^n p_kT^k(x)\right).$$

Taking the limit as $n\to\infty$ gives

$$\widetilde{E}(x)=\widetilde{E}(E_T(x))=E_T(x).$$

The implication (iii)⇒(iv) is obvious. Let us consider (iv)⇒(i). Let ψ be any state on $\mathfrak{A}^T$, then $\psi\circ E_T$ is a T-invariant extension of ψ to $\mathfrak{A}$. Assume that ϕ is any T-invariant, linear extension of ψ. Then

$$\phi(x)=\frac{1}{\sum_{k=1}^n p_k}\sum_{k=1}^n p_k\phi(T^k(x))=\phi\left(\frac{1}{\sum_{k=1}^n p_k}\sum_{k=1}^n p_kT^k(x)\right).$$

Now taking the limit from both sides of the last equality as $n\to\infty$ one gives

$$\phi(x)=\phi(E_T(x))=\psi(E_T(x)),$$

so $\phi=\psi\circ E_T$. □

Remark 3.3. If we choose $p_n=1$ for all $n\in\mathbb{N}$ then it is clear that the condition (3.2) is satisfied, hence we infer that unique ergodicity relative to $\mathfrak{A}^T$ is equivalent to the norm convergence of the mean averages, i.e.

$$\frac{1}{n}\sum_{k=1}^{n} T^k(x),$$

which recovers the result of [2].

Remark 3.4. If the condition (2.3) is satisfied then condition (3.2) is valid as well. This means that unique ergodicity would remain true if Cesaro summation is replaced by a weaker. Theorem 3.2 extends a result of Mukhamedov and Temir [18].

Example. If we define $p_n = n^\alpha$ with $\alpha > 0$, then one can see that $\{p_n\}$ is an increasing sequence and condition (3.2) is also satisfied. This provides a concrete example of weights.

Remark 3.5. Note that some nontrivial examples of uniquely ergodic quantum dynamical systems based on automorphisms, has been given in [2]. Namely, it was proved that free shifts based on reduced C^*-algebras of RD-groups (including the free group on infinitely many generators), and amalgamated free product C^*-algebras, are uniquely ergodic relative to the fixed-point subalgebra. In [11] it has been proved that such shifts possess a stronger property called F-strict weak mixing (see also [18]).

Observation. We note that, in general, the projection E_T is not a conditional expectation, but when T is an automorphism then it is so. Now we are going to provide an example of Markov operator which is uniquely ergodic relative to its fixed point subspace for which the projector E_T is not a conditional expectation.

Consider the algebra $M_d(\mathbb{C})$ – $d \times d$ matrices over $\mathbb{C}$. For a matrix $\mathbf{x} = (x_{ij})$ by $\mathbf{x}^t$ we denote its transpose matrix, i.e. $\mathbf{x}^t = (x_{ji})$. Define a mapping $\phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ by $\phi(\mathbf{x}) = \mathbf{x}^t$. Then it is known [20] that such a mapping is positive, but not completely positive. One can see that ϕ is a Markov operator. Due to the equality

$$\mathbf{x} = \frac{\mathbf{x}+\mathbf{x}^t}{2} + \frac{\mathbf{x}-\mathbf{x}^t}{2}$$

condition (ii) of Theorem 3.2 is satisfied, so ϕ is uniquely ergodic with respect to $M_d(\mathbb{C})^\phi$. Hence, the corresponding projection E_ϕ is given by $E_\phi(\mathbf{x}) = (\mathbf{x}+\mathbf{x}^t)/2$, which is not completely positive. Moreover, $M_d(\mathbb{C})^\phi$ is the set of all symmetric matrices, which do not form an algebra. So, E_ϕ is not a conditional expectation.

4. A uniquely ergodic entangled Markov operator

In recent developments of quantum information many people have discussed the problem of finding a satisfactory quantum generalization of classical random walks. Motivating this in [3,10] a new class of quantum Markov chains was constructed which are at the same time purely generated and uniquely determined by a corresponding classical Markov chain. Such a class of Markov chains was constructed by means of entangled Markov operators. In one's turn they were associated with Schur multiplication. In that paper, ergodicity and weak clustering properties of such chains were established. In this section we are going to provide entangled Markov operator which is uniquely ergodic relative to its fixed point subspace, but which is not ergodic.

Let us recall some notations. To define Schur multiplication, we choose an orthonormal basis $\{e_j\}$, $j = 1, \dots, d$ in a d-dimensional Hilbert space H_d which is kept fixed during the analysis. In such a way, we have the natural identification H_d with C^d. The corresponding system of matrix units $e_{ij} = e_i \otimes e_j$ identifies $B(H_d)$ with $M_d(\mathbb{C})$. Then, for $\mathbf{x} = \sum_{i,j=1}^{d} x_{ij}e_{ij}$, $\mathbf{y} = \sum_{i,j=1}^{d} y_{ij}e_{ij}$ elements of $M_d(\mathbb{C})$, we define *Schur multiplication* in $M_d(\mathbb{C})$ as usual,

$$\mathbf{x} \diamond \mathbf{y} = \sum_{i,j=1}^{d} (x_{ij}y_{ij})e_{ij}, \tag{4.1}$$

that is, componentwise, $(\mathbf{x} \diamond \mathbf{y})_{ij} := x_{ij}y_{ij}$.

A linear map $P : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ is said to be *Schur identity-preserving* if its diagonal projection is the identity, i.e. $\mathbb{1} \diamond P(\mathbb{1}) = \mathbb{1}$. It is called an *entangled* Markov operator if, in addition, $P(\mathbb{1}) \neq \mathbb{1}$.

The entangled Markov operator (see [3]) associated to a stochastic matrix $\Pi = (p_{ij})_{i,j=1}^d$ and to the canonical systems of matrix units $\{e_{ij}\}_{i,j=1}^d$ of $M_d(\mathbb{C})$ is defined by

$$P(\mathbf{x})_{ij} := \sum_{k,l=1}^{d} \sqrt{p_{ik}p_{jl}}x_{kl}, \tag{4.2}$$

where as before $\mathbf{x} = \sum_{i,j=1}^d x_{ij}e_{ij}$.

Define a Markov operator $\Psi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ by

$$\Psi(\mathbf{x}) = \mathbb{1} \diamond P(\mathbf{x}), \quad \mathbf{x} \in M_d(\mathbb{C}). \tag{4.3}$$

Given a stochastic matrix $\Pi = (p_{ij})$ put

$$Fix(\Pi) = \{\psi \in \mathbb{C}^d : \Pi\psi = \psi\}.$$

To every vector $a = (a_1, \ldots, a_d) \in \mathbb{C}^d$ corresponds a diagonal matrix $\mathbf{x}_a$ in $M_d(\mathbb{C})$ defined by

$$\mathbf{x}_a = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_d \end{pmatrix}. \tag{4.4}$$

Lemma 4.1. *For a Markov operator given by (4.3) one has*

$$M_d(\mathbb{C})^{\Psi} = \{\mathbf{x}_\psi : \psi \in Fix(\Pi)\}$$

Proof. Let $\mathbf{x} = (x_{ij}) \in M_d(\mathbb{C})^{\Psi}$, i.e. $\Psi(\mathbf{x}) = \mathbf{x}$. From (4.1) and (4.3) we conclude that $x_{ij} = 0$ if $i \neq j$. Therefore, due to (4.2) one finds

$$\sum_{j=1}^{d} \sqrt{p_{ij}p_{ij}}x_{jj} = x_{ii}$$

which implies that $(x_{11}, \ldots, x_{dd}) \in Fix(\Pi)$. □

Furthermore, we assume that the dimension of $Fix(\Pi)$ is greater or equal than 2, i.e. $dim(Fix(\Pi)) \geqslant 2$. Hence, according to Lemma 4.1 we conclude that $M_d(\mathbb{C})^{\Psi}$ is a nontrivial commutative subalgebra of $M_d(\mathbb{C})$.

Theorem 4.2. *Let Π be a stochastic matrix such that $dim(Fix(\Pi)) \geqslant 2$. Then the corresponding Markov operator Ψ given by (4.3) is uniquely ergodic w.r.t. $M_d(\mathbb{C})^{\Psi}$.*

Proof. To prove the statement, it is enough to establish condition (ii) of Theorem 3.2. Take any $\mathbf{x} = (x_{ij}) \in M_d(\mathbb{C})$. Now we are going to show that it can be represented as follows:

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2, \tag{4.5}$$

where $\mathbf{x}_1 \in M_d(\mathbb{C})^{\Psi}$ and $\mathbf{x}_2 \in \{\mathbf{y} - \Psi(\mathbf{y}) : \mathbf{y} \in M_d(\mathbb{C})\}$.

Due to Lemma 4.1 there is a vector $\psi \in Fix(\Pi)$ such that $\mathbf{x}_1 = \mathbf{x}_\psi$, and hence, from (4.3), (4.5) one finds that

$$\mathbf{x}_2 = \begin{pmatrix} \varphi_{11} & x_{12} & \cdots & x_{1d} \\ x_{21} & \varphi_{22} & \cdots & x_{2d} \\ \cdots & \cdots & \cdots & \cdots \\ x_{d1} & x_{d2} & \cdots & \varphi_{dd} \end{pmatrix}, \tag{4.6}$$

where

$$\varphi_{ii} = \xi_i - \sum_{j=1}^{d} p_{ij}\xi_j - \sum_{k,l=1 k\neq j}^{d} \sqrt{p_{ik}p_{il}} x_{kl}. \tag{4.7}$$

The existence of the vectors $\psi = (\psi_1, \ldots, \psi_d)$ and $(\xi_1, \ldots, \xi_d)$ follows immediately from the following relations:

$$\psi_i + \xi_i - \sum_{j=1}^{d} p_{ij}\xi_j = x_{ii} + \sum_{\substack{k,l=1\\ k\neq j}}^{d} \sqrt{p_{ik}p_{il}} x_{kl}, \quad i = 1, \ldots, d, \tag{4.8}$$

since the number of unknowns is greater than the number of equations. Note that the equality (4.8) comes from (4.3)–(4.7). Hence, one concludes that the equality

$$M_d(\mathbb{C})^{\Psi} + \{\mathbf{x} - \Psi(\mathbf{x}) : \ \mathbf{x} \in M_d(\mathbb{C})\} = M_d(\mathbb{C}),$$

which completes the proof. □

Let us provide a more concrete example.

Example. Consider on $M_3(\mathbb{C})$ the following stochastic matrix Π_0 defined by:

$$\Pi_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & u & v \end{pmatrix}, \tag{4.9}$$

here $u, v \geqslant 0, u + v = 1$.

One can immediately find that

$$Fix(\Pi_0) = \{(x, y, y) : x, y \in \mathbb{C}\}. \tag{4.10}$$

Then for the corresponding Markov operator Ψ_0, given by (4.3), (4.2), due to Lemma 4.1 one has

$$M_3(\mathbb{C})^{\Psi_0} = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{pmatrix} : x, y \in \mathbb{C} \right\}. \tag{4.11}$$

So, $M_3(\mathbb{C})^{\Psi_0}$ is a nontrivial commutative subalgebra of $M_3(\mathbb{C})$ having dimension 2.

So, according to Theorem 4.2 we see that Ψ_0 is uniquely ergodic relative to $M_3(\mathbb{C})^{\Psi_0}$. But (4.11) implies that Ψ_0 is not ergodic. Note that ergodicity of entangled Markov chains has been studied in [3].

Acknowledgments

The second named author (F.M.) thanks Prof. L. Accardi for kind hospitality at “Università di Roma Tor Vergata” during 18–22 June of 2006. Finally, the authors also would like to thank to the referee for his useful suggestions which allowed them to improve the text of the paper.

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