Transversality of Homoclinic Orbits and Exponential Dichotomies for Parabolic Equations*

Zeng Weiyao²

Research Institute of Sciences, Changsha Railway University, Changsha, Hunan, 410075, People’s Republic of China

Submitted by William F. Ames

Received December 10, 1996

In this paper we discuss the existence of exponential dichotomies on \( \mathbb{R} \) of linear parabolic equations depending on small parameters and provide a tool of proving the transversality of the homoclinic orbits of parabolic equations, and by making use of the results on exponential dichotomies of this paper, we investigate the transversality of homoclinic orbits for parabolic equations.

1. INTRODUCTION

Let \( F \) be a diffeomorphism on a two dimensional manifold. It is well known that the existence of one transversal homoclinic point for \( F \) implies the existence of the Smale horseshoe (that is, the chaotic dynamics) from the Smale–Birkhoff homoclinic theorem. Recently, Palmer [1] and Meyer and Sell [2, 3] have shown that the same results in \( \mathbb{R}^n \) can be obtained by using elementary ideas from the theory of exponential dichotomies and the shadowing lemma. In the infinite case the problem was considered by Hale and Lin [4] for functional differential equations and by Blazquez [7] for parabolic equations. It is well known that one of the best tools for detecting the existence of transversal homoclinic orbits of differential equations is the Melnikov method. The Melnikov method has been extended since the paper of Melnikov [8], therefore we have a vast literature. We refer to Wiggins [9], Hale and Chow [10], Palmer [1, 11], Zeng [6], Meyer and Sell [2], Hale and Lin [4], and Blazquez [7].

* This work is partially supported by NSF of China.
² E-mail address: wyzeng@csru.edu.cn.

0022-247X/97 $25.00
Copyright © 1997 by Academic Press
All rights of reproduction in any form reserved.

The main purpose of this paper is to generalize the result in Palmer [11] on the existence of the exponential dichotomies depending on small parameters on $R$ to the linear parabolic equations by using the Liapunov–Schmidt method and the Melnikov integral. Also we use the results we obtained on the exponential dichotomies to show rigorously the transversality of the homoclinic orbits for parabolic equations.

We want to use the following notations. We let $L(X^\beta, X) = \{ L:X^\beta \to X \}$, $L(X) = \{ L:X \to X \}$, $C^\beta_0(R, X) = \{ u: R \to X, \text{ is continuous and bounded in } X \text{ for } t \in R \}$, $C^\beta_0(R, X) = \{ u: R \to X, \text{ is continuously differentiable and } u, u' \text{ are bounded in } X \text{ for } t \in R \}$. $C^\beta_0(R, X)$ has the same meaning. For any operator $L$ we denote by $R(L)$ the range of $L$. In this paper, the definitions of the solution of parabolic equations in $X$, and the derivative of functions in $X$ are the same as in Henry [13].

If $A$ is a sectorial operator in a Banach space $X$ then there is a real number $a$ such that $A_1 = A + aI$ implies $\Re \sigma(A_1) > 0$. Let $0 \leq \beta \leq 1$. One can define a fractional power $A^\beta_1$ of $A_1$. Let $X^\beta = D(A^\beta_1)$ with the graph norm $||x||_\beta = ||A^\beta_1 x||$ for $x \in X^\beta$ (refer to Henry [13]). The space $X^\beta$ is a Banach space with the norm $||\cdot||_\beta$ for $0 \leq \beta \leq 1$ and $X^\beta$ is a dense subspace of $X^\tau$ with continuous inclusion for $0 \leq \tau \leq \beta \leq 1$ (refer to [13, Theorem 1.4.8]). Suppose $A$ is a sectorial operator on $X$. If $t \to B(t):[t_0, t_1] \to L(X^\beta, X)$ is Hölder continuous, then, for any $u_0 \in X^\beta$ and $t_0 \leq \tau \leq t_1$, there exists a unique solution $u(t) = u(t, \tau, u_0)$ of the
\[ \dot{u} + Au = B(t)u \]

on \( \tau \leq t \leq t_2 \), \( u(\tau) = u_0 \) and \( u_0 \to u(t, \tau, u_0) \) is linear and bounded in \( X \). We can define an evolution operator \( \{T(t, \tau) : t_0 \leq \tau \leq t \} \) by \( u(t, \tau, u_0) = T(t, \tau)u_0, t \geq \tau \).

Suppose the evolution operator \( T(t, s) \in L(X) \) for the equation
\[ \dot{u} + A(t)u = 0 \tag{1.1} \]
is defined on a real interval \( J \). Equation (1.1) is said to have an exponential dichotomy on \( J \) with the exponent \( \beta > 0 \) and the bound \( M \) if there exist projections \( P(t), Q(t) = I - P(t), t \in J \), such that
(a) \( T(t, s)P(s) = P(t)T(t, s), t \geq s \) in \( J \).
(b) The restriction \( T(t, s)|_{R(Q(t))}, t \geq s \) is an isomorphism onto \( R(Q(t)) \). We define the inverse map from \( R(Q(t)) \) to \( R(Q(s)) \) by \( T(s, t) \).
(c) \( \|T(t, s)P(s)\| \leq Me^{-\beta(s-t)} \) for \( t \geq s \) in \( J \).
(d) \( \|T(t, s)Q(s)\| \leq Me^{-\beta(t-s)} \) for \( s \geq t \) in \( J \).

The theory of exponential dichotomies plays a very important role in studying the global bifurcations; we refer to Palmer [1], Meyer and Sell [2], Zeng [6], and Sandstede [18]. On the theory of exponential dichotomies for finite dimensional systems, we refer to Palmer [1], Meyer and Sell [2], Coppel [5], and Sacker and Sell [19]; for infinite dimensional systems, we refer to Sackel and Sell [16], Henry [13], and Chow and Leiva [17]. On the other related results on dynamics of parabolic equations, we refer to Shen and Yi [21, 22] and Fiedler and Rocha [23].

If \( A \) is a sectorial operator, \( A(t) - A : R \to L(X^B, X) \) is locally Hölder continuous, and \( T(t, s) \in L(X^B, X) \) is the evolution operator, by Henry's Theorem 7.3.1 in [13], we can define the adjoint operator \( T^*(s, t) = T(t, s)^* \) by
\[ \langle y, T(t, s)x \rangle = \langle T^*(s, t)y, x \rangle, \quad x \in X, y \in X^*, \tag{1.2} \]
and \( T^*(s, t) \in L(X^*), \) locally continuous in \( s < t \) (but weak*-continuous at \( s = t \)), such that \( y(s) = T^*(s, t_0)y_0 \) satisfies the equation adjoint to Eq. (1.1)
\[ \frac{dy}{ds}(s) = A^*(s)y(s). \tag{1.3} \]

If Eq. (1.1) has an exponential dichotomy on \( R_+ \) and \( R_- \) with projections \( P_+(t) \) and \( P_-(t) \), respectively, then Hale and Lin [4] and Blazquez [7] proved that the adjoint equation (1.3) also has an exponential dichotomy.
on $R_+$ and $R_-$ with the same constants and with projections $I - (P_+(t))^*$ and $I - (P_-(t))^*$, respectively.

Let $O$ be an open subset of $X^\beta$ and let $f: R \times O \to X$ be a continuous $T$-periodic function in $t$ with continuous partial derivatives $f_u(t, u)$. We denote by $\Phi(t, \xi)$ the unique solution of the equation

$$\dot{u} + Au = f(t, u)$$

(1.4)

with $\Phi(0, \xi) = \xi$ and by

$$F: X^\beta \to X^\beta$$

$$F(\xi) = \Phi(T, \xi)$$

the period map for Eq. (1.4). Blazquez [7] proved the following result: $y \in O$ is a transversal homoclinic point with respect to a hyperbolic fixed point $\xi_0$ if and only if

$$\| \Phi(t, y) - \Phi(t, \xi_0) \| \to 0 \quad \text{as } |t| \to \infty$$

(1.5)

and the variational equation

$$\dot{u} + Au = f_u(t, \Phi(t, y))u$$

(1.6)

has an exponential dichotomy on $R$.

So in studying the transversality of homoclinic orbits which bifurcate from the nontransversal homoclinic orbits of parabolic equations, we want to investigate the existence of exponential dichotomies on $R$ of the linear parabolic equation depending on a small parameter

$$\dot{u} + A(t, \varepsilon)u = 0, \quad (1.7)$$

where $A(t, \varepsilon) - A: R \times (-l_0, l_0) \to L(X^\beta, X)$ is locally Hölder and $A$ is a sectorial operator and $l_0 > 0$ a sufficiently small constant. We want to consider the following problem: Suppose when $\varepsilon = 0$ the linear parabolic equation

$$\dot{u} + A(t, 0)u = 0$$

(1.8)

admits an exponential dichotomy on both $R_+$ and $R_-$ but does not have exponential dichotomies on $R$. Under what conditions can we guarantee that the linear parabolic equation (1.7) has an exponential dichotomy on $R$ for $\varepsilon \neq 0$ sufficiently small? In this paper, we want to solve the above problem and to use the result to prove the transversality of the homoclinic orbits of parabolic equations. The result in Palmer [11] is generalized to the parabolic equations. This paper is organized as follows: Section 1 is an introduction. In Section 2 we give a sufficient condition which assures the
existence of exponential dichotomies on $R$ for Eq. (1.7) for $\varepsilon \neq 0$ sufficiently small. In Section 3 we apply the result in Section 2 to investigate the transversality of the homoclinic orbits.

2. THE EXISTENCE OF EXPONENTIAL DICHOTOMIES

In this section, we mainly study the existence of exponential dichotomies on $R$ of the linear parabolic equation depending on a small parameter

$$\dot{u} + A(t, \varepsilon)u = 0,$$

(2.1)

where $x \in X^\beta$, $A(t, \varepsilon) - A: R \times (-l_0, l_0) \to L(X^\beta, X)$ is locally Hölder continuous, $A$ is a sectorial operator, $\varepsilon$ is a small parameter, and $0 \leq \beta \leq 1$.

We need a lemma.

**Lemma 1.** If $A$ is a sectorial operator, $A(t) - A: R \to L(X^\beta, X)$ is locally Hölder continuous, such that the equation

$$\dot{u} + A(t)u = 0$$

(2.2)

has an exponential dichotomy on $R_+$ and $R_-$ with projections $P_+(t), P_-(t)$, respectively, then for the linear operator

$$L: C^1_b(R, X^\beta) \to C^0_b(R, X)$$

defined by

$$(Lu)(t) = \dot{u}(t) + A(t)u(t)$$

we have

(a) $L$ is a Fredholm operator.

(b) $N(L) = \{g \in C^1_b(R, X): \dot{g}(t) + A(t)g(t) = 0, \ g(0) \in R(P_+(0)) \cap R(I - P_-(0))\};$

(c) $R(L) = \{f \in C^0_b(R, X): \int_{-\infty}^{\infty} \langle \psi(t), f(t) \rangle \ dt = 0 \text{ for all } \psi \in C^2_b(R, X^\beta), \text{satisfying the adjoint equation } \dot{u} - A^*(u) = 0 \text{ and } \text{ind}(L) = \dim RQ_-(0) - \dim RQ_+(0).$ 

Lemma 1 is due to Blazquez [7], also refer to Palmer [1], Hale and Lin [4], Weinian [14], Sacker and Sell [16], and Chow and Leiva [17].

For Eq. (2.1), we assume

(H1) The derivatives $A(t, \varepsilon)$ and $A_\varepsilon(t, \varepsilon)$ exist continuous and bounded on $R \times (-l_0, l_0), \text{where } l_0 > 0 \text{ a sufficient small constant. We let } I = (-l_0, l_0).$
The equation
\[ u(t) + A(t,0)u = 0 \]  
(2.3)

admits an exponential dichotomy on \( R_+ \) and \( R_- \) with projections \( P_+(t) \) and \( P_-(t) \), and \( \dim R(Q_+(0)) = \dim R(Q_-(0)) \), respectively, and \( \dim R(Q_+(t)) \) and \( \dim R(Q_-(t)) \) are finite, and has a unique, up to a scalar multiple, nontrivial bounded solution \( \varphi(t) \) on \( R \) in \( X \).

For the linear equation (2.1) we define an operator \( L_\varepsilon \) by
\[
L_\varepsilon : C^{\beta}_b(\mathbb{R}, X^\beta) \rightarrow C^0_b(\mathbb{R}, X)
\]

\[
(Lu)(t) = \dot{u}(t) + A(t,\varepsilon)u(t).
\]

We say the operator \( L_\varepsilon \) is the operator associated with the linear parabolic equation (2.1). Since Eq. (2.3) admits an exponential dichotomy on \( R_+ \) and \( R_- \) with projections \( P_+(t) \) and \( P_-(t) \), we have that \( L_0 \) is a Fredholm operator and \( \text{ind}(L_0) = 0 \) from Lemma 1. It follows from the roughness of exponential dichotomies (refer to HALE and LIN [4], CHOW and LEIVA [17], and SACKER and SELL [16]), that for \( \varepsilon \neq 0 \) sufficiently small, Eq. (2.1) also has an exponential dichotomy on \( R_+ \) and \( R_- \) with projections \( P^*_+(t) \) and \( P^*_-(t) \), respectively. So it follows from Lemma 1 that the operator \( L_\varepsilon \) is Fredholm and \( \text{ind}(L_\varepsilon) = 0 \) because the index of the Fredholm operator persists under small perturbations.

Since
\[
\text{ind}(L_0) = 0 = \dim N(L_0) - \text{codim} R(L_0) = \dim R(P_+(0)) \cap R(I - P_-(0)) - \dim R(P_-(0)^*) \cap R(I - P_+((0))^*),
\]
from condition (H2), we have
\[
\dim R(P_-(0)^*) \cap R(I - P_+((0))^*) = \dim R(P_+(0)) \cap R(I - P_-(0)) = 1.
\]

Since \( R(P_-(0)^*) \cap R(I - P_+((0))^*) \) and \( R(P_+(0)) \cap R(I - P_-(0)) \) are the spaces of the initial values of bounded solutions on \( R \) in \( X \) of Eq. (2.3) and the adjoint equation of Eq. (2.3), respectively, the adjoint equation of Eq. (2.3)
\[
\dot{u}(t) - A^*(t,0)u(t) = 0
\]  
(2.4)
also has and only has a unique, up to a scalar multiple, nontrivial bounded solution, denoted by \( \psi(t) \in X \) for \( t \in R \).
Now we consider Eq. (2.1). We let $T_t(t,s) \in L(X)$ be the evolution operator for Eq. (2.1). The main result of this paper is as follows:

**Theorem 1.** Suppose $A$ is a sectorial operator, $A(t,\varepsilon) - A: R \times (-l_0, l_0) \rightarrow L(X^{\theta}, X)$ is locally Hölder continuous, and conditions $(H1)$ and $(H2)$ are satisfied. If

$$\int_{-\infty}^{+\infty} \langle \psi(t), A_s(t,0) \varphi(t) \rangle \, dt \neq 0$$

then for $\varepsilon \neq 0$ sufficiently small Eq. (2.1)

$$\dot{u} + A(t, \varepsilon)u = 0$$

admits an exponential dichotomy on $R$.

**Remark.** Theorem 1 generalizes the result of Palmer [11] to the infinite dimensional systems.

**Proof of Theorem 1.** For any $f \in C_0^2(R, X)$, we consider the equation

$$\dot{u} + A(t, \varepsilon)u = \varepsilon^2 f(t). \quad (2.5)$$

We assume, without loss of generality,

$$\int_{-\infty}^{+\infty} \psi^2(t) \, dt = 1.$$  

We make a change of variable $y(t) = u(t) - \varphi(t)\gamma$ for Eq. (2.5) where $\gamma \in R$ is a new parameter. Then Eq. (2.5) becomes

$$\dot{y} + A(t,0)y = [A(t,0) - A(t, \varepsilon)] [y + \varphi(t)\gamma] + \varepsilon^2 f(t). \quad (2.6)$$

Since the linear parabolic equation $\dot{y} + A(t,0)y = 0$ admits an exponential dichotomy on both $R_+$ and $R_-$, using the Liapunov–Schmidt method (refer to Lemma 1), we see that Eq. (2.6) is equivalent to the two equations

$$\dot{y} + A(t,0)y = [A(t,0) - A(t, \varepsilon)] [y + \varphi(t)\gamma] + \varepsilon^2 f(t) \quad (2.7)$$

We can prove that for $\varepsilon$ sufficiently small Eq. (2.7) has a unique solution $y = y(t, \gamma, \varepsilon) \in C_0^1(R, X^{\theta})$ for $t \in R$, and satisfies $y(t, \gamma, 0) = 0$. More-
over, \( y(t, \gamma, \varepsilon) \) is second order continuously differentiable in \((\gamma, \varepsilon)\). Substituting \( y = y(t, \gamma, \varepsilon) \) into Eq. (2.8), we obtain the bifurcative equation

\[
B(\gamma, \varepsilon) = \int_{-\infty}^{+\infty} \langle \psi(t), [A(t,0) - A(t, \varepsilon)] \left[ y(t, \gamma, \varepsilon) + \varphi(t) \gamma \right] + \varepsilon^2 f(t) \rangle \, dt. \tag{2.9}
\]

Since \( y(t, \gamma, 0) = 0 \) and \( y(t, \gamma, \varepsilon) \) is second order continuously differentiable in \( \gamma, \varepsilon \), we have that \( B(\gamma, \varepsilon) \) is also second order continuously differentiable in \( \gamma, \varepsilon \) and

\[
B(\gamma, 0) = 0.
\]

We define a function \( H(\gamma, \varepsilon) \) by

\[
H(\gamma, \varepsilon) = \begin{cases} 
\frac{B(\gamma, \varepsilon)}{\varepsilon}, & \varepsilon \neq 0 \\
B_\varepsilon(\gamma, 0), & \varepsilon = 0.
\end{cases}
\]

Since \( B(\gamma, 0) = 0 \) and \( B(\gamma, \varepsilon) \) is second order continuously differentiable in \( \gamma, \varepsilon \), the function \( H(\gamma, \varepsilon) \) is continuously differentiable in \( \gamma, \varepsilon \). Then

\[
H(\gamma, 0) = B_\varepsilon(\gamma, 0)
\]

\[
= \int_{-\infty}^{+\infty} \langle \psi(t), A_\varepsilon(t,0) \left[ y(t, \gamma, 0) + \varphi(t) \gamma \right] \rangle \, dt
\]

\[
= \int_{-\infty}^{+\infty} \langle \psi(t), A_\varepsilon(t,0) \left[ \varphi(t) \right] \rangle \, dt \cdot \gamma
\]

Hence

\[
H(0, 0) = 0
\]

and

\[
H_\gamma(0, 0) = \int_{-\infty}^{+\infty} \langle \psi(t), A_\varepsilon(t,0) \varphi(t) \rangle \, dt.
\]

Since

\[
\int_{-\infty}^{+\infty} \langle \psi(t), A_\varepsilon(t,0) \varphi(t) \rangle \, dt \neq 0,
\]

it follows from the implicit theorem that there exists a continuous function \( \gamma = \gamma(\varepsilon) \), satisfying \( \gamma(0) = 0 \), such that

\[
H(\gamma(\varepsilon), \varepsilon) = 0
\]
holds for $\varepsilon$ sufficiently small. Hence for $\varepsilon \neq 0$ sufficiently small the equation
\[ \dot{u} + A(t, \varepsilon)u = \varepsilon^2 f(t) \quad (2.10) \]
has a bounded solution $u(t, \varepsilon) = y(t, \gamma(\varepsilon), \varepsilon) + \varphi(t)\gamma(\varepsilon) \in C^1_C(R, X^\#)$. So for any $f \in C^0_C(R, X)$ and $\varepsilon \neq 0$ sufficiently small the equation
\[ \dot{u} + A(t, \varepsilon)u = f(t) \quad (2.11) \]
has a bounded solution $u(t, \varepsilon)/\varepsilon^2 = [y(t, \gamma(\varepsilon), \varepsilon) + \varphi(t)\gamma(\varepsilon)]/\varepsilon^2 \in C^1_C(R, X^\#)$. Thus we have for, $\varepsilon \neq 0$ sufficiently small, $R(L_{\varepsilon}) = C^0_C(R, X)$, that is, $\text{codim } R(L_{\varepsilon}) = 0$. Since $\text{ind}(L_{\varepsilon}) = 0$ for $\varepsilon$ sufficiently small, we have
\[ 0 = \dim N(L_{\varepsilon}) - \text{codim } R(L_{\varepsilon}) = \dim N(L_{\varepsilon}) \]
and hence
\[ N(L_{\varepsilon}) = R(P^\varepsilon_+(0)) \cap R(Q^\varepsilon_-(0)) = \{0\}. \]
From condition (H2) we see
\[
\begin{align*}
\dim\left( R(P^\varepsilon_+(0)) + R(Q^\varepsilon_-(0)) \right) \\
= \dim R(P^\varepsilon_+(0)) + \dim R(Q^\varepsilon_-(0)) - \dim (R(P^\varepsilon_+(0)) \cap R(Q^\varepsilon_-(0))) \\
= \dim R(P^\varepsilon_+(0)) + \dim R(Q^\varepsilon_-(0)) \\
= \dim R(P^\varepsilon_+(0)) \cup \dim R(Q^\varepsilon_+(0)) \\
= n.
\end{align*}
\]
Hence
\[ R(P^\varepsilon_+(0)) + R(Q^\varepsilon_-(0)) = X. \quad (2.12) \]
We define two closed subsets, $X_1(t)$ and $X_2(t)$, of $X$ for $\varepsilon \neq 0$ sufficiently small by
\[
X_1(t) = \begin{cases} 
R(P^\varepsilon_+(t)), & t \geq 0 \\
\{ \varphi \mid T_{\varepsilon}(0, t) \varphi \in R(P^\varepsilon_+(0)) \}, & t \leq 0 
\end{cases}
\]
\[
X_2(t) = \begin{cases} 
R(Q^\varepsilon_-(t)), & t \leq 0 \\
\{ T_{\varepsilon}(t, 0) \varphi \mid \varphi \in R(Q^\varepsilon_-(0)) \}, & t \geq 0 
\end{cases}
\]
Then from (2.12) we have
\[ X_1(0) \oplus X_2(0) = X. \quad (2.13) \]
We claim that

\[ X_1(t) \oplus X_2(t) = X. \] (2.14)

Now we want to prove (2.14) by the following two steps.

Step 1. We show \( X_1(t) \cap X_2(t) = \emptyset \) for \( t \in R \).

When \( t \geq 0 \), if \( \varphi \in X_1(t) \cap X_2(t) \), then there exists \( \varphi_0 \in X_2(0) \) such that \( \varphi = T_x(0,0)\varphi_0 \in R(P^+_z(t)) \). Hence \( \varphi_0 \in R(P^+_z(0)) \). So \( \varphi_0 \in X_1(0) \cap X_2(0) \). From (2.13), we have \( \varphi_0 = 0 \) and so \( \varphi = 0 \).

When \( t < 0 \), if \( \varphi \in X_1(t) \cap X_2(t) \), then \( T_x(0,0)\varphi \in X_1(0) \cap X_2(0) \). Thus, \( T_x(0,0)\varphi = 0 \). Since \( T_x(0,0) \) is an isomorphism from \( X_1(t) \) to \( X_2(0) \) we have \( \varphi = 0 \). Hence for \( t \in R \) we have

\[ X_1(t) \cap X_2(t) = \emptyset. \] (2.15)

Step 2. We show \( X_1(t) + X_2(t) = X \) for \( t \in R \).

When \( t \leq 0 \), since (2.13) holds, for any \( \varphi \in X \), we have

\[ T(0, t) \varphi = \varphi_1 + \varphi_2, \quad \text{where } \varphi_1 \in X_1(0) \text{ and } \varphi_2 \in X_2(0). \]

Hence there exists a \( \varphi_1 \in X_1(t) \) such that \( \varphi_1 = T_x(0, t)\varphi_1 \), and so \( \varphi_2 = T_x(0, t)(\varphi - \varphi_1) \in X_2(0) = R(Q^z(0)) \). Hence \( \varphi - \varphi_1 \in R(Q^z(t)) \), and so there exists a \( \varphi_2 \in R(Q^z(t)) \) satisfying \( \varphi = \varphi_1 + \varphi_2 \). Hence

\[ X_1(t) + X_2(t) = X \quad (t \leq 0). \]

When \( t \geq 0 \),

\[ X_1(t) = R(P^z_x(t)), \]
\[ X_2(t) = \{ T_x(0, t) \varphi \mid \varphi \in R(Q^z(0)) \}. \]

Since \( T_x(0, t) \) is an isomorphism of \( R(Q^z(0)) \) onto \( R(Q^z(t)) \), we have \( \dim X_1(t) = \dim X_2(0) = m \) \( (t \geq 0) \). Since

\[ X_1(0) \oplus X_2(0) = X, \]

we have \( m = \text{codim } X_1(0) = \text{codim } R(P^z_x(0)) = \text{codim } R(P^z_x(t)) = \dim X_1(t) \) \( (t \geq 0) \). Hence we have

\[ X_1(t) + X_2(t) = X \quad (t \geq 0). \]

So we have proved that

\[ X_1(t) + X_2(t) = X \quad (t \in R). \]
From (2.15) we obtain

\[ X_1(t) \circ X_2(t) = X \quad (t \in R). \]

Hence for \( \varepsilon \neq 0 \) sufficiently small Eq. (2.1) admits an exponential dichotomy on \( R \) because Eq. (2.1) has an exponential dichotomy on \( R_+ \) and \( R_- \) with projections \( P_+(t) \) and \( P_-(t) \), respectively. This completes the proof of Theorem 1.

---

3. THE TRANSVERSALITY OF HOMOCLINIC ORBITS

In this section we consider the perturbed parabolic equation

\[ \dot{u} + Au = g(u) + \varepsilon h(t, u, \varepsilon), \quad (3.1) \]

where \( g \in C^k(X^\beta, X) \), \( k \geq 1 \), and \( A \) is a sectorial operator in \( X \), \( \varepsilon \) is a small parameter, and \( h(t, u, \varepsilon) \in C^k(R \times X^\beta \times R, X) \) is bounded for \( t \in R \). We have the following result.

**Theorem 2.** Let \( g \in C^k(X^\beta, X) \) be defined in an open subset \( X \) of \( X^\beta \), such that the unperturbed equation

\[ \dot{u} + Au = g(u) \]

has a bounded solution \( r(t) \) with the bound of the orbit contained in \( X \) on \( R \). Suppose the variational equation along the orbit \( r(t) \)

\[ \dot{u} + Au = g_u(r(t))u \quad (3.2) \]

has an exponential dichotomy on \( R_+ \) and \( R_- \) with projections \( P_+(t) \), \( P_-(t) \) satisfying \( \dim R(Q_+(0)) = \dim R(Q_-(0)) \) and \( r(t) \) is the unique, up to a scalar multiple, nontrivial bounded solution, in \( X \), of Eq. (3.2) on \( R \) then its adjoint equation

\[ \dot{u} = A^x u - g_u^x(r(t))u \]

also has a unique, up to a scalar multiple, nontrivial bounded solution \( \psi(t) \in X \) on \( R \).

If the Melnikov function \( M(\alpha) \) defined by

\[ M(\alpha) = \int_{-\infty}^{+\infty} \langle \psi(t), h(t + \alpha, r(t), 0) \rangle dt \]

has a simple zero \( \alpha_0 \), then there exists a continuously differentiable function \( \alpha = \alpha(\varepsilon) \) such that for \( \varepsilon \neq 0 \) sufficiently small Eq. (3.1) has a unique bounded solution \( u(t, \varepsilon) \) on \( R \) satisfying

\[ \| u(t + \alpha(\varepsilon), \varepsilon) - r(t) \|_\beta = O(\varepsilon). \]
Moreover, for $\varepsilon \neq 0$ sufficiently small the variational equation of Eq. (3.1) along the bounded solution $u(t, \varepsilon)$

$$\dot{\xi} + A\xi = \{g_u(u(t, \varepsilon)) + \varepsilon h_u(t, u(t, \varepsilon), \varepsilon)\}\xi$$  \hspace{1cm} (3.3)

admits an exponential dichotomy on $\mathbb{R}$.

**Remark.** Theorem 2 of this paper generalizes Theorem 3.3 of Blazquez [7]. Now we give an important application of Theorem 2.

**Corollary 3.** Let the conditions of Theorem 2 for $A$, $g$, and $f$ be satisfied, $h$ is periodic in $t$ with period $T$. In addition, we assume Eq. (3.2) has a hyperbolic equilibrium $u_0 \in X$ and a homoclinic orbit $r(t) \in X$ connecting $u_0$ (that is, $\|r(t) - u_0\|_\beta \to 0$ as $t \to \pm\infty$).

Then for $\varepsilon$ sufficiently small Eq. (3.1) has a unique hyperbolic $T$-periodic solution $u_0(t, \varepsilon) \in X$ satisfying $u_0(t, 0) = u_0$ and for $\varepsilon \neq 0$ sufficiently small there exists a bounded solution $u^\varepsilon(t, \varepsilon) \in X$ such that

$$\lim_{|t| \to \infty} \|u^\varepsilon(t, \varepsilon) - u_0(t, \varepsilon)\|_\beta = 0$$

and the variational equation along $u^\varepsilon(t, \varepsilon)$

$$\dot{\xi} + A\xi = \{g_u(u^\varepsilon(t, \varepsilon)) + \varepsilon h_u(t, u^\varepsilon(t, \varepsilon), \varepsilon)\}\xi$$  \hspace{1cm} (3.4)

admits an exponential dichotomy on $\mathbb{R}$.

The proof of Corollary 3 follows from Theorem 2.

**Remark.** The existence of the exponential dichotomy on $\mathbb{R}$ of (3.4) implies the bounded solution $u^\varepsilon(t, \varepsilon)$ is a transversal orbit homoclinic to the hyperbolic $T$-periodic solution $u_0(t, \varepsilon)$ (refer to Blazquez [7]). The transversality of the homoclinic orbit was not proved rigorously in Blazquez [7]. Moreover, if the conditions of Corollary 3 are satisfied Blazquez [7] proved that Eq. (3.1) admits a Bernoulli bundle under the condition of transversality. We can also show the existence of a Bernoulli bundle (chaotic dynamics) by using the method of Meyer and Sell [2].

**Proof of Theorem 2.** The proof of the existence of the bounded solution $u(t, \varepsilon)$ is almost the same as the one in Blazquez [7]; we only give the outline of the proof because we will need the form of $u(t, \varepsilon)$ in proving that Eq. (3.4) has an exponential dichotomy on $\mathbb{R}$. Let $y(t) = u(t + \alpha) - r(t)$, where $\alpha \in \mathbb{R}$ is a new parameter. Then Eq. (3.1) reads

$$\dot{y} + Ay = g_u(r(t))y + \varepsilon h(t + \alpha, y + r(t), \varepsilon)$$

$$- g(r(t)) + g(y + r(t)) - g_u(r(t))y.$$  \hspace{1cm} (3.5)
For simplicity, we let

\[ G(t, y, a, \varepsilon) = \varepsilon h(t + a, y + r(t), \varepsilon) - g(r(t)) \]
\[ + g(y + r(t)) - g_a(r(t))y. \]  \hspace{1cm} (3.6)

Then Eq. (3.5) can be written as

\[ \dot{y} + Ay = g_u(r(t))y + G(t, y, a, \varepsilon). \]  \hspace{1cm} (3.7)

Using the Liapunov–Schmidt method (refer to Blazquez [7]), we can prove that if the Melnikov function

\[ M(\alpha) = \int_{-\infty}^{+\infty} \langle \psi(t), h(t + \alpha, r(t), 0) \rangle \, dt \]

has a simple zero \( \alpha = \alpha_0 \) (that is, \( M(\alpha_0) = 0 \neq M'(\alpha_0) \)) then for \( \varepsilon \neq 0 \) sufficiently small Eq. (3.5) has a unique bounded solution \( y(t, \varepsilon) \in X \) satisfying \( y(t, 0) = 0 \). Hence for \( \varepsilon \neq 0 \) sufficiently small Eq. (3.1) has a bounded solution \( u(t, \varepsilon) = y(t - \alpha(\varepsilon)) + r(t - \alpha(\varepsilon)) \in X \) for \( t \in \mathbb{R} \).

Now we want to prove, by making use of Theorem 1, that for \( \varepsilon \neq 0 \) sufficiently small Eq. (3.4) admits an exponential dichotomy on \( \mathbb{R} \). Equivalently, we want to prove that for \( \varepsilon \neq 0 \) sufficiently small the equation

\[ \dot{\xi} + A\xi = \left[ g_u(y(t, \varepsilon)) + \varepsilon h_u(t + \alpha(\varepsilon), y(t, \varepsilon) + r(t), \varepsilon) \right] \xi \]  \hspace{1cm} (3.8)

admits an exponential dichotomy on \( \mathbb{R} \). We let

\[ A(t, \varepsilon) = -A + g_u(y(t, \varepsilon) + r(t)) + \varepsilon h_u(t + \alpha(\varepsilon), y(t, \varepsilon) + r(t), \varepsilon). \]

Then we have

\[ A(t, 0) = -A + g_u(r(t)), \]  \hspace{1cm} (3.9)
\[ A_{\varepsilon}(t, 0) = g_{uu}(r(t))y_u(t, 0) + h_u(t, r(t), 0). \]  \hspace{1cm} (3.10)

Since

\[ \dot{y}(t, \varepsilon) + Ay(t, \varepsilon) = g(y(t, \varepsilon) + r(t)) + \varepsilon h(t + a, y(t, \varepsilon)) \]
\[ + r(t), \varepsilon) - g(r(t)), \]

differentiating both sides of the above equation with respect to \( \varepsilon \) and setting \( \varepsilon = 0 \), we have

\[ \ddot{y}_\varepsilon(t, 0) + A_{\varepsilon}(t, 0) = g_u(r(t))y_u(t, 0) + h(t + a_0, r(t), 0). \]  \hspace{1cm} (3.11)
Differentiating both sides of the above equation with respect to \( t \), we obtain

\[
\begin{align*}
\ddot{y}_x(t, 0) + A\dot{y}_x(t, 0) &= g_u(r(t))\dot{y}_x(t, 0) + h_x(t + \alpha_0, r(t), 0) \\
&\quad + \left[ g_uu(r(t))y_x(t, 0) + h_u(t + \alpha_0, r(t), 0) \right] \dot{r}(t),
\end{align*}
\]

(3.12)

hence \( \dot{y}_x(t, 0) \) is the bounded solution in \( X \) on \( R \) of the equation

\[
\dot{y} + Ay = g_u(r(t))y + h_x(t + \alpha_0, r(t), 0) \\
+ \left[ g_uu(r(t))y_x(t, 0) + h_u(t + \alpha_0, r(t), 0) \right] \dot{r}(t).
\]

It follows from Lemma 1 that

\[
\int_{-\infty}^{+\infty} \left\langle \psi(t), h_x(t + \alpha_0, r(t), 0) \\
+ \left[ g_u(r(t))y_x(t, 0) + h_u(t + \alpha_0, r(t), 0) \right] \dot{r}(t) \right\rangle \, dt = 0.
\]

Hence

\[
\int_{-\infty}^{+\infty} \left\langle \psi(t), A_x(t, 0)\dot{r}(t) \right\rangle \, dt
= \int_{-\infty}^{+\infty} \left\langle \psi(t), \left[ g_u(r(t))y_x(t, 0) + h_u(t + \alpha_0, r(t), 0) \right] \dot{r}(t) \right\rangle \, dt
= -\int_{-\infty}^{+\infty} \left\langle \psi(t), h_x(t + \alpha_0, r(t), 0) \right\rangle \, dt
= -M'(\alpha_0) \neq 0,
\]

so we follow from Theorem 1 that for \( \varepsilon \neq 0 \) sufficiently small the variational equation (3.4) admits an exponential dichotomy on \( R \). The proof of Theorem 2 is complete.

**Acknowledgments**

We thank Professor Sell, Professor Palmer, Professor Y. Yi, Professor Z. Xia, and the referees for their help.
REFERENCES