Riesz transform and integration by parts formulas for random variables

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Abstract

We use integration by parts formulas to give estimates for the L^p norm of the Riesz transform. This is motivated by the representation formula for conditional expectations of functionals on the Wiener space already given in Malliavin and Thalmaier (2006) [13]. As a consequence, we obtain regularity and estimates for the density of non-degenerated functionals on the Wiener space. We also give a semi-distance which characterizes the convergence to the boundary of the set of the strict positivity points for the density.
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1. Introduction

The starting point of this paper is the representation theorem for densities and conditional expectations of random variables based on the Riesz transform, recently given by Malliavin and Thalmaier in [13]. Let us recall it. Let F and G denote random variables taking values on $\mathbb{R}^d$ and $\mathbb{R}$ respectively and consider the following integration by parts formula: there exist some integrable random variables $H_i(F, G)$ such that for every test function $f \in C_c^\infty(\mathbb{R}^d)$

$$IP_i(F, G) \quad \mathbb{E}(\partial_i f(F)G) = -\mathbb{E}(f(F)H_i(F, G)), \quad i = 1, \ldots, d.$$

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Malliavin and Thalmaier proved that if $IP_i(F, 1), i = 1, \ldots, d$ hold and the law of F has a continuous density p_F, then

$$p_F(x) = -\sum_{i=1}^{d} \mathbb{E}(\partial_i Q_d(F - x) H_i(F, 1))$$

where Q_d denotes the Poisson kernel on $\mathbb{R}^d$, that is the fundamental solution of the Laplace operator. Moreover, they proved also that if $IP_i(F, G), i = 1, \ldots, d$, a similar representation formula holds also for the conditional expectation of G with respect to F. The interest of Malliavin and Thalmaier in these representations come from numerical reasons – this allows one to simplify the computation of densities and conditional expectations using a Monte Carlo method. This is crucial in order to implement numerical algorithms for solving non-linear PDE's or optimal stopping problems – for example for pricing American options. But there is a difficulty: the variance of the estimators produced by such a representation formula is infinite. Roughly speaking, this comes from the blowing up of the Poisson kernel around zero: $\partial_i Q_d \in L^p$ for $p < d/(d-1)$, so that $\partial_i Q_d \notin L^2$ for every $d \geq 2$. So estimates of $\mathbb{E}(|\partial_i Q_d(F - x)|^p)$ are crucial in this framework and this is the central point of interest in our paper. In [8,9], Kohasu-Higa and Yasuda proposed a solution to this problem using some cut off arguments. And in order to find the optimal cut off level they used the estimates of $\mathbb{E}(|\partial_i Q_d(F - x)|^p)$ which we prove in this paper (actually, they used a former version given in the preprint [2]).

So our central result concerns estimates of $\mathbb{E}(|\partial_i Q_d(F - x)|^p)$. It turns out that, in addition to the interest in numerical problems, such estimates represent a suitable instrument in order to obtain regularity of the density of functionals on the Wiener space—for which Malliavin calculus produces integration by parts formulas. Before going further let us mention that one may also consider integration by parts formulas of higher order, that is

$$IP_\alpha(F, G) \quad E(\partial_\alpha f(F)) = E(f(F) H_\alpha(F, G))$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$. We say that an integration by parts formula of order k holds if this is true for every $\alpha \in \{1, \ldots, d\}^k$. Now, a first question is: which is the order k of integration by parts that one needs in order to prove that the law of F has a continuous density p_F? If one employs a Fourier transform argument (see [14]) or the representation of the density by means of the Dirac function (see [1]) then one needs d integration by parts if $F \in \mathbb{R}^d$. In [11] Malliavin proves that integration by parts of order one is sufficient in order to obtain a continuous density, the dimension d does not matter (he employs some harmonic analysis arguments). A second problem concerns estimates of the density p_F (and of its derivatives) and such estimates involve the L^p norms of the weights $H_\alpha(F, 1)$. In the approach using the Fourier transform or the Dirac function, $\|H_\alpha(F, 1)\|_p, |\alpha| \leq d$ are involved if one estimates $\|p_F\|_\infty$. But in [15] Shigekawa obtains estimates of $\|p_F\|_\infty$ depending only on $\|H_i(F, 1)\|_p$, so on the weights of order one (and similarly for derivatives). In order to do it, he needs some Sobolev inequalities that he proves using a representation formula based on the Green function and some estimates of modified Bessel functions. Our program and our results are similar but the instrument used in our paper is the Riesz transform and the estimates of the Poisson kernel mentioned above.

Let us be more precise. Notice that $IP_i(F, G)$ may also be written as

$$IP_i(F, G) \quad \int \partial_i f(x) g(x) \mu_F(dx) = -\int f(x) \partial_i^{\mu_F} g(x) \mu_F(dx)$$

where μ_F is the law of F, $g(x) := \mathbb{E}(G \mid F = x)$ and $\partial^{\mu_F} g(x) := \mathbb{E}(H(F, G) \mid F = x)$. This suggests that we can work in the abstract framework of Sobolev spaces with respect to the probability measure μ_F (instead of the usual Lebesgue measure). More precisely for a probability measure μ we denote by $W_\mu^{1,p}$ the space of the functions $\phi \in L^p(d\mu)$ for which there exist some functions $\theta_i \in L^p(d\mu)$, $i = 1, \ldots, d$ such that $\int \phi \partial_i f d\mu = -\int \theta_i f d\mu$ for every test function $f \in C_c^\infty(\mathbb{R}^d)$. If F is a random variable of law μ then the above duality relation reads $\mathbb{E}(\phi(F) \partial_i f(F)) = -\mathbb{E}(\theta_i(F) f(F))$ and these are the usual integration by parts formulas in the probabilistic approach – for example $-\theta_i(F)$ is connected to the weight produced by Malliavin calculus for a functional F on the Wiener space. But one may consider other frameworks – as the Malliavin calculus on the Wiener Poisson space for example. This formalism has already been used by Shigekawa in [15] and a slight variant appears in the book of Malliavin and Thalmaier [13] (the so-called covering vector fields).

In Section 2 we prove the following result: if $1 \in W_\mu^{1,p}$ (or equivalently $IP_i(F, 1), i = 1, \ldots, d$ hold) for some $p > d$ then

$$\sup_{x \in \mathbb{R}^d} \sum_{i=1}^d \int |\partial_i Q_d(y - x)|^{p/(p-1)} \mu(dy) \le C_{d,p} \|1\|_{W_\mu^{1,p}}^{\ell_{d,p}},$$

Moreover $\mu(dx) = p_\mu(x)dx$, with p_μ Hölder continuous of order $1 - d/p$, and the following representation formula holds:

$$p_\mu(x) = \sum_{i=1}^d \int \partial_i Q_d(y - x) \partial_i^\mu 1(y) \mu(dy).$$

More generally, let $\mu_\phi(dx) := \phi(x)\mu(dx)$. If $\phi \in W_\mu^{1,p}$ then $\mu_\phi(dx) = p_{\mu_\phi}(x)dx$ and p_{μ_ϕ} is Hölder continuous. This last generalization is important from a probabilistic point of view because it produces a representation formula and regularity properties for the conditional expectation. We introduce in a straightforward way higher order Sobolev spaces $W_\mu^{m,p}$, $m \in \mathbb{N}$ and we prove that if $1 \in W_\mu^{m,p}$ then p_μ is $m - 1$ times differentiable and the derivatives of order $m - 1$ are Hölder continuous. And the analogous result holds for $\phi \in W_\mu^{m,p}$. So if we are able to iterate m times the integration by parts formula we obtain a density in C^{m-1}. These results are already obtained by Shigekawa. In our paper we get some supplementary information about the queues of the density function and we develop more the applications to conditional expectations.

Furthermore, we prove an alternative representation formula. Suppose that F satisfies integration by parts formulas in order to get that its law μ has a C^1 density p_μ. We set $U_\mu = \{p_\mu > 0\}$ and for $x, y \in U_\mu$, $A_{x,y} = \{\varphi : [0, 1] \to U_\mu ; \varphi \in C^1, \varphi_0 = x, \varphi_1 = y\}$. Then for any $\varphi \in A_{x,y}$ one has

$$p_\mu(y) = p_\mu(x) \exp\left(\int_0^1 \langle \partial^\mu 1(\varphi_t), \dot{\varphi}_t \rangle dt\right),$$

a formula which generalizes the one given by Bell [3] (he assumes $U_\mu = \mathbb{R}^d$ and takes φ as the straight line). The above formula suggests to introduce the following Riemannian semi-distance on U_μ: setting $C_\mu^{ij} = \partial_i^\mu 1 \partial_j^\mu 1$, $i, j = 1, \ldots, d$, for $x, y \in U_\mu$ one defines

$$d_\mu(x, y) = \inf \left\{ \int_0^1 \langle C_\mu(\varphi_t)\dot{\varphi}_t, \dot{\varphi}_t \rangle^{1/2} dt; \ \varphi \in A_{x,y} \right\}.$$

Such a distance is of interest in the following framework. Suppose that F is a non-degenerate and smooth r.v. on the Wiener space, so that integration by parts formulas hold and F has a smooth probability density. If F is one-dimensional then Fang [6] proved that $\overline{U}_\mu$ is connected and the interior of $\overline{U}_\mu$ is given by U_μ (so the density is itself strictly positive in the interior of the support of the law). But this is false in the multi-dimensional case, as shown by Nualart [14] through a counterexample. Then Malliavin suggested that one has to replace the Euclidean distance by the intrinsic distance associated to the Dirichlet form of F (see [7] for details). And he conjectured that if $d(\cdot, \cdot)$ is such a distance then for any sequence $\{x_n\}_n \subset U_\mu$ such that $p_\mu(x_n) \to 0$ then $d(x_n, x_1) \to \infty$, as $n \to \infty$. But Hirsch and Song [7] provided a counterexample which shows that it is false as long as the intrinsic distance is taken into account. We prove here that the Malliavin's conjecture is true but with the distance d_μ defined above (and in fact, we prove the equivalence).

The paper is organized as follows. In Section 2 we develop the main results in the abstract Sobolev spaces framework. In Section 3 we translate these results in probabilistic terms and in Section 4 we give their applications on the Wiener space.

2. Sobolev spaces associated to a probability measure and Riesz transform

2.1. Definitions and main objects

We consider a probability measure μ on $\mathbb{R}^d$ (with the Borel σ-field) and we denote by $L_\mu^p = \{\phi : \int |\phi(x)|^p \mu(dx) < \infty\}$ and we put $\|\phi\|_{L_\mu^p} = (\int |\phi(x)|^p \mu(dx))^{1/p}$. We also denote by $W_\mu^{1,p}$ the space of the functions $\phi \in L_\mu^p$ for which there exist some functions $\theta_i \in L_\mu^p, i = 1, \ldots, d$ such that, for every test function $f \in C_c^\infty(\mathbb{R}^d)$, one has

$$\int \partial_i f(x)\phi(x)\mu(dx) = -\int f(x)\theta_i(x)\mu(dx).$$

We denote $\partial_i^\mu \phi = \theta_i$. And we define the norm

$$\|\phi\|_{W_\mu^{1,p}} = \|\phi\|_{L_\mu^p} + \sum_{i=1}^d \|\partial_i^\mu \phi\|_{L_\mu^p}.$$

We similarly define the Sobolev spaces of higher order. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \{1, \ldots, d\}^m$ be a multi-index. We denote by $|\alpha| = m$ the length of α and for a function $f \in C^m(\mathbb{R}^d)$ we denote by $\partial_\alpha f = \partial_{\alpha_m} \cdots \partial_{\alpha_1} f$ the standard derivative corresponding to the multi-index α. Then we define $W_\mu^{m,p}$ to be the space of the functions $\phi \in L_\mu^p$ such that for every multi-index α with $|\alpha| \leq m$ there exist some functions $\theta_\alpha \in L_\mu^p$ such that

$$\int \partial_\alpha f(x)\phi(x)\mu(dx) = (-1)^{|\alpha|} \int f(x)\theta_\alpha(x)\mu(dx) \quad \forall f \in C_c^\infty(\mathbb{R}^d).$$

We denote $\partial_\alpha^\mu \phi = \theta_\alpha$ and we define the norm

$$\|\phi\|_{W_\mu^{m,p}} = \|\phi\|_{L_\mu^p} + \sum_{|\alpha| \leq m} \|\partial_\alpha^\mu \phi\|_{L_\mu^p}.$$

We will use the notation L^p, $W^{m,p}$ for the spaces associated to the Lebesgue measure (instead of μ), which are the standard L^p and the standard Sobolev spaces which are used in the literature. If $D \subset \mathbb{R}^d$ is an open set we denote by $W_\mu^{m,p}(D)$ the space of the functions ϕ which verify the

integration by parts formula for test functions f which have a compact support included in D (so $W_\mu^{m,p} = W_\mu^{m,p}(\mathbb{R}^d)$). The same for $W^{m,p}(D)$, $L^p(D)$, $L_\mu^p(D)$.

Our aim is to study the link between $W_\mu^{m,p}$ and $W^{m,p}$ and the main tool is the Riesz transform that we introduce now. The solution Q_d of the equation $\Delta Q_d = \delta_0$ in $\mathbb{R}^d$ has the following explicit form:

$$Q_2(x) = a_2^{-1} \ln |x| \quad \text{and} \quad Q_d(x) = -a_d^{-1} |x|^{2-d}, \quad d > 2 \tag{1}$$

where a_d is the area of the unit sphere in $\mathbb{R}^d$. For $f \in C_c^1(\mathbb{R}^d)$ one has

$$f = (\nabla Q_d) * \nabla f.$$

In Theorem 4.22 of [13], this representation for the function f is called the Riesz transform of f and is employed in order to obtain representation formulas for the conditional expectation. Moreover, some analogous representation formulas for functions on the sphere and on the ball are used by Malliavin and Nualart in [12] in order to obtain lower bounds for the density of a strongly non-degenerated random variable.

Setting $A_2 = 1$ and for $d > 2$, $A_d = d - 2$, we have

$$\partial_i Q_d(x) = a_d^{-1} A_d \frac{x_i}{|x|^d}. \tag{2}$$

By using polar coordinates, one has

$$\int_{|x|\leq 1} |\partial_i Q_d(x)|^{1+\delta} \, dx \leq A_d^{1+\delta} \int_0^1 \left|\frac{r}{r^d}\right|^{1+\delta} r^{d-1} dr = A_d^{1+\delta} \int_0^1 \frac{1}{r^{\delta(d-1)}} dr \tag{3}$$

which is finite for any $\delta < \frac{1}{d-1}$. But $\left|\partial_i^2 Q_d(x)\right| \sim |x|^{-d}$ and so $\int_{|x|\leq 1} \left|\partial_i^2 Q_d(x)\right| dx = \infty$. This is the reason for which we have to integrate by parts once and to remove one derivative, but we may keep the other derivative.

In order to include the one dimensional case we set $Q_1(x) = \max\{x, 0\}$, $a_1 = A_1 = 1$ and we have

$$\frac{dQ_1(x)}{dx} = 1_{(0,\infty)}(x).$$

In this case the above integral is finite for every $\delta > 0$.

2.2. An absolute continuity criterion

For a function $\phi \in L_\mu^1$ we denote by μ_ϕ the signed finite measure defined by

$$\mu_\phi(dx) := \phi(x)\mu(dx).$$

We prove now the following theorem, which is starting point of our next results.

Theorem 1. A. *Let* $\phi \in W_\mu^{1,1}$. *Then*

$$\int \left|\partial_i Q_d(y - x)\partial_i^\mu \phi(y)\right| \mu(dy) < \infty$$

for a.e. $x \in \mathbb{R}^d$ and $\mu_\phi(dx) = p_{\mu_\phi}(x)dx$ with

$$p_{\mu_\phi}(x) = -\sum_{i=1}^{d} \int \partial_i Q_d(y-x)\partial_i^\mu \phi(y)\mu(dy). \tag{4}$$

B. *If $\phi \in W_\mu^{m,1}$ for some $m \geq 2$ then $p_{\mu_\phi} \in W^{m-1,1}$ and for each multi index α of length less or equal to $m - 1$, the associated weak derivative can be represented as*

$$\partial_\alpha p_{\mu_\phi}(x) = -\sum_{i=1}^{d} \int \partial_i Q_d(y-x)\partial_{(\alpha,i)}^\mu \phi(y)\mu(dy). \tag{5}$$

If in addition $1 \in W_\mu^{1,1}$, the following alternative representation formula holds:

$$\partial_\alpha p_{\mu_\phi}(x) = p_\mu(x)\partial_\alpha^\mu \phi(x). \tag{6}$$

In particular, taking $\phi = 1$ and $\alpha = \{i\}$ one has

$$\partial_i^\mu 1 = 1_{\{p_\mu > 0\}}\partial_i \ln p_\mu. \tag{7}$$

Proof. A. We take $f \in C_c^1(\mathbb{R}^d)$ and we write $f = \Delta(Q_d * f) = \sum_{i=1}^{d}(\partial_i Q_d) * (\partial_i f)$. Then

$$\int f d\mu_\phi = \int f\phi d\mu = \sum_{i=1}^{d} \int \mu(dx)\phi(x)\int \partial_i Q_d(z)\partial_i f(x-z)dz$$

$$= \sum_{i=1}^{d} \int \partial_i Q_d(z)\int \mu(dx)\phi(x)\partial_i f(x-z)dz$$

$$= -\sum_{i=1}^{d} \int \partial_i Q_d(z)\int \mu(dx)f(x-z)\partial_i^\mu \phi(x)dz$$

$$= -\sum_{i=1}^{d} \int \mu(dx)\partial_i^\mu \phi(x)\int \partial_i Q_d(z)f(x-z)dz$$

$$= -\sum_{i=1}^{d} \int \mu(dx)\partial_i^\mu \phi(x)\int \partial_i Q_d(x-y)f(y)dy$$

$$= \int f(y)\left(-\sum_{i=1}^{d} \int \partial_i Q_d(x-y)\partial_i^\mu \phi(x)\mu(dx)\right)dy$$

which proves the representation formula (4).

In the previous computations we have used the Fubini theorem several times so we need to prove that some integrability properties hold. Let us suppose that the support of f is included in $B_R(0)$ for some $R > 1$. We denote $C_R(x) = \{y : |x| - R \leq |y| \leq |x| + R\}$ and we have $B_R(x) \subset C_R(x)$. First of all

$$|\phi(x)\partial_i Q_d(z)\partial_i f(x-z)| \leq \|\partial_i f\|_\infty |\phi(x)| |\partial_i Q_d(z)1_{C_R(x)}(z)|$$

and

$$\int_{C_R(x)} |\partial_i Q_d(z)| dz \leq A_d \int_{|x|-R}^{|x|+R} \frac{r}{r^d} \times r^{d-1}dr = 2RA_d.$$

So

$$\int \int |\phi(x)\partial_i Q_d(z)\partial_i f(x-z)| \, dz\mu(dx) \leq 2RA_d\|\partial_i f\|_\infty \int |\phi(x)| \, \mu(dx) < \infty.$$

Similarly

$$\int \int |\partial_i^\mu \phi(x)\partial_i Q_d(z) f(x-z)| \, dz\mu(dx) = \int \int |\partial_i^\mu \phi(x)\partial_i Q_d(x-y) f(y)| \, dy\mu(dx)$$

$$\leq 2RA_d\|f\|_\infty \int |\partial_i^\mu \phi(x)| \, \mu(dx) < \infty$$

so all the needed integrability properties hold and our computation is correct. In particular we have checked that $\int dy f(y) \int |\partial_i^\mu \phi(x)\partial_i Q_d(x-y)| \, \mu(dx) < \infty$ for every $f \in C_c^1(\mathbb{R}^d)$ so $\int |\partial_i^\mu \phi(x)\partial_i Q_d(x-y)| \, \mu(dx)$ is finite dy almost surely.

B. In order to prove (5), we write $\partial_\alpha f = \sum_{i=1}^d \partial_i Q_d * \partial_i \partial_\alpha f$. Now, we use the same chain of equalities as above and we obtain

$$\int \partial_\alpha f(x) p_{\mu_\phi}(x) dx = \int \partial_\alpha f \, d\mu_\phi$$

$$= (-1)^{|\alpha|} \int f(y) \left(-\sum_{i=1}^d \int \partial_i Q_d(x-y)\partial_{(\alpha,i)}^\mu \phi(x)\mu(dx) \right) dy$$

so that $\partial_\alpha p_{\mu_\phi}(y) = -\sum_{i=1}^d \int \partial_i Q_d(x-y)\partial_{(\alpha,i)}^\mu \phi(x)\mu(dx)$. As for (6), we have

$$\int \partial_\alpha f(x) p_{\mu_\phi}(x) dx = \int \partial_\alpha f(x)\mu_\phi(dx) = \int \partial_\alpha f(x)\phi(x)\mu(dx)$$

$$= (-1)^{|\alpha|} \int f(x)\partial_\alpha^\mu \phi(x)\mu(dx)$$

$$= (-1)^{|\alpha|} \int f(x)\partial_\alpha^\mu \phi(x) p_\mu(x) dx. \quad \square$$

Remark 2. Notice that if $1 \in W_\mu^{1,1}$ then for any $f \in C_c^\infty$ one has

$$\left| \int \partial_i f \, d\mu \right| \leq c_i \|f\|_\infty \quad \text{with } c_i = \|\partial_i^\mu 1\|_{L_\mu^1}, \ i = 1, \ldots, d.$$

Now, it is known that the above condition implies the existence of the density, as proved by Malliavin in [10] (see also [14], Lemma 2.1.1), and Theorem 1 gives a new proof including the representation formula in terms of the Riesz transform.

2.3. Estimate of the Riesz transform

As we will see later on, an important fact is to be able to control the quantities $\partial_i Q_d$, and more precisely $\Theta_p(\mu)$ defined by

$$\Theta_p(\mu) = \sup_{a\in\mathbb{R}^d} \sum_{i=1}^d \left(\int_{\mathbb{R}^d} |\partial_i Q_d(x-a)|^{\frac{p}{p-1}} \mu(dx) \right)^{\frac{p-1}{p}}. \tag{8}$$

This is the main content of Theorem 5. We begin with two preparatory lemmas.

For a probability measure μ and probability density ψ (a non negative measurable function ψ with $\int_{\mathbb{R}^d} \psi(x)dx = 1$) we define the probability measure $\psi * \mu$ by

$$\int f(x)(\psi * \mu)(dx) := \int \psi(x) \int f(x + y)\mu(dy)dx.$$

Lemma 3. *Let $p \geq 1$. If $1 \in W_\mu^{1,p}$ then $1 \in W_{\psi*\mu}^{1,p}$ and $\|1\|_{W_{\psi*\mu}^{1,p}} \leq \|1\|_{W_\mu^{1,p}}$.*

Proof. On a probability space $(\Omega, \mathcal{F}, P)$ we consider two independent random variables F and Δ such that $F \sim \mu(dx)$ and $\Delta \sim \psi(x)dx$. Then $F + \Delta \sim (\psi * \mu)(dx)$. We define $\theta_i(x) = \mathbb{E}(\partial_i^\mu 1(F) \mid F + \Delta = x)$ and we claim that $\partial_i^{\psi*\mu} 1 = \theta_i$. In fact, for $f \in C_c^1(\mathbb{R}^d)$ one has

$$-\int \partial_i f(x)(\psi * \mu)(dx) = -\int dx\, \psi(x) \int \partial_i f(x + y)\mu(dy)$$

$$= -\int dx\, \psi(x) \int f(x + y)\partial_i^\mu 1(y)\mu(dy)$$

$$= \mathbb{E}(f(F + \Delta)\partial_i^\mu 1(F)) = \mathbb{E}(f(F + \Delta)\mathbb{E}(\partial_i^\mu 1(F) \mid F + \Delta))$$

$$= \mathbb{E}(f(F + \Delta)\theta_i(F + \Delta)) = \int f(x) \times \theta_i(x)(\psi * \mu)(dx)$$

so $\partial_i^{\psi*\mu} 1 = \theta_i$. Moreover

$$\int |\theta_i(x)|^p (\psi * \mu)(dx) = \mathbb{E}(|\theta_i(F + \Delta)|^p) = \mathbb{E}\left(\left|\mathbb{E}(\partial_i^\mu 1(F) \mid F + \Delta)\right|^p\right)$$

$$\leq \mathbb{E}\left(\left|\partial_i^\mu 1(F)\right|^p\right) = \int \left|\partial_i^\mu 1(x)\right|^p \mu(dx)$$

so $1 \in W_{\psi*\mu}^{1,p}$ and $\|1\|_{W_{\psi*\mu}^{1,p}} \leq \|1\|_{W_\mu^{1,p}}$. $\square$

Lemma 4. *Let $p_n, n \in \mathbb{N}$ be a sequence of probability densities such that $\sup_n \|p_n\|_\infty = C_\infty < \infty$. Suppose also that the sequence of probability measures $\mu_n(dx) = p_n(x)dx, n \in \mathbb{N}$ converges weakly to a probability measure μ. Then $\mu(dx) = p(x)dx$ and $\|p\|_\infty \leq C_\infty$.*

Proof. Since $\int p_n^2(x)dx \leq C_\infty$ the sequence p_n is bounded in $L^2(\mathbb{R}^d)$ and so it is weakly relative compact. Passing to a subsequence (which we still denote by p_n) we may find $p \in L^2(\mathbb{R}^d)$ such that $\int p_n(x)f(x)dx \to \int p(x)f(x)dx$ for every $f \in L^2(\mathbb{R}^d)$. But, if $f \in C_c(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ then $\int p_n(x)f(x)dx \to \int f(x)\mu(dx)$ so that $\mu(dx) = p(x)dx$.

Let us now check that p is bounded. Using Mazur's theorem we may construct a convex combination $q_n = \sum_{i=1}^{m_n} \lambda_i^n p_{n+i}$, with $\lambda_n^i \geq 0, \sum_{i=1}^{m_n} \lambda_i^n = 1$, such that $q_n \to p$ strongly in $L^2(\mathbb{R}^d)$. Then, passing to a subsequence, we may assume that $q_n \to p$ almost everywhere. It follows that $p(x) \leq \sup_n q_n(x) \leq C_\infty$ almost everywhere. And we may change p on a set of null measure. $\square$

We are now able to give our basic estimate of $\Theta_p(\mu)$.

Theorem 5. *Let $p > d$ and let μ be a probability measure on $\mathbb{R}^d$ such that $1 \in W_\mu^{1,p}$. So by Theorem 1 $\mu(dx) = p_\mu(x)dx$ with*

$$p_\mu(x) = \sum_{i=1}^{d} \int \partial_i Q_d(y - x) \partial_i^\mu 1(y) \mu(dy). \tag{9}$$

Then

(i) $\Theta_p(\mu) \leq d K_{d,p} \|1\|_{W_\mu^{1,p}}^{k_{d,p}}$

(ii) $\|p_\mu\|_\infty \leq 2d K_{d,p} \|1\|_{W_\mu^{1,p}}^{k_{d,p}+1} .$ (10)

with

$$k_{d,p} = \frac{(d-1)p}{p-d} \quad and \quad K_{d,p} = 1 + 2A_d^{\frac{p}{p-1}} \left(\frac{p-1}{p-d} \cdot 2d \, A_d^{\frac{p}{p-1}} \right)^{k_{d,p}}.$$

Remark 6. The inequality in (10) (i) gives estimates of the kernels $\partial_i Q_d, i = 1, \ldots, d$ which appear in the Riesz transform. This is the crucial point in our approach. In [12], the authors use the Riesz transform on the sphere and they give estimates of the L^p norms of the corresponding kernels (which are of course different).

Proof. We will first prove the theorem under the supplementary assumption:

(H) p_μ is bounded.

We take $\rho > 0$ and notice that if $|x - a| > \rho$ then $|\partial_i Q_d(x - a)| \leq A_d \rho^{-(d-1)}$. Since $p > d$, for any $a \in \mathbb{R}^d$ we have

$$\int |\partial_i Q_d(x-a)|^{\frac{p}{p-1}} \mu(dx) \leq A_d^{\frac{p}{p-1}} \rho^{-(d-1)\frac{p}{p-1}} + \int_{|x-a| \leq \rho} |\partial_i Q_d(x-a)|^{\frac{p}{p-1}} p_\mu(x) dx$$

$$\leq A_d^{\frac{p}{p-1}} \left[\rho^{-(d-1)\frac{p}{p-1}} + \|p_\mu\|_\infty \int_0^\rho \frac{dr}{r^{\frac{d-1}{p-1}}} \right].$$

This gives

$$\int |\partial_i Q_d(x-a)|^{\frac{p}{p-1}} \mu(dx) \leq A_d^{\frac{p}{p-1}} \left[\rho^{-(d-1)\frac{p}{p-1}} + \|p_\mu\|_\infty \frac{p-1}{p-d} \rho^{\frac{p-d}{p-1}} \right] < \infty. \tag{11}$$

We use the representation formula (9) and Hölder's inequality and we obtain

$$p_\mu(x) = -\sum_{i=1}^{d} \int \partial_i Q_d(y-x) \partial_i^\mu 1(y) \mu(dy)$$

$$\leq \|1\|_{W_\mu^{1,p}} \sum_{i=1}^{d} \left(\int |\partial_i Q_d(y-x)|^{\frac{p}{p-1}} \mu(dy) \right)^{\frac{p-1}{p}}$$

$$\leq \|1\|_{W_\mu^{1,p}} \left(d + \sum_{i=1}^{d} \int |\partial_i Q_d(y-x)|^{\frac{p}{p-1}} \mu(dy) \right). \tag{12}$$

By using (11), we obtain

$$\|p_\mu\|_\infty \le d \ \|1\|_{W_\mu^{1,p}} \left(A_d^{\frac{p}{p-1}} \left(\rho^{-(d-1)\frac{p}{p-1}} + \|p_\mu\|_\infty \frac{p-1}{p-d} \rho^{\frac{p-d}{p-1}} \right) + 1 \right).$$

Now choose $\rho = \rho_*$, with ρ_* such that

$$d \ A_d^{\frac{p}{p-1}} \ \|1\|_{W_\mu^{1,p}} \frac{p-1}{p-d} \rho_*^{\frac{p-d}{p-1}} = \frac{1}{2}$$

that is,

$$\rho_* = \left(\frac{p-1}{p-d} \cdot 2d \ A_d^{\frac{p}{p-1}} \ \|1\|_{W_\mu^{1,p}} \right)^{-\frac{p-1}{p-d}}.$$

Then

$$\|p_\mu\|_\infty \le 2d \ \|1\|_{W_\mu^{1,p}} (A_d^{\frac{p}{p-1}} \rho_*^{-(d-1)\frac{p}{p-1}} + 1).$$

Since $\frac{p-1}{p-d} \rho_*^{\frac{p-d}{p-1}} = (2dA_d^{p/(p-1)} \ \|1\|_{W_\mu^{1,p}})^{-1}$, by using (11) we obtain

$$\int |\partial_i Q_d(x-a)|^{\frac{p}{p-1}} \mu(dx) \le 1 + 2A_d^{\frac{p}{p-1}} \rho_*^{-(d-1)\frac{p}{p-1}}$$

$$= 1 + 2A_d^{\frac{p}{p-1}} \left(\frac{p-1}{p-d} \cdot 2d \ A_d^{\frac{p}{p-1}} \right)^{\frac{(d-1)p}{p-d}} \cdot \|1\|_{W_\mu^{1,p}}^{\frac{(d-1)p}{p-d}}$$

$$\le \left(1 + 2A_d^{\frac{p}{p-1}} \left(\frac{p-1}{p-d} \cdot 2d \ A_d^{\frac{p}{p-1}} \right)^{\frac{(d-1)p}{p-d}} \right) \|1\|_{W_\mu^{1,p}}^{k_{d,p}}$$

$$= K_{d,p} \cdot \|1\|_{W_\mu^{1,p}}^{k_{d,p}}$$

and finally

$$\sum_{i=1}^d \int |\partial_i Q_d(y-x)|^{\frac{p}{p-1}} \mu(dy) \le d(1 + K_{d,p}) \|1\|_{W_\mu^{1,p}}^{k_{d,p}} \le 2dK_{d,p} \|1\|_{W_\mu^{1,p}}^{k_{d,p}}.$$

Using (12) this gives

$$\|p_\mu\|_\infty \le 2dK_{d,p} \|1\|_{W_\mu^{1,p}}^{k_{d,p}+1}.$$

So the theorem is proved under the supplementary assumption (H). We now remove this assumption. We consider a non-negative and continuous function ψ such that $\int \psi = 1$ and $\psi(x) = 0$ for $|x| \ge 1$. Then we define $\psi_n(x) = n^d \psi(nx)$ and $\mu_n = \psi_n * \mu$. We have $\mu_n(dx) = p_n(x)dx$ with $p_n(x) = \int \psi_n(x-y)\mu(dy)$. Using Lemma 3 we have $1 \in W_{\mu_n}^{1,p}$ and $\|1\|_{W_{\mu_n}^{1,p}} \le \|1\|_{W_\mu^{1,p}} < \infty$. Since p_n is bounded, μ_n verifies assumption (H) and so, using the first part of the proof, we obtain

$$\|p_n\|_\infty \le 2dK_{d,p} \|1\|_{W_{\mu_n}^{1,p}}^{k_{d,p}+1} \le 2dK_{d,p} \|1\|_{W_\mu^{1,p}}^{k_{d,p}+1}.$$

Clearly $\mu_n \to \mu$ weakly so, using Lemma 4 we may find p such that $\mu(dx) = p(x)dx$ and p is bounded. So μ itself satisfies (H) and the proof is completed. $\square$

2.4. Regularity of the density

Theorem 1 says that μ_ϕ has a density as soon as $\phi \in W^{1,1}_\mu$—and this does not depend on the dimension d of the space. But if we want to obtain a continuous or a derivable density, we need more regularity for ϕ. The main instrument in order to obtain such properties is the classical theorem of Morrey which we recall now (see Corollary IX.13 in [5]).

Theorem 7. *Let $u \in W^{1,p}(\mathbb{R}^d)$. If $1 - \frac{d}{p} > 0$ then u is Hölder continuous of exponent $q = 1 - \frac{d}{p}$. Furthermore suppose that $u \in W^{m,p}(\mathbb{R}^d)$ and $m - \frac{d}{p} > 0$. Let $k = [m - \frac{d}{p}]$ be the integer part of $m - \frac{d}{p}$ and $q = \{m - \frac{d}{p}\}$ the fractional part. If $k = 0$ then u is Hölder continuous of exponent q and if $k \geq 1$ then $u \in C^k$ and for any multi-index α with $|\alpha| \leq k$ the derivative $\partial_\alpha u$ is Hölder continuous of exponent q: for any $x, y \in \mathbb{R}^d$,*

$$|\partial_\alpha u(x) - \partial_\alpha u(y)| \leq C_{d,p} \|u\|_{W^{m,p}(\mathbb{R}^d)} |x - y|^q$$

$C_{d,p}$ being dependent on d and p only.

It is clear from Theorem 7 that there are two ways to improve the regularity of u: one has to increase m or/and p. If $\phi \in W^{m,p}_\mu$ for a sufficiently large m then Theorem 1 already gives us a differentiable density p_{μ_ϕ}. But if we want to keep m low we have to increase p. And in order to be able to do it the key point is the estimate for $\Theta_p(\mu)$ given in Theorem 5. This is done in Theorem 8, where we use the following natural notation: we allow a multi-index to be equal to the empty set and for $\alpha = \emptyset$, we set $|\alpha| = 0$ and $\partial_\alpha f := f$.

Theorem 8. *We consider some $p > d$ and we suppose that $1 \in W^{1,p}_\mu$. For $m \geq 1$, let $\phi \in W^{m,p}_\mu$, so that $\mu_\phi(dx) = p_{\mu_\phi}(x)dx$. Then the following statements hold.*
A. *We have $p_{\mu_\phi} \in W^{m,p}$ and*

$$\left\| p_{\mu_\phi} \right\|_{W^{m,p}} \leq (2dK_{d,p})^{1-1/p} \|1\|_{W^{1,p}_\mu}^{k_{d,p}(1-1/p)} \|\phi\|_{W^{m,p}_\mu}. \tag{13}$$

Moreover, for any multi-index α such that $0 \leq |\alpha| = \ell \leq m - 1$, we have

$$\left\| \partial_\alpha p_{\mu_\phi} \right\|_\infty \leq dK_{d,p} \|1\|_{W^{1,p}_\mu}^{k_{d,p}} \|\phi\|_{W^{\ell+1,p}_\mu}. \tag{14}$$

B. *We have $p_{\mu_\phi} \in C^{m-1}$. Moreover, for any multi-index β such that $0 \leq |\beta| = k \leq m - 2$, $\partial_\beta p_{\mu_\phi}$ is Lipschitz continuous: for any $x, y \in \mathbb{R}^d$,*

$$|\partial_\beta p_{\mu_\phi}(x) - \partial_\beta p_{\mu_\phi}(y)| \leq d^2 K_{d,p} \|1\|_{W^{1,p}_\mu}^{k_{d,p}} \|\phi\|_{W^{k+2,p}_\mu} |x - y|.$$

And for any multi-index β such that $|\beta| = m - 1$, $\partial_\beta p_{\mu_\phi}$ is Hölder continuous of exponent $1 - d/p$: for any $x, y \in \mathbb{R}^d$,

$$\left| \partial_\beta p_{\mu_\phi}(x) - \partial_\beta p_{\mu_\phi}(y) \right| \leq C_{d,p} \left\| p_{\mu_\phi} \right\|_{W^{m,p}} |x - y|^{1-d/p}$$

$C_{d,p}$ being dependent on d and p only.
C. *We have $W^{m,p}_\mu \subset \bigcap_{\delta > 0} W^{m,p}(\{p_\mu > \delta\})$.*

Proof. A. We use (6) (with the notation $\partial_\alpha^\mu \phi := \phi$ if $\alpha = \emptyset$) and we obtain

$$\int \left| \partial_\alpha p_{\mu_\phi}(x) \right|^p dx = \int \left| \partial_\alpha^\mu \phi(x) \right|^p \left| p_\mu(x) \right|^p dx \leq \| p_\mu \|_\infty^{p-1} \int \left| \partial_\alpha^\mu \phi(x) \right|^p p_\mu(x) dx.$$

So $\left\| \partial_\alpha p_{\mu_\phi} \right\|_{L^p} \leq \| p_\mu \|_\infty^{1-1/p} \left\| \partial_\alpha^\mu \phi \right\|_{L^p_\mu}$ and by using (10) we obtain (13). Now, using the representation formula (5), Hölder's inequality and Theorem 5 we get

$$\left| \partial_\alpha p_{\mu_\phi}(x) \right| \leq \Theta_p(\mu) \sum_{i=1}^d \| \partial_{(\alpha,i)}^\mu \phi \|_{L^p_\mu} \leq d K_{d,p} \| 1 \|_{W^{1,p}_\mu}^{k_{d,p}} \| \phi \|_{W^{\ell+1,p}_\mu}$$

and (14) is proved.

B. The fact that $p_{\mu_\phi} \in C^{m-1}(\mathbb{R}^d)$ and the Hölder property are standard consequences of $p_{\mu_\phi} \in W^{m,p}$, as stated in Theorem 7. As for the Lipschitz property, it immediately follows from (14) and the fact that if $f \in C^1$ with $\| \nabla f \|_\infty < \infty$ then $|f(x) - f(y)| \leq \sum_{i=1}^d \| \partial_i f \|_\infty |x - y|$.

C. We have $p_{\mu_\phi}(x) dx = \phi(x) \mu(dx) = \phi(x) p_\mu(x) dx$ so $\phi(x) = p_{\mu_\phi}(x)/p_\mu(x)$ if $p_\mu(x) > 0$. And since $p_{\mu_\phi}, p_\mu \in W^{m,p}(\mathbb{R}^d)$ we obtain $\phi \in W^{m,p}(\{p_\mu > \delta\})$. $\quad\square$

2.5. Estimate of the tails of the density

In order to study the behavior of the tails of the density, we need the following computational rules.

Lemma 9. A. *If* $\phi \in W^{1,p}_\mu$ *and* $\psi \in C^1_b(\mathbb{R}^d)$ *then* $\psi\phi \in W^{1,p}_\mu$ *and*

$$\partial_i^\mu (\psi\phi) = \psi \partial_i^\mu \phi + \partial_i \psi \phi. \tag{15}$$

In particular, if $1 \in W^{1,p}_\mu$ *then for any* $\psi \in C^1_b(\mathbb{R}^d)$ *one has* $\psi \in W^{1,p}_\mu$ *and*

$$\partial_i^\mu \psi = \psi \partial_i^\mu 1 + \partial_i \psi. \tag{16}$$

B. *Suppose that* $1 \in W^{1,p}_\mu$. *If* $\psi \in C^1_b(\mathbb{R}^m)$ *and if* $u = (u_1, \ldots, u_m) : \mathbb{R}^d \to \mathbb{R}^m$ *is such that* $u_j \in C^1_b(\mathbb{R}^d)$, $j = 1, \ldots, m$, *then* $\psi \circ u \in W^{1,p}_\mu$ *and*

$$\partial_i^\mu \psi \circ u = \sum_{j=1}^m (\partial_j \psi) \circ u \, \partial_i^\mu u_j + T_\psi \circ u \, \partial_i^\mu 1$$

where

$$T_\psi(x) = \psi(x) - \sum_{j=1}^d \partial_j \psi(x) x_j.$$

Proof. A. Since ψ and $\partial_i \psi$ are bounded, $\psi\phi, \psi\partial_i^\mu\phi, \partial_i\psi\phi \in L^p_\mu$. So we just have to check the integration by parts formula. We have

$$\int \partial_i f \psi\phi \, d\mu = \int \partial_i (f\psi)\phi \, d\mu - \int f \partial_i \psi\phi \, d\mu = - \int f\psi \partial_i^\mu \phi \, d\mu - \int f \partial_i \psi\phi \, d\mu$$

and the statement holds.

B. By using (16), we have

$$\partial_i^\mu(\psi \circ u) = \psi \circ u\, \partial_i^\mu 1 + \partial_i(\psi \circ u) = \psi \circ u\, \partial_i^\mu 1 + \sum_{j=1}^m (\partial_j \psi) \circ u\, \partial_i u_j.$$

The formula now follows by inserting $\partial_i u_j = \partial_i^\mu u_j - u_j \partial_i^\mu 1$, as given by (16). $\square$

We now give a result which allows us to estimate the queues of p_μ.

Proposition 10. *Let $\phi \in C_b^1(\mathbb{R}^d)$ be a function such that $1_{B_1(0)} \leq \phi \leq 1_{B_2(0)}$. We set $\phi_x(y) = \phi(x - y)$ and we assume that $1 \in W_\mu^{1,p}$ with $p > d$, so, in view of Lemma 9, $\phi_x \in W_\mu^{1,p}$. Then we have the representation*

$$p_\mu(x) = \sum_{i=1}^d \int \partial_i Q_d(y - x) \partial_i^\mu \phi_x(y) 1_{\{|y-x|<2\}} \mu(dy).$$

As a consequence, for any positive $a < \frac{1}{d} - \frac{1}{p}$ one has

$$p_\mu(x) \leq \Theta_{\overline{p}}(\mu)\big(d + \|1\|_{W_\mu^{1,p}}\big) \mu\big(B_2(x)\big)^a \tag{17}$$

where $\overline{p} = 1/(a + \frac{1}{p})$. In particular,

$$\lim_{|x| \to \infty} p_\mu(x) = 0. \tag{18}$$

Proof. By Lemma 9, $\phi_x \in W_\mu^{1,p}$ and $\partial_i^\mu \phi_x = \phi_x \partial_i^\mu 1 + \partial_i \phi_x$, so that $\partial_i^\mu \phi_x = \partial_i^\mu \phi_x 1_{B_2(x)}$. Now, for $f \in C_c^1(B_1(x))$ we have

$$\int f(y)\mu(dy) = \int f(y)\phi_x(y)\mu(dy) = \int f(y) p_{\mu_{\phi_x}}(y)dy$$

$$= -\int f(y) \sum_{i=1}^d \int \partial_i Q_d(z - y) \partial_i^\mu \phi_x(z)\mu(dz)dy$$

$$= -\int f(y) \sum_{i=1}^d \int \partial_i Q_d(z - y) \partial_i^\mu \phi_x(z) 1_{B_2(x)}(z)\mu(dz)dy.$$

It follows that for $y \in B_1(x)$ we have

$$p_\mu(y) = -\sum_{i=1}^d \int \partial_i Q_d(z - y) \partial_i^\mu \phi_x(z) 1_{B_2(x)}(z)\mu(dz).$$

We now consider $y = x$ and we take $a \in (0, \frac{1}{d} - \frac{1}{p})$. Using Hölder's inequality we obtain

$$p_\mu(x) \leq \mu(B_2(x))^a \sum_{i=1}^d I_i \quad \text{with } I_i = \left(\int \left| \partial_i Q_d(z - x) \partial_i^\mu \phi_x(z) \right|^{\frac{1}{1-a}} \mu(dz) \right)^{1-a}.$$

Notice that $1 < d(1 - a)/(1 - da) < p(1 - a)$. We take β such that $d(1 - a)/(1 - da) < \beta < p(1 - a)$ and we denote by α the conjugate of β. Again using Hölder's inequality we

obtain

$$I_i \leq \left(\int |\partial_i Q_d(z - x)|^{\frac{\alpha}{1-a}} \mu(dz) \right)^{(1-a)/\alpha} \left(\int |\partial_i^\mu \phi_x(z)|^{\frac{\beta}{1-a}} \mu(dz) \right)^{(1-a)/\beta}$$

$$\leq \left(\int |\partial_i Q_d(z - x)|^{\frac{\alpha}{1-a}} \mu(dz) \right)^{(1-a)/\alpha} \|\partial_i^\mu \phi_x\|_{L_\mu^p}.$$

We let $\beta \uparrow p(1 - a)$ so that

$$\frac{\alpha}{1 - a} = \frac{\beta}{(\beta - 1)(1 - a)} \rightarrow \frac{p}{p(1 - a) - 1} = \frac{\overline{p}}{\overline{p} - 1}.$$

So we obtain

$$I_i \leq \Theta_{\overline{p}}(\mu) \left\| \partial_i^\mu \phi_x \right\|_{L_\mu^p}$$

and then

$$p_\mu(x) \leq \Theta_{\overline{p}}(\mu) \|\phi_x\|_{W_\mu^{1,p}} \mu(B_2(x))^a.$$

Now, since $\partial_i^\mu \phi_x = \phi_x \partial_i^\mu 1 + \partial_i \phi_x$, we have $\|\phi_x\|_{W_\mu^{1,p}} \leq c_\phi d + \|1\|_{W_\mu^{1,p}}$ where $c_\phi > 1$ is such that $\|\nabla \phi\|_\infty \leq c_\phi$, so that

$$p_\mu(x) \leq \Theta_{\overline{p}}(\mu)\big(c_\phi d + \|1\|_{W_\mu^{1,p}}\big)\mu(B_2(x))^a.$$

(17) now follows by taking a sequence $\{\phi_n\}_n \subset C_b^1$ such that $1_{B_1(0)} \leq \phi_n \leq 1_{B_2(0)}$ with $\|\nabla \phi_n\|_\infty \leq 1 + 1/n$ and by letting $n \to \infty$.

Finally, $1_{B_2(x)} \to 0$ a.s. when $|x| \to \infty$ and by using the Lebesgue dominated convergence theorem, one has $\mu\big(B_2(x)\big) = \int 1_{B_2(x)}(y)\mu(dy) \to 0$. By applying (17), one obtains (18). □

2.6. On the set of strict positivity for the density

Suppose that $1 \in W_\mu^{2,1}$ and set $U_\mu = \{p_\mu > 0\}$. We define the matrix field $C_\mu : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ through

$$C_\mu^{ij}(x) = \partial_i^\mu 1(x)\partial_j^\mu 1(x), \quad i, j = 1, \ldots, d.$$

For $x, y \in \mathbb{R}^d$, we set

$$A_{x,y}^\mu = \left\{ \varphi \in C^1([0, 1], U_\mu); \ \varphi_0 = x, \varphi_1 = y \right\}$$

$$d_\mu(x, y) = \inf \left\{ \int_0^1 \langle C_\mu(\varphi_t)\dot{\varphi}_t, \dot{\varphi}_t \rangle^{1/2} dt; \ \varphi \in A_{x,y}^\mu \right\} \tag{19}$$

with the understanding $d_\mu(x, y) = +\infty$ if $A_{x,y} = \emptyset$. Notice that $d_\mu(x, y) = +\infty$ if x and y belong to two different connected components of the open set U_μ, if they exist. Moreover, it is easy to see that $d_\mu(x, y)$ does not define in general a distance but only a semi-distance. In fact, as an example, take p_μ as a smooth probability density on $\mathbb{R}$ which is constant on some interval (a, b), $a < b$. Then, $\partial^\mu 1 = \partial \ln p_\mu \equiv 0$ on (a, b), so that $d_\mu(x, y) = 0$ for any $x, y \in (a, b)$.

Then we have the following representation formula and estimates for the density.

Proposition 11. *Let* $1 \in W_{\mu}^{2,p}$ *with* $p > d$, *and let* $x, x_0 \in U_{\mu}$ *be such that* $A_{x_0,x}^{\mu} \neq \emptyset$. *Then, for any* $\varphi \in A_{x_0,x}^{\mu}$ *one has*

$$p_{\mu}(x) = p_{\mu}(x_0) \exp \left(\int_0^1 \langle \partial^{\mu} 1(\varphi_t), \dot{\varphi}_t \rangle \, dt \right). \tag{20}$$

As a consequence,

$$p_{\mu}(x_0) e^{-d_{\mu}(x_0,x)} \leq p_{\mu}(x) \leq p_{\mu}(x_0) e^{d_{\mu}(x_0,x)}. \tag{21}$$

Proof. If $1 \in W_{\mu}^{1,p}$ with $p > d$, the density p_{μ} exists and is continuous, so that $U_{\mu} = \{p_{\mu} > 0\}$ is open. Now, for $\varphi \in A_{x_0,x}^{\mu}$, one has $\partial_i \ln p_{\mu}(\varphi_{x_0,x}(t)) = \partial_i^{\mu} 1(\varphi_{x_0,x}(t))$ so that

$$d \ln p_{\mu}(\varphi(t)) = \sum_{i=1}^{d} \partial_i^{\mu} 1(\varphi(t)) \dot{\varphi}^i(t) dt = \langle \partial^{\mu} 1(\varphi_t), \dot{\varphi}_t \rangle$$

and (20) follows by integrating over [0, 1]. Now, for any $\varphi \in A_{x_0,x}^{\mu}$ one has

$$\left| \ln \frac{p(x)}{p(x_0)} \right| = \left| \int_0^1 \langle \partial^{\mu} 1(\varphi_t), \dot{\varphi}_t \rangle \, dt \right| \leq \int_0^1 |\langle \partial^{\mu} 1(\varphi_t), \dot{\varphi}_t \rangle| \, dt = \int_0^1 \langle C_{\mu}(\varphi_t) \dot{\varphi}_t, \dot{\varphi}_t \rangle^{1/2} \, dt.$$

By taking the inf over $A_{x_0,x}^{\mu}$ one proves (21). $\quad\square$

We can now state the main result of this section.

Proposition 12. *Suppose that* $1 \in W_{\mu}^{2,p}$ *with* $p > d$. *Then the following statements hold.*
(i) *If* $p_{\mu}(x_n) \to 0$ *then* $d_{\mu}(x_n, x_1) \to \infty$.
(ii) *If* U_{μ} *is connected, the converse of* (i) *holds: if* $d_{\mu}(x_n, x_1) \to \infty$ *then* $p_{\mu}(x_n) \to 0$.
(iii) $\partial^{\mu} 1$ *is locally bounded (that is, bounded on compact sets of* $\mathbb{R}^d$) *if and only if* $\{p_{\mu} > 0\} = \mathbb{R}^d$.

Proof. (i) The statement immediately follows by the first inequality in (21).

(ii) By contradiction, we assume that $p_{\mu}(x_n) \not\to 0$: there exist $c > 0$ and a subsequence $\{x_{n_k}\}_k$ such that $p_{\mu}(x_{n_k}) \geq c$ for any k. By Proposition 10, and in particular (18), there exists $R > 0$ such that $p_{\mu}(x) < c$ if $|x| > R$. This gives that the sequence $\{x_{n_k}\}_k$ is bounded and then there exists a further subsequence $\{x_{n_{k_\ell}}\}_\ell$ converging to some point $\bar{x}$. Now, since p_{μ} is continuous and $p_{\mu}(\bar{x}) \geq c$, there exists $r > 0$ such that $p_{\mu} \geq \frac{c}{2}$ in the ball $B(\bar{x}, r)$, which of course contains the points $x_{n_{k_\ell}}$ for any large ℓ. This means that the path φ^{ℓ} joining $\bar{x}$ to $x_{n_{k_\ell}}$ at a constant speed belong to $A_{\bar{x}, x_{n_{k_\ell}}}$ for any large ℓ. Therefore,

$$\int_0^1 |\langle \partial^{\mu} 1(\varphi_t^{\ell}), \dot{\varphi}_t^{\ell} \rangle| \, dt \leq \int_0^1 |\partial^{\mu} 1(\varphi_t^{\ell})| |\dot{\varphi}_t^{\ell}| \, dt \leq C \times |\bar{x} - x_{n_{k_\ell}}| \leq C \times r$$

in which we have used the fact that $\partial^{\mu} 1$ is bounded on $B(\bar{x}, r)$. It follows that for some ℓ_0,

$$\sup_{\ell \geq \ell_0} d_{\mu}(\bar{x}, x_{n_{k_\ell}}) \leq C \times r.$$

Now,

$$\sup_{\ell \geq \ell_0} d_{\mu}(x_1, x_{n_{k_\ell}}) \leq d_{\mu}(x_1, \bar{x}) + C \times r$$

and $d_{\mu}(x_1, \bar{x}) < \infty$ because U_{μ} is connected, and this gives a contradiction.

(iii) If $\partial^\mu 1$ is bounded on compact sets of $\mathbb{R}^d$, then for $x \in U_\mu$ by (20) we get

$$p_\mu(x) \le p_\mu(x_0) \exp\left(C \sup_{t \in [0,1]} |\dot\varphi_{x_0,x}|\right)$$

where $x_0 \in U_\mu$ is such that $A^\mu_{x_0,x} \ne \emptyset$. Now, if $\partial U_\mu \ne \emptyset$, we can let x_0 tend to the boundary of U_μ and in such a case we obtain $p_\mu(x) = 0$, which is a contradiction. Therefore, $p_\mu > 0$ everywhere. On the contrary, it is sufficient to recall that $\partial^\mu 1(x) = \partial \ln p_\mu$ is continuous on $U_\mu = \mathbb{R}^d$. $\square$

Remark 13. Parts (i) and (ii) of the above Proposition 12 allow us to discuss the Malliavin conjecture about the set of the strict positivity points of the density, as already described in the Introduction at page section. For further details, we address Proposition 27.

Furthermore, (iii) says that if $1 \in W^{1,p}_\mu$ with $\partial^\mu_i 1$ locally bounded then we can take $x_0 = 0$ and $\varphi_{x_0,x}(t) = tx$, so that

$$p_\mu(x) = p_\mu(0) \exp\left(\int_0^1 \sum_{i=1}^d x_i \partial^\mu_i 1(tx)\, dt\right).$$

Such a representation formula has already been given by Bell in [3].

Remark 14. It is easy to see that all the results of this section hold if the semi-distance d_f is replaced by the square root of the energy associated to the matrix field C_μ, which is defined by

$$\overline{d}_\mu(x, y) = \inf\left\{\left(\int_0^1 \langle C_\mu(\varphi_t)\dot\varphi_t, \dot\varphi_t\rangle dt\right)^{1/2} ; \varphi \in A_{x,y}\right\} \tag{22}$$

with $\overline{d}_\mu(x, y) = +\infty$ if $A_{x,y} = \emptyset$. Again, $\overline{d}_\mu(x, y)$ defines only a semi-distance and one has $d_\mu(x, y) \le \overline{d}_\mu(x, y)$.

2.7. Local integration by parts formulas

The assumptions in the previous sections are global – and this may fail in many interesting cases – for example for diffusion processes living in a region of the space or, as a more elementary example, for the exponential distribution. So in this section we give a hint about the localized version of the results presented above.

An open domain $D \subset \mathbb{R}^d$ is given. We recall that $L^p_\mu(D) = \{f : \int_D |f(x)|^p d\mu(x) < \infty\}$ and $W^{1,p}_\mu(D)$ is the space of the functions $\phi \in L^p_\mu(D)$ which verify the integration by parts formula $\int \phi \partial_i f d\mu = -\int \theta_i f d\mu$ for test functions f which have a compact support included in D. And $\partial^{\mu,D}_i \phi := \theta_i \in L^p_\mu(D)$. The space $W^{m,p}_\mu(D)$ is similarly defined. Our aim is to give sufficient conditions in order that μ has a smooth density on D, that means that we look for a smooth function p such that $\int_D f(x) d\mu(x) = \int_D f(x) p(x) dx$. And we want to give estimates for p and its derivatives in terms of the Sobolev norms of $W^{m,p}_\mu(D)$.

The main step in our approach is a truncation argument that we present now. Given $-\infty \le a \le b \le \infty$ and $\varepsilon > 0$ we define $\psi_{\varepsilon,a,b} : \mathbb{R} \to \mathbb{R}_+$ by

$$\psi_{\varepsilon,a,b}(x) = 1_{(a-\varepsilon,a]}(x) \exp\left(1 - \frac{\varepsilon}{x + \varepsilon - a}\right) + 1_{(a,b)}(x)$$

$$+ 1_{[b,b+\varepsilon)}(x) \exp\left(1 - \frac{\varepsilon}{x - b - \varepsilon}\right)$$

with the convention $1_{(a-\varepsilon,a)} = 0$ if $a = -\infty$ and $1_{(b,b+\varepsilon)} = 0$ if $b = \infty$. Notice that $\psi_{\varepsilon,a,b} \in C_b^\infty(\mathbb{R})$ and

$$\sup_{x \in (a-\varepsilon,b+\varepsilon)} \left| \partial_x \ln \psi_{\varepsilon,a,b}(x) \right|^p \psi_{\varepsilon,a,b}(x) \le \varepsilon^{-p} \sup_{y>0} y^{2p} e^{-y}.$$

For $x = (x^1, \ldots, x^d)$ and $i \in \{1, \ldots, d\}$ we denote $\widehat{x}_i = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^d)$ and for $y \in \mathbb{R}$ we put $(\widehat{x}_i, y) = (x^1, \ldots, x^{i-1}, y, x^{i+1}, \ldots, x^d)$. Then we define

$$a(\widehat{x}_i) = \inf_{y \in \mathbb{R}} \left\{ y : d((\widehat{x}_i, y), D^c) > 2\varepsilon \right\}, \qquad b(\widehat{x}_i) = \sup_{y \in \mathbb{R}} \left\{ y : d((\widehat{x}_i, y), D^c) > 2\varepsilon \right\}$$

with the convention $a(\widehat{x}_i) = b(\widehat{x}_i) = 0$ if $\{y : d((\widehat{x}_i, y), D^c) > 2\varepsilon\} = \emptyset$. Finally we define

$$\Psi_{D,\varepsilon}(x) = \prod_{i=1}^d \psi_{\varepsilon,a(\widehat{x}_i),b(\widehat{x}_i)}(x_i).$$

We denote $D_\varepsilon = \{x : d(x, D^c) \ge \varepsilon\}$ so that $1_{D_{2\varepsilon}} \le \Psi_{D,\varepsilon} \le 1_{D_\varepsilon}$. And we also have

$$\sup_{x \in D_\varepsilon} \left| \partial_x \ln \Psi_{D,\varepsilon}(x) \right|^p \Psi_{D,\varepsilon}(x) \le d\varepsilon^{-p} \sup_{y>0} y^{2p} e^{-y}. \tag{23}$$

We are now able to give the main result in this section. The symbol $\nu|_D$ denotes the measure ν restricted to the open set D.

Theorem 15. A. *Suppose that $\phi \in W_\mu^{1,1}(D)$. Then $\mu_\phi|_D(dx) = p_{\mu_\phi}(x)dx$.*

B. *Suppose that $1 \in W_\mu^{1,p}(D)$ for some $p > d$. Then for each $\varepsilon > 0$*

$$\sup_{x \in \mathbb{R}^d} \sum_{i=1}^d \int_{D_{2\varepsilon}} |\partial_i Q_d(y-x)|^{p/(p-1)} \mu(dy) \le C_{d,p} \varepsilon^{-p} \|1\|_{W_\mu^{1,p}(D)}.$$

C. *Suppose that $1 \in W_\mu^{1,p}(D)$ for some $p > d$. Then for $\phi \in W_\mu^{m,p}(D)$ we have $\mu_\phi|_D(dx) = p_{\phi,D}(x)dx$ and*

$$p_{\phi,D}(x) = -\sum_{i=1}^d \int \partial_i Q_d(y-x)(\Psi_{D,\varepsilon} \partial_i^\mu \phi + \phi \partial_i \Psi_{D,\varepsilon}) \mu(dx) \quad \text{for } x \in D_\varepsilon.$$

Moreover $p_{\phi,D} \in \bigcap_{\varepsilon>0} W^{m,p}(D_\varepsilon)$ and

$$\|p_{\phi,D}\|_{W^{m,p}(D_\varepsilon)} \le C_{p,d}(1 + \varepsilon^{-1})(1 \vee \|1\|_{W_\mu^{1,p}(D)})^{k_{d,p}(1-1/p)} \|\phi\|_{W_\mu^{m,p}(D)}.$$

Finally, $p_{\phi,D}$ is $m-1$ times differentiable on D and for every multi-index α of length $0 \le \ell \le m-1$ one has

$$\|\partial_\alpha p_{\phi,D}\|_\infty \le C_{p,d}(1 + \varepsilon^{-\ell})(1 \vee \|1\|_{W_\mu^{1,p}(D)})^{k_{d,p}(1-1/p)} \|\phi\|_{W_\mu^{\ell+1,p}(D)}.$$

Proof. We denote $\mu_{D,\varepsilon}(dx) = \Psi_{D,\varepsilon}(x)\mu(dx)$. Let us first show that if $\phi \in W_\mu^{1,p}(D)$ then $\phi \in W_{\mu_{D,\varepsilon}}^{1,p}(\mathbb{R}^d)$. In fact, if $f \in C_c^\infty(\mathbb{R}^d)$ then $f\Psi_{D,\varepsilon} \in C_c^\infty(D)$ and similarly to what developed

in Lemma 9, one has

$$\int \partial_i f \phi d\mu_{D,\varepsilon} = \int \partial_i f \phi \, \Psi_{D,\varepsilon} d\mu = -\int \partial_i^\mu (\Psi_{D,\varepsilon} \phi) f d\mu$$

$$= -\int (\Psi_{D,\varepsilon} \partial_i^\mu \phi + \phi \partial_i \Psi_{D,\varepsilon}) f d\mu = -\int (\partial_i^\mu \phi + \phi \partial_i \ln \Psi_{D,\varepsilon}) f d\mu_{D,\varepsilon}$$

so that $\partial_i^{\mu_{D,\varepsilon}} \phi = \partial_i^\mu \phi + \phi \partial_i \ln \Psi_{D,\varepsilon}$. And by using (23) we have $\partial_i^{\mu_{D,\varepsilon}} \phi \in L^p_{\mu_{D,\varepsilon}}(\mathbb{R}^d)$:

$$\sum_{i=1}^d \int |\partial_i^{\mu_{D,\varepsilon}} \phi|^p d\mu_{D,\varepsilon} = \sum_{i=1}^d \int |\partial_i^\mu \phi + \phi \partial_i \ln \Psi_{D,\varepsilon}|^p \, \Psi_{D,\varepsilon} d\mu$$

$$\leq \sum_{i=1}^d \int_D |\partial_i^\mu \phi + \phi \partial_i \ln \Psi_{D,\varepsilon}|^p \, \Psi_{D,\varepsilon} d\mu \leq C_p \varepsilon^{-p} \|\phi\|^p_{W^{1,p}_\mu(D)}.$$

It follows that

$$\|\phi\|_{W^{1,p}_{\mu_{D,\varepsilon}}(R^d)} \leq C_p \varepsilon^{-1} \|\phi\|_{W^{1,p}_\mu(D)}.$$

Setting $\mu_{D,\varepsilon,\phi}(dx) := \phi \mu_{D,\varepsilon}(dx)$, we can use Theorem 8 and we obtain $\mu_{D,\varepsilon,\phi}(dx) = p_{D,\varepsilon,\phi}(x)dx$ with $p_{D,\varepsilon,\phi} \in W^{1,p}(\mathbb{R}^d)$. Similarly we prove that if $\phi \in W^{m,p}_\mu(D)$ then $p_{D,\varepsilon,\phi} \in W^{m,p}(\mathbb{R}^d)$. We notice that for a function f with the support included in $D_{2\varepsilon}$ we have $\int f \phi d\mu = \int f \phi d\mu_{D,\varepsilon}$. It follows that $\mu_\phi|_{D_{2\varepsilon}}(dx) = p_{D,\varepsilon}(x)dx$.

Now, statement **A.** immediately follows from the above arguments.

B. Suppose now that $1 \in W^{1,p}_\mu(D)$ for some $p > d$. Then

$$\int_{D_{2\varepsilon}} |\partial_i Q_d(x-y)|^{p/(p-1)} \mu(dy) \leq \int |\partial_i Q_d(x-y)|^{p/(p-1)} \Psi_{D,\varepsilon}(y) \mu(dy)$$

$$= \int |\partial_i Q_d(x-y)|^{p/(p-1)} \mu_{D,\varepsilon}(dy)$$

$$\leq d K_{d,p} (1 \vee \|1\|_{W^{1,p}_{\mu_{D,\varepsilon}}})^{k_{d,p}} \leq C_{d,p} \varepsilon^{-p} \|1\|_{W^{1,p}_\mu(D)}.$$

The upper bounds for the density and its derivatives are proved in a similar way.

Finally, **C** follows similarly as in (iii) of Proposition 12. $\square$

3. Integration by parts formulas for random variables and Riesz transform

Let $(\Omega, \mathcal{F}, P)$ ba a probability space and let F and G be two random variables taking values in $\mathbb{R}^d$ and $\mathbb{R}$ respectively.

Definition 16. Given a multi-index α and a power $p \geq 1$, we say that the integration by parts formula $IP_{\alpha,p}(F, G)$ holds if there exists a random variable $H_\alpha(F; G) \in L^p$ such that

$$IP_{\alpha,p}(F, G) \quad \mathbb{E}(\partial_\alpha f(F)G) = (-1)^{|\alpha|} \mathbb{E}(f(F)H_\alpha(F; G)), \quad \forall f \in C_c^{|\alpha|}(\mathbb{R}^d). \tag{24}$$

We define $W^{m,p}_F$ to be the space of the random variables $G \in L^p$ such that $IP_{\alpha,p}(F, G)$ holds for every multi-index α with $|\alpha| \leq m$. For $G \in W^{m,p}_F$ we define

$$\partial_\alpha^F G = \mathbb{E}(H_\alpha(F; G) \mid F).$$

We denote by μ_F the law of F and $\mu_{F,G}(f) := \mathbb{E}(f(F)G)$. So $\mu_F = \mu_{F,1}$. Moreover we denote $\phi_G(x) = \mathbb{E}(G \mid F = x)$. Then it is easy to check that

$$W_F^{m,p} = \{G \in L^p : \phi_G \in W_{\mu_F}^{m,p}\} \quad \text{and} \quad \partial_\alpha^F G = \partial_\alpha^{\mu_F} \phi_G(F).$$

We also define the norms

$$\|G\|_{W_F^{m,p}} = \|G\|_{L^p} + \sum_{1 \le |\alpha| \le m} (\mathbb{E}(|\partial_\alpha^F G|^p))^{1/p}.$$

It is easy to see that $(W_F^{m,p}, \|\cdot\|_{W_F^{m,p}})$ is a Banach space.

Remark 17. Notice that $\mathbb{E}(|\partial_\alpha^F G|^p) \le \mathbb{E}(|H_\alpha(F; G)|^p)$ so that $\partial_\alpha^F G$ is the weight of the minimal L^p norm which verifies $IP_{\alpha,p}(F; G)$. In particular

$$\|G\|_{W_F^{m,p}} \le \|G\|_{L^p} + \sum_{1 \le |\alpha| \le m} \|H_\alpha(F; G)\|_{L^p}$$

and this last quantity is the one which naturally appears in concrete computations.

We can resume the result of Section 2 as follows. As for the density, we obtain

Theorem 18. A. *Suppose that* $1 \in W_F^{1,p}$ *for some* $p > d$. *Then* $\mu_F(dx) = p_F(x)dx$ *and* $p_F \in C_b(\mathbb{R}^d)$. *Moreover*

$$\Theta_F(p) := \sup_{a \in \mathbb{R}^d} \sum_{i=1}^d \left(\mathbb{E}(|\partial_i Q_d(F - a)|^{p/(p-1)})\right)^{(p-1)/p} \le K_{d,p} \|1\|_{W_F^{1,p}}^{k_{d,p}},$$

$$\|p_F\|_\infty \le K_{d,p} \|1\|_{W_F^{1,p}}^{1+k_{d,p}}$$

and

$$p_F(x) = \sum_{i=1}^d \mathbb{E}(\partial_i Q_d(F - x)\partial_i^F 1) = \sum_{i=1}^d \mathbb{E}(\partial_i Q_d(F - x)H_i(F; 1)).$$

B. *For any positive* $a < \frac{1}{d} - \frac{1}{p}$ *one has*

$$p_F(x) \le \Theta_F(\overline{p})(d + \|1\|_{W_F^{1,p}}) \mathbb{P}(F \in B_2(x))^a$$

where $\overline{p} = 1/(a + \frac{1}{p})$.

Now we give the representation formula and the estimates for the conditional expectation.

Theorem 19. *Suppose* $1 \in W_F^{1,p}$. *Let* $m \ge 1$ *and* $G \in W_F^{m,p}$.
A. *We have* $\mu_{F,G}(dx) = p_{F,G}(x)dx$ *and*

$$\phi_G(x) = \mathbb{E}(G \mid F = x) = 1_{\{p_F(x)>0\}} \frac{p_{F,G}(x)}{p_F(x)}$$

with

$$p_{F,G}(x) = \sum_{i=1}^d \mathbb{E}(\partial_i Q_d(F - x)\partial_i^F G) = \sum_{i=1}^d \mathbb{E}(\partial_i Q_d(F - x)H_i(F; G)).$$

B. *We have* $p_{F,G} \in W^{m,p}$ *and* $\phi_G \in \bigcap_{\delta>0} W^{m,p}(\{p_F > \delta\})$. *Moreover,*

(i) $\quad \|p_{F,G}\|_\infty \leq K_{d,p} \|1\|_{W_F^{1,p}}^{k_{d,p}} \|G\|_{W_F^{1,p}}$

(ii) $\quad \|p_{F,G}\|_{W^{m,p}} \leq (2dK_{d,p})^{1-1/p} \|1\|_{W_F^{1,p}}^{k_{d,p}(1-1/p)} \|G\|_{W_F^{m,p}}.$

We also have the representation formula

$$\partial_\alpha p_{F,G}(x) = \sum_{i=1}^d \mathbb{E}(\partial_i Q_d(F-x)\partial_{(\alpha,i)}^F G) = \sum_{i=1}^d \mathbb{E}(\partial_i Q_d(F-x)H_{(\alpha,i)}(F;G))$$

for any α *with* $0 \leq |\alpha| \leq m-1$. *Furthermore,* $p_{F,G} \in C^{m-1}(\mathbb{R}^d)$ *and for any multi-index* α *with* $|\alpha| = k \leq m-2$, $\partial_\alpha p_{F,G}$ *is Lipschitz continuous with Lipschitz constant*

$$L_\alpha = d^2 \|1\|_{W_F^{1,p}}^{k_{d,p}} \|G\|_{W_F^{k+2,p}}.$$

And for any multi index α *with* $|\alpha| = m-1$, $\partial_\alpha p_{F,G}$ *is Hölder continuous of exponent* $1 - d/p$ *and Hölder constant*

$$L_\alpha = C_{d,p} \|p_{F,G}\|_{W^{m,p}},$$

$C_{d,p}$ *being dependent on* d *and* p *only.*

Finally we give a stability property.

Proposition 20. *Let* $F_n, G_n, n \in \mathbb{N}$ *be two sequences of random variables such that* $(F_n, G_n) \to (F, G)$ *in probability. Suppose that* $G_n \in W_{F_n}^{m,p}$ *and* $\sup_n(\|G_n\|_{W_{F_n}^{m,p}} + \|F_n\|_{L^p}) < \infty$ *for some* $m \in \mathbb{N}$. *Then* $G \in W_F^{m,p}$ *and* $\|G\|_{W_F^{m,p}} \leq \sup_n \|G_n\|_{W_{F_n}^{m,p}}$.

Remark 21. Suppose that we are in the framework of Malliavin calculus and think that F is a functional on the Wiener space which is non-degenerate and sufficiently smooth in the Malliavin sense. And G is another functional which is sufficiently smooth in the Malliavin sense. Then the Malliavin calculus produces integration by parts formulas and so permits us to prove that $G \in W_F^{m,p}$. But we may proceed in a different way: we start by taking a sequence $F_n, n \in \mathbb{N}$ of simple functionals such that $F_n \to F$ and a sequence $G_n, n \in \mathbb{N}$ such that $G_n \to G$ and then we may use standard finite dimensional integration by parts formulas in order to prove that $G_n \in W_{F_n}^{m,p}$. If we are able to check that $\sup_n \|G_n\|_{W_{F_n}^{m,p}} < \infty$ then using the above stability property we conclude that $G \in W_F^{m,p}$.

Proof. We denote $Q_n = (F_n, G_n, \partial_\alpha^{F_n} G_n, |\alpha| \leq m)$. Since $p \geq 2$ and $\sup_n(\|G_n\|_{W_{F_n}^{m,p}} + \|F_n\|_{L^p}) < \infty$ it follows that the sequence $Q_n, n \in \mathbb{N}$ is bounded in L^2 and consequently weakly relative compact. Let Q be a limit point. Using Mazur's theorem we construct a sequence of convex combinations $\overline{Q}_n = \sum_{i=1}^{k_n} \lambda_i^n Q_{n+i}$, (with $\sum_{i=1}^{k_n} \lambda_i^n = 1$ and $\lambda_i^n \geq 0$) such that $\overline{Q}_n \to Q$ strongly in L^2. And passing to a subsequence we may assume that the convergence holds almost surely as well. Since $(F_n, G_n) \to (F, G)$ in probability it follows that $Q = (F, G, \theta_\alpha, |\alpha| \leq m)$. And using the integration by parts formulas $IP_{\alpha,p}(F_n, G_n)$ and the almost sure convergence it is easy to see that $IP_{\alpha,p}(F, G)$ holds with $H_\alpha(F; G) = \theta_\alpha$ so $\theta_\alpha = \partial_\alpha^F G$. Moreover again using the almost sure convergence and the convex combinations

one checks that $\|G\|_{W_F^{m,p}} \leq \sup_n \|G\|_{W_{F_n}^{m,p}}$. In all the above arguments we have to use the Lebesgue dominated convergence theorem so the almost sure convergence is not sufficient. But a straightforward truncation argument which we do not develop here permits us to handle this difficulty. $\quad\square$

4. Functionals on the Wiener space

In this section we consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a Brownian motion $W = (W^1, \ldots, W^n)$ and we use the Malliavin calculus in order to obtain integration by parts formulas. We refer to [14] for notation and basic results. We denote by $\mathbb{D}^{k,p}$ the space of the random variables which are k times differentiable in Malliavin sense in L^p and for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \{1, \ldots, n\}^m$ we denote by $D^\alpha F$ the Malliavin derivative of F corresponding to the multi-index α. Moreover, for any multi-index α with length $|\alpha| = m$ we set

$$|D^{(m)}F|^2 = \sum_{|\alpha|=m} |D^\alpha F|^2.$$

We also consider the norms

$$\|F\|_{m,p}^p = \|F\|_p^p + \sum_{k=1}^m \sum_{|\alpha|=k} \mathbb{E}\left(\left(\int_{[0,\infty)^k} |D_{s_1,\ldots,s_k}^\alpha F|^2 \, ds_1 \ldots ds_k\right)^{p/2}\right).$$

So $\mathbb{D}^{m,p}$ is the closure of the space of the simple functionals with respect to the norm $\|\cdot\|_{m,p}$. Moreover, for $F = (F^1, \ldots, F^d)$, $F^i \in \mathbb{D}^{1,2}$, one denotes by σ_F the Malliavin covariance matrix associated to F:

$$\sigma_F^{i,j} = \langle DF^i, DF^j \rangle = \sum_{k=1}^n \int_0^\infty D_s^k F^i D_s^k F^j ds, \quad i,j = 1, \ldots, d.$$

We will assume the non-degeneracy condition

$$(\det \sigma_F)^{-1} \in \bigcap_{p \in \mathbb{N}} L^p. \tag{25}$$

Under this assumption the matrix σ_F is invertible and we denote by $\widehat{\sigma}_F$ the inverse matrix. We also denote by δ the divergence operator (Skorohod integral) and by L the Ornstein Uhlembeck operator and we recall that if $F \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{2,p}$ then $F \in \text{Dom}(L)$. The following proposition gives the classical integration by parts formula from Malliavin calculus.

Proposition 22. (i) *Let* $F = (F^1, \ldots, F^d)$ *with* $F^1, \ldots, F^d \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{2,p}$ *and* $G \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{1,p}$. *Assume that* (25) *holds. Then for every function* $f \in C_b^1(\mathbb{R}^d) \to \mathbb{R}$ *and every* $i = 1, \ldots, d$ *one has*

$$\mathbb{E}(\partial_i f(F)G) = -\mathbb{E}(f(F)H_i(F,G)) \quad \text{with}$$

$$H_i(F,G) = -\sum_{j=1}^d \delta(G\widehat{\sigma}_F^{ji} DF^j) = -\sum_{j=1}^d (G\widehat{\sigma}_F^{ji} L(F^j) + \langle DF^j, D(\widehat{\sigma}_F^{ji} \times G)\rangle) \tag{26}$$

and $H_i(F; G) \in \bigcap_{p \in \mathbb{N}} L^p$.

(ii) *Suppose that* $F^1, \ldots, F^d \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{k+1,p}$ *and* $G \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{k,p}$ *for some* $k \in \mathbb{N}$. *Then for every multi-index* $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{1, \ldots, d\}^k$ *one has*

$$\mathbb{E}(\partial_\alpha f(F) G) = \mathbb{E}(f(F) H_\alpha(F, G)) \quad \text{with } H_\alpha(F, G) = H_{\alpha_k}(F, H_{(\alpha_1, \ldots, \alpha_{k-1})}(F, G)) \quad (27)$$

and $H_\alpha(F; G) \in \bigcap_{p \in \mathbb{N}} L^p$.

Notice that with the notation from the previous section we have

$$\partial_i^F G = -\mathbb{E}(\delta(G(\widehat{\sigma}_F DF)^i) \mid F).$$

So Proposition 22 says that if (25) holds and $F^1, \ldots, F^d \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{k+1,p}$ and $G \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{k,p}$ then $G \in \bigcap_{p \in \mathbb{N}} W_F^{k,p}$.

As an immediate consequence of Proposition 22 and Theorem 18 we obtain the following result.

Proposition 23. (i) *Let* $F = (F^1, \ldots, F^d)$ *with* $F^1, \ldots, F^d \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{2,p}$. *Assume that* (25) *holds. Then the law of* F *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^d$ *and the density* p_F *can be represented as*

$$p_F(x) = \mathbb{E}\left(\sum_{i=1}^d \partial_i Q_d(F - x) H_i(F; 1)\right). \quad (28)$$

Moreover p_F *is Hölder continuous of any exponent* $q < 1$. *And there exists some universal constants* C_d *and* $p_i, i = 1, \ldots, 5$, *depending on d such that*

$$p_F(x) \le C_d \mathbb{E}((\det \sigma_F)^{-p_1})^{p_2} \|F\|_{2,p_4}^{p_3} (P(|F - x| \le 2))^{1/p_5}. \quad (29)$$

In particular, $\lim_{x \to \infty} |x|^p p_F(x) = 0$ *for every* $p \in \mathbb{N}$.

(ii) *Suppose that* $F^1, \ldots, F^d \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{k+1,p}$. *Then* $p_F \in C^{k-1}(\mathbb{R}^d)$ *and for every multi-index* α *with* $|\alpha| \le k$ *one has*

$$\partial_\alpha p_F(x) = \mathbb{E}\left(\sum_{i=1}^d \partial_i Q_d(F - x) H_{(\alpha,i)}(F; 1)\right). \quad (30)$$

Moreover, for $|\alpha| \le k - 1$, $\partial_\alpha p_F$ *is Hölder continuous of any exponent* $q < 1$. *And there exists some universal constants* C_d *and* $p_i, i = 1, \ldots, 5$, *depending on d such that*

$$|\partial_\alpha p_F(x)| \le C_d \mathbb{E}((\det \sigma_F)^{-p_1})^{p_2} \|F\|_{k+1,p_4}^{p_3} (P(|F - x| \le 2))^{1/p_5}. \quad (31)$$

In particular, if $F \in \bigcap_{p \in \mathbb{N}} L^p$ *then* $\lim_{x \to \infty} |x|^p |\partial_\alpha p_F(x)| = 0$ *for every* $p \in \mathbb{N}$.

Remark 24. The gain with respect to the classical result concerns the regularity (in the Malliavin sense) required for F: recall that in the standard statement of this criterion one needs that $F^1, \ldots, F^d \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{d+1,p}$ in order to obtain the existence of a continuous density and $F^1, \ldots, F^d \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{d+k+1,p}$ in order that the density is k times differentiable. Moreover, notice that the estimate given in (29) depends on the Sobolev norms of order two – and the same estimates involve Sobolev norms of order $d + 1$ if one applies the standard criterion.

Remark 25. The absolute continuity criterion of Bouleau and Hirsh [4] asserts that if $F_j \in \mathbb{D}^{1,2}, j = 1, \ldots, d$ and $\sigma_F \ne 0$ a.s. then the law of F is absolutely continuous with respect to

the Lebesgue measure. The results presented in this section do not permit to prove this criterion because we need at least one integration by parts and then we need that $F_j \in \mathbb{D}^{2,2}$, $j = 1, \ldots, d$. But if this stronger regularity assumption holds, and moreover, if the stronger non degeneracy assumption (25) holds as well, we obtain a density which is Hölder continuous and not only measurable.

Remark 26. The representation formula (28) has been used by Kohatsu Higa and Yasuda in [8] and [9] in order to provide numerical approximation schemes for the density of the law of a diffusion process. Notice that $\mathbb{E}(|\partial_i Q_d(F - x)|^2) = \infty$. Then a direct use of the representation based on the Riesz transform leads to approximation schemes of infinite variance which consequently are not implementable by Monte Carlo methods. This is why they used a truncation argument and gave an estimate of the error due to truncation. For this estimate they used an old version of the present paper (namely [2]).

Finally we give a result concerning the strict positivity set $U_F = \{p_F > 0\}$. We define the matrix field

$$C_F^{ij}(x) = \mathbb{E}(H_i(F; 1)|F = x)\mathbb{E}(H_j(F; 1)|F = x)$$

and the distance

$$d_F(x, y) = \inf \left\{ \int_0^1 \langle C_F(\varphi_t)\dot{\varphi}_t, \dot{\varphi}_t \rangle^{1/2} \, dt; \ \varphi \in A_{x,y}^F \right\}$$

where $A_{x,y}^F = \{\varphi \in C^1([0, 1], U_F) : \varphi_0 = x, \varphi_1 = y\}$. Then,

Proposition 27. *Let $F = (F^1, \ldots, F^d)$ with $F^1, \ldots, F^d \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{3,p}$, and assume that (25) holds. Then, for any sequence $\{x_n\}_n \subset U_F$, $\lim_{n \to \infty} p_F(x_n) = 0$ if and only if $\lim_{n \to \infty} d_F(x_n, x_1) = \infty$.*

Proof. By recalling that Hirsch and Song [7] proved that U_F is a connected set, the statement immediately follows by applying parts (i) and (ii) of Proposition 12.   □

We give now the representation theorem for the conditional expectation.

Proposition 28. *Let $F = (F^1, \ldots, F^d)$ be such that $F^1, \ldots, F^d \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{2,p}$ and let $G \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{1,p}$. Assume that (25) holds. Then*

$$\phi_G(x) := \mathbb{E}(G \mid F = x) = 1_{\{p_F > 0\}} \frac{p_{F,G}(x)}{p_F(x)} \quad with$$

$$p_{F,G}(x) = \mathbb{E}\left(\sum_{i=1}^d \partial_i Q_d(F - x) H_i(F; G)\right). \tag{32}$$

Moreover $\phi_G \in \bigcap_{\delta > 0} \bigcap_{p \in \mathbb{N}} W^{1,p}(p_F > \delta)$ and it is locally Hölder continuous of any exponent $q < 1$ on $\{p_F > 0\}$. And if $F^1, \ldots, F^d \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{k+2,p}$ and $G \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{k+1,p}$ then $\phi_G \in C^k(\{p_F > 0\})$.

Remark 29. The representation formula (32) has been already obtained by Malliavin and Thalmaier in [13] and was the starting point in our work. But they need more regularity, namely $F^1, \ldots, F^d \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{d+2,p}$. This is because they need to know that a bounded density exists and they use the standard criterion for this.

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