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# Expansive homeomorphisms and plane separating continua

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**Abstract**

A homeomorphism $h: X \to X$ is *expansive* provided that there exists a constant $c > 0$ and for every $x, y \in X$ there exists an integer $n$, dependent only on $x$ and $y$, such that $\mathrm{d}(h^n(x), h^n(y)) > c$. It is shown that if $X$ is a 1-dimensional continuum that separates the plane into 2 pieces, then $h$ cannot be expansive.



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*Keywords:* Expansive homeomorphism; 1-cyclic continuum; Plane separating continuum

## 1. Introduction

A *continuum* is a compact, connected metric space. A continuum is a *plane continuum* if it can be embedded in the plane. A homeomorphism $h: X \to X$ is called *expansive* provided that there exists a constant $c > 0$ and for every $x, y \in X$ there exists an integer $n$ such that $\mathrm{d}(h^n(x), h^n(y)) > c$. Expansive homeomorphisms exhibit sensitive dependence on initial conditions in the strongest sense in that no matter how close any two points are, their images will eventually be distant in a certain distance.

One problem of interest is the classification of plane continua that admit (or do not admit) expansive homeomorphisms. The Plykin attractor [9] is a 1-dimensional plane continuum that admits an expansive homeomorphism. A 2-dimensional plane continuum that admits an expansive homeomorphisms has been constructed in [7]. It is known that tree-like continua and hence, 1-dimensional non-separating plane continua, do not admit expansive homeomorphisms [6]. A *n-separating* plane continuum is a continuum separating the plane into $n$ complementary domains. The Plykin attractor is a 4-separating plane continuum. The main result of this paper will show that 1-dimensional 2-separating plane continua do not admit expansive homeomorphisms. The corresponding result is still unknown for 3-separating plane continua. For more background information see [4] and [5].

In order for a homeomorphism to be expansive, stretching of subcontinua by the homeomorphism must occur. In compact spaces, this means that subcontinua must either be stretched and wrapped or stretched and folded. The dyadic solenoid [11] and the Plykin attractor are examples of continua that admit expansive homeomorphisms that wrap subcontinua. On the other hand, when subcontinua are stretched and folded, some points do move closer together. Under this type of action, it appears that the homeomorphism will not be expansive. This is evident in the result that

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tree-like continua do not admit expansive homeomorphisms. Similarly, homeomorphisms that stretch subcontinua of 2-separating plane continua must have some folding. This is the idea in the proof of the main result.

## 2. Characterization of 1-dimensional 2-separating plane continua

We begin with several important definitions. Let $\mathrm{d}(x, y)$ be the Euclidean metric in the plane. If $A$ and $B$ are subsets of $\mathbb{R}^2$, define the *distance between $A$ and $B$* as $\mathrm{d}(A, B) = \inf\{d(x, y) \mid x \in A \text{ and } y \in B\}$ and the *Hausdorff distance between* $A$ and $B$ by $\mathrm{d}_H(A, B) = \max\{\sup_{x\in A} \mathrm{d}(x, B),\ \sup_{y\in B} \mathrm{d}(A, y)\}$.

Let $\mathcal{U}$ be an open cover. The *mesh* of $\mathcal{U}$ is defined by

$$\mathrm{mesh}(\mathcal{U}) = \sup\{\mathrm{diam}(U) \mid U \in \mathcal{U}\}.$$

For $U \in \mathcal{U}$, the *core* of $U$ is defined as

$$\mathrm{core}(U) = \bigcap\{U - \overline{V} \mid V \in \mathcal{U} - \{U\}\}.$$

A cover is *taut* if $\overline{U} \cap \overline{V} = \emptyset$ for all disjoint $U, V \in \mathcal{U}$. Throughout the paper, we will assume that all covers are taut. It follows that if $\mathcal{U}$ is a taut open cover of a compact space, then $\mathrm{core}(U) \neq \emptyset$ for each $U \in \mathcal{U}$ since we are only considering compact spaces. A cover $\mathcal{V}$ *refines* $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$; $\mathcal{V}$ *closure refines* $\mathcal{U}$ if $\overline{V} \subset U$; and $\mathcal{V}$ 2-*refines* $\mathcal{U}$ if for every $V_i, V_j \in \mathcal{V}$ such that $V_i \cap V_j \neq \emptyset$, there exists a $U \in \mathcal{U}$ such that $V_i \cup V_j \subset U$. $\mathcal{U}'$ is an *amalgamation* of $\mathcal{U}$ if each element of $\mathcal{U}'$ is the union of elements of $\mathcal{U}$. Define the *star* of $\mathcal{U}$ as

$$\mathcal{U}^* = \bigcup_{U\in\mathcal{U}} U.$$

A *chain* $[C_1, C_2, \ldots, C_n]$ is a collection of open sets such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leqslant 1$. A *circle-chain* $[C_1, C_2, \ldots, C_n]_\circ$ is a collection of open sets such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leqslant 1$ or $|i - j| = n - 1$. A collection of open sets $\mathcal{U}$ is *connected* if for every $U, U' \in \mathcal{U}$, there exists a chain from $U$ to $U'$ in $\mathcal{U}$.

A open cover $\mathcal{U}$ is 1-*dimensional* if every element of $X$ is in at most 2 elements of $\mathcal{U}$. $X$ is 1-*dimensional* if for every $\varepsilon > 0$ there exists a finite 1-dimensional open cover $\mathcal{U}$ of $X$ with $\mathrm{mesh}(\mathcal{U}) < \varepsilon$. The *nerve* of cover $\mathcal{U}$, denoted $N(\mathcal{U})$, is a geometric simplicial complex (a graph if $\mathcal{U}$ is 1-dimensional) where each element $U_i \in \mathcal{U}$ is represented by a vertex $u_i \in N(\mathcal{U})$ and there exists an arc (edge) in $N(\mathcal{U})$ from $u_i$ to $u_j$ if and only if $U_i \cap U_j \neq \emptyset$. Suppose that $\mathcal{U}$ and $\mathcal{V}$ are taut 1-dimensional covers such that $\mathcal{V}$ refines $\mathcal{U}$. Then there is an *induced vertex map* $f : N(\mathcal{V}) \to N(\mathcal{U})$ defined in the following way: Let $v_i$ be the vertex of $N(\mathcal{V})$ that corresponds to element $V_i$ of $\mathcal{V}$ and let $u_j$ be the vertex of $N(\mathcal{U})$ that corresponds to element $U_i$ of $\mathcal{U}$. Then construct $f$ in the following way:

(1) If $V_i \cap \mathrm{core}(U_j) \neq \emptyset$, then let $f(v_i) = u_j$.
(2) If $V_i \subset U_j \cap U_{j'}$, $V_i \cap V_{i'} \neq \emptyset$ and $V_{i'} \cap \mathrm{core}(U_j) \neq \emptyset$ (or $V_{i'} \cap \mathrm{core}(U_{j'}) \neq \emptyset$), then let $f(v_i) = (\frac{3}{4})u_j + (\frac{1}{4})u_{j'}$ (or $f(v_i) = (\frac{3}{4})u_{j'} + (\frac{1}{4})u_j$).
(3) If $V_i \subset U_j \cap U_{j'}$ but $V_{i'} \cap \mathrm{core}(U_j) = \emptyset$ and $V_{i'} \cap \mathrm{core}(U_{j'}) = \emptyset$ whenever $V_i \cap V_{i'} \neq \emptyset$, let $f(v_i) = (\frac{1}{2})u_{j'} + (\frac{1}{2})u_j$.

Notice that $f$ maps adjacent vertices of $N(\mathcal{V})$ to either the same or to adjacent endpoints of quarter subdivisions of an edge of $N(\mathcal{U})$, i.e., the same or adjacent vertices of the second barycentric subdivision of $N(\mathcal{V})$. Extend $f$ linearly onto the edges of $N(\mathcal{V})$ to produce a simplicial map $N(\mathcal{V})$ onto the second barycentric subdivision of $N(\mathcal{V})$.

A finite open cover $\mathcal{U}$ is a *tree-cover* if the nerve $N(\mathcal{U})$ is a tree. A finite open cover $\mathcal{U}$ is 1-*cyclic* if the nerve $N(\mathcal{U})$ is a graph that contains exactly 1 simple closed curve. A continuum $X$ is *tree-like* (1-*cyclic*) if given any $\varepsilon > 0$, there is a tree (respectively, 1-cyclic) cover $\mathcal{U}$ of $X$ such that $\mathrm{mesh}(\mathcal{U}) < \varepsilon$. Equivalently, $X$ is 1-cyclic if it is the inverse limit of 1-cyclic graphs.

In this section, it will be shown that 1-dimensional 2-separating plane continua are 1-cyclic continua with "degree 1" nested covers. The proof of this is a generalization of the one due to Bing [1] of the theorem that all circle-like continua that can be embedded in the plane are the nested intersections of refining circle covers with degree 1.

Let $S$ be a simple closed curve in the plane. Then the *interior* of $S$ is its bounded complementary domain and the *exterior* is its unbounded complementary domain. If $Y$ is a subset of the plane, let $Y^c$ denote the complement of $Y$. A closed (not necessarily compact) connected subset $W$ of the plane is an *unbounded plane continuum* provided that $W$ is closed, connected and $W^c$ is bounded.

**Theorem 1.** *Each* 1*-dimensional* 2*-separating continuum* $X$ *has the following properties*:

(1) $X$ *is* 1*-cyclic.*
(2) *For every* $\varepsilon > 0$ *there exists a* 1*-cyclic cover* $\mathcal{U}_\varepsilon$ *of* $X$ *with mesh less than* $\varepsilon$ *such that each member of* $\mathcal{U}_\varepsilon$ *is the interior of a disk.*
(3) $X$ *is the nested intersection of annuli.*

**Proof.** Given $\varepsilon > 0$, let $\mathcal{U}_\varepsilon$ be a finite collection of interiors of simple closed curves that covers $X$ with mesh less than $\varepsilon$ such that no point is in more than 2 elements of $\mathcal{U}_\varepsilon$. Also let

$$M = X \cup \{\text{the bounded complementary domain of } X\}$$

and

$$P = X \cup \{\text{the unbounded complementary domain of } X\}.$$

Since $M$ does not separate the plane, there exists an unbounded plane continuum $W_\varepsilon$ which does not intersect $M$ but with a simple-closed-curve-boundary $S_{W_\varepsilon}$ and is covered by $\mathcal{U}_\varepsilon$. Let $\mathcal{U}$ be the collection of all interiors $\{U_\alpha\}_{\alpha\in\Omega}$ of closed curves such that each $U_\alpha$ is a component of the intersection of $W_\varepsilon^c$ and an element of $\mathcal{U}_\varepsilon$. Let $\mathcal{U}' \subset \mathcal{U}$ be a minimal finite subcover of $X$. Notice that no point is in more than 2 elements of $\mathcal{U}'$.

Likewise, there exists a continuum $Q_\varepsilon$ which does not intersect $P$ such that the boundary is a simple closed curve $S_{Q_\varepsilon}$ which is covered by $\mathcal{U}'$. Let $\mathcal{V}$ be the collection of all interiors $\{V_\beta\}_{\beta\in\Gamma}$ of closed curves such that each $V_\beta$ is a component of the intersection of $Q_\varepsilon^c$ and an element of $\mathcal{U}'$. Let $\mathcal{V}' \subset \mathcal{V}$ be a minimal finite subcover of $X$. Again, no point is in more than 2 elements of $\mathcal{V}'$. Additionally, each member of $\mathcal{V}'$ is the interior of a disk. Also, notice that $\mathcal{U}_\varepsilon^* \cup W_\varepsilon \cup Q_\varepsilon = \mathbb{R}^2$.

Now we show that $\mathcal{V}'$ is a 1-cyclic cover of $X$ by contradiction. Suppose $\mathcal{C}_0 = [C_1^0, \ldots, C_n^0]_\circ$ and $\mathcal{C}_1 = [C_1^1, \ldots, C_m^1]_\circ$ are distinct circle-chains of $\mathcal{V}'$. Let $S_0 = p_1p_2\ldots p_np_1$ and $S_1 = r_1r_2\ldots r_mr_1$ be simple closed curves such that $p_ip_{i+1}$, $p_np_1$, $r_jr_{j+1}$ and $r_mr_1$ are arcs contained $C_i^0$, $C_n^0$, $C_j^1$ and $C_m^1$, respectively. Without loss of generality, we can assume that $S_{Q_\varepsilon}$ is in the interior of $S_1$ and $S_0$ is not in the interior of $S_1$.

Since $\mathcal{C}_0$ and $\mathcal{C}_1$ are distinct, there is an element $C_i^0 \in \mathcal{C}_0$ that is not in $\mathcal{C}_1$. So there exists a point $p$ that is on the boundary of $C_i^0$, $C_{i-1}^0$ and $C_i^0 \cap C_{i-1}^0$ and that is in the interior of $S_0$ (being in the interior of $S_{W_\varepsilon}$) but in the exterior of $S_{Q_\varepsilon}$ (being separated by $S_1$). Thus, $p \notin W_\varepsilon \cup Q_\varepsilon$ and it follows that $p \in \mathcal{U}_\varepsilon^*$. Hence, there exist distinct $U_{i-1}, U_i \in \mathcal{U}_\varepsilon$ such that $C_i^0 \subset U_i$, $C_{i-1}^0 \subset U_{i-1}$. It now follows from the construction of $\mathcal{V}'$ that $p$ is on the boundary of $U_i$, $U_{i-1}$ and $U_i \cap U_{i-1}$. But then there exists a $U \in \mathcal{U}_\varepsilon$, distinct from $U_i$ and $U_{i-1}$, such that $p \in U$. However, that implies that $U \cap U_i \cap U_{i-1} \neq \emptyset$ which contradicts the fact that no point is in more than 2 elements of $\mathcal{U}_\varepsilon$. Therefore, $\mathcal{V}$ must have at most 1 cycle.

To show that $X$ is the nested intersection of annuli, first let $A_\varepsilon = \overline{W_\varepsilon^c} - Q_\varepsilon^\circ$. Then $X \subset A_\varepsilon$ and $\mathrm{d}_H(X, A_\varepsilon) < \varepsilon$. Let $\varepsilon_1 = 1$ and inductively let

$$\varepsilon_{n+1} < (1/2)\min\bigl\{1/(n+1), \mathrm{d}(A_{\varepsilon_n}, X)\bigr\}.$$

Then $A_{\varepsilon_{n+1}} \subset A_\varepsilon$. Hence $X = \bigcap_{n=1}^\infty A_{\varepsilon_n}$. □

Next, we will derive a necessary condition for a 1-cyclic continuum to be embeddable in the plane. Let $\mathcal{C} = [C_0, \ldots, C_{n-1}]_\circ$ be the circle chain of 1-cyclic cover $\mathcal{U}$. The *branch* of $C_i \in \mathcal{C}$ is the subset:

$$\mathcal{B}(C_i) = \bigl\{U \in \mathcal{U} \bigm| \text{if } \mathcal{D} \text{ is a chain from } U \text{ to } C_i \text{ then } \mathcal{D} \cap \mathcal{C} = \{C_i\}\bigr\}.$$

Let $\mathcal{U}$ and $\mathcal{V}$ be 1-cyclic covers such that

(1) $\mathcal{V}$ refines $\mathcal{U}$,
(2) $\mathcal{C} = [C_0, \ldots, C_{n-1}]_\circ$ is the circle chain cover of $\mathcal{U}$,
(3) $\mathcal{C}' = [C_0', \ldots, C_{m-1}']_\circ$ is the circle chain cover of $\mathcal{V}$,
(4) both $C_0'$ and $C_{m-1}'$ intersect the core of $C_0$.

For $V \in \mathcal{V}$, define $\Gamma_{\mathcal{U}}^{\mathcal{V}}(V) = j$ if there exists $U \in \mathcal{B}(C_j)$ such that $V \subset U \in \mathcal{B}(C_j)$. Note that there could be two choices for $\Gamma_{\mathcal{U}}^{\mathcal{V}}(V)$, but it does not matter which one is picked. Next define

$$\Delta_{\mathcal{U}}^{\mathcal{V}} : \{0, 1, \dots, m-1\} \to \mathbb{Z},$$

such that $\Delta_{\mathcal{U}}^{\mathcal{V}}(0) = 0$ and then continue inductively by

$$\Delta_{\mathcal{U}}^{\mathcal{V}}(i) = \begin{cases} \Delta_{\mathcal{U}}^{\mathcal{V}}(i-1) & \text{if } \Gamma_{\mathcal{U}}^{\mathcal{V}}(C'_i) = \Gamma_{\mathcal{U}}^{\mathcal{V}}(C'_{i-1}), \\ \Delta_{\mathcal{U}}^{\mathcal{V}}(i-1)+1 & \text{if } \Gamma_{\mathcal{U}}^{\mathcal{V}}(C'_i) = \Gamma_{\mathcal{U}}^{\mathcal{V}}(C'_{i-1})+1, \text{ or } \Gamma_{\mathcal{U}}^{\mathcal{V}}(C'_i) = 0 \text{ and } \Gamma_{\mathcal{U}}^{\mathcal{V}}(C'_{i-1}) = n-1, \\ \Delta_{\mathcal{U}}^{\mathcal{V}}(i-1)-1 & \text{if } \Gamma_{\mathcal{U}}^{\mathcal{V}}(C'_i) = \Gamma_{\mathcal{U}}^{\mathcal{V}}(C'_{i-1})-1, \text{ or } \Gamma_{\mathcal{U}}^{\mathcal{V}}(C'_i) = n-1 \text{ and } \Gamma_{\mathcal{U}}^{\mathcal{V}}(C'_{i-1}) = 0. \end{cases}$$

Define the *degree* of $\mathcal{V}$ in $\mathcal{U}$ by

$$\deg_{\mathcal{U}}(\mathcal{V}) = \frac{|\Delta_{\mathcal{U}}^{\mathcal{V}}(m-1) - \Delta_{\mathcal{U}}^{\mathcal{V}}(0)|}{n}.$$

Notice that the degree of $\mathcal{V}$ in $\mathcal{U}$ is an integer that measures the number of times for which $\mathcal{C}'$ "essentially circles" $\mathcal{C}$. Also, since both $C'_0$ and $C'_{m-1}$ intersect the core of $C_0$, this value is independent of our choice for $\Gamma_{\mathcal{U}}^{\mathcal{V}}(V)$.

The following definitions and results can be found in [8]: Given a topological space $Y$ let $H_p(Y)$, $\tilde{H}_p(Y)$, $H^p(Y)$ be the $p$th-dimensional homology group, reduced homology group and cohomology group for $Y$ (with integral coefficients), respectively. If $f : X \to Y$ is a continuous function on topological spaces $X$ and $Y$, then let $f_* : H_p(X) \to H_p(Y)$ and $f^* : H^p(X) \to H^p(Y)$ to be the induced homology and cohomology homomorphisms, respectively. Suppose that $Y = \bigcap_{i=1}^{\infty} D_i = \bigcap_{i=1}^{\infty} \mathcal{U}_i^*$ where $D_i$ is a nested sequence of polyhedra and $\mathcal{U}_i$ is a sequence open covers of $Y$ such that $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$. Then the $p$th-*dimensional Čech cohomology of* $Y$ is defined by

$$\check{H}^p(Y) = \varinjlim_{i\to\infty} \left(H^p(D_i), g_i^*\right) = \varinjlim_{i\to\infty} \left(H^p\left(N(\mathcal{U}_i)\right), f_i^*\right)$$

where $g_i : D_i \to D_{i+1}$ is the inclusion map and $f_i : N(\mathcal{U}_i) \to N(\mathcal{U}_{i+1})$ is the induced simplicial map of the nerves.

Let $X$ and $Y$ be homotopic to $S^1$. Given generators $\alpha$ of $H_1(X)$ and $\beta$ of $H_1(Y)$, the *degree of a map* $f : X \to Y$ is defined as the integer given by $f_*(\alpha) = \deg(f)\beta$. Since $H^1(S^1)$ is infinite cyclic, it also follows that $f^*(\alpha) = \alpha \deg(f)$. Notice that if $\mathcal{U}$ and $\mathcal{V}$ are 1-cyclic covers of $X$ such that $\mathcal{V}$ refines $\mathcal{U}$, then $\deg_{\mathcal{U}}(\mathcal{V}) = \deg(f)$ where $f$ is the simplicial vertex map from $N(\mathcal{V})$ to $N(\mathcal{U})$ which is induced from the refinement.

The next theorem is the main result of this section:

**Theorem 2.** *If $X$ is a one-dimensional* 2*-separating plane continuum, then $X = \bigcap_{i=1}^{\infty} \mathcal{U}_i^*$ where $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a sequence of* 1*-cyclic covers of $X$ with the following properties*:

(1) $\mathcal{U}_{i+1}$ *refines* $\mathcal{U}_i$,
(2) $\mathrm{mesh}(\mathcal{U}_i) \to 0$ *as* $i \to \infty$,
(3) $\deg_{\mathcal{U}_i}(\mathcal{U}_{i+1}) = 1$.

**Proof.** Since $S^2 - X$ has 2 path connected components, $\tilde{H}_0(S^2 - X) \cong \mathbb{Z}$. So by the Alexander–Pontryagin Duality Theorem, we have $\check{H}^1(X) \cong \tilde{H}_0(S^2 - X) \cong \mathbb{Z}$. Also, since each $N(\mathcal{U}_i)$ is homotopic to $S^1$, it follows that $H^1(N(\mathcal{U}_i)) \cong \mathbb{Z}$. Thus, $f_i^* : H^1(N(\mathcal{U}_i)) \to H^1(N(\mathcal{U}_{i+1}))$ must be the identity for all but a finitely many $i$'s. Therefore, the simplicial vertex map $f_i : N(\mathcal{U}_{i+1}) \to N(\mathcal{U}_i)$ must have degree 1. Thus, $\deg_{\mathcal{U}_i}(\mathcal{U}_{i+1}) = 1$. □

## 3. Wrapping tree-like subcontinua

In this section, we construct a method to measure how a tree-like subcontinuum "wraps" in a 1-cyclic cover. Again, let $\mathcal{U}$ and $\mathcal{V}$ be 1-cyclic covers such that

(1) $\mathcal{V}$ refines $\mathcal{U}$,
(2) $\mathcal{C} = [C_0, \dots, C_{n-1}]_\circ$ is the circle chain cover of $\mathcal{U}$,

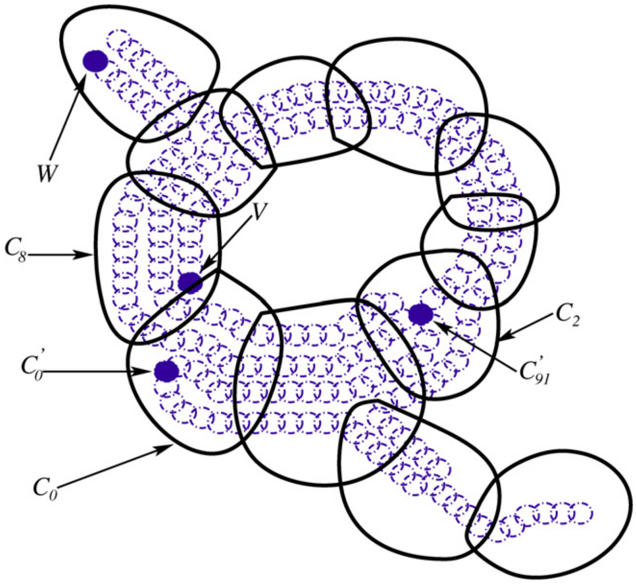


Fig. 1. $\Delta_{\mathcal{U}}^{\mathcal{V}}(91) = 11$, $\Theta_{\mathcal{U}}^{\mathcal{V}}(V) = 6$, $\Theta_{\mathcal{U}}^{\mathcal{V}}(W) = -5$, $\Omega_{\mathcal{U}}^{\mathcal{V}}(V) = 17$ and $\Omega_{\mathcal{U}}^{\mathcal{V}}(W) = 6$. The positive direction is taken to be counterclockwise rotation.

(3) $\mathcal{C}' = [C_0', \ldots, C_{m-1}']_\circ$ is the circle chain cover of $\mathcal{V}$,
(4) both $C_0'$ and $C_{m-1}'$ intersect the core of $C_0$.

Next, for each $C_j' \in \mathcal{C}'$ define $\Theta_{\mathcal{U}}^{\mathcal{V}}(C_j') = 0$. Let $V \in \mathcal{B}(C_j') - \{C_j'\}$ and let $p(V)$ be the unique element of the chain from $V$ to $C_j'$ in $\mathcal{B}(C_j')$ that intersects $V$ (i.e. $p(V)$ precedes $V$ in the chain). Then we can define inductively:

$$\Theta_{\mathcal{U}}^{\mathcal{V}}(V) = \begin{cases} \Theta_{\mathcal{U}}^{\mathcal{V}}(p(V)) & \text{if } \Gamma_{\mathcal{U}}^{\mathcal{V}}(V) = \Gamma_{\mathcal{U}}^{\mathcal{V}}(p(V)), \\ \Theta_{\mathcal{U}}^{\mathcal{V}}(p(V)) + 1 & \text{if } \Gamma_{\mathcal{U}}^{\mathcal{V}}(V) = \Gamma_{\mathcal{U}}^{\mathcal{V}}(p(V)) + 1, \text{ or } \Gamma_{\mathcal{U}}^{\mathcal{V}}(V) = 0 \text{ and } \Gamma_{\mathcal{U}}^{\mathcal{V}}(p(V)) = n-1, \\ \Theta_{\mathcal{U}}^{\mathcal{V}}(p(V)) - 1 & \text{if } \Gamma_{\mathcal{U}}^{\mathcal{V}}(V) = \Gamma_{\mathcal{U}}^{\mathcal{V}}(p(V)) - 1, \text{ or } \Gamma_{\mathcal{U}}^{\mathcal{V}}(V) = n-1 \text{ and } \Gamma_{\mathcal{U}}^{\mathcal{V}}(p(V)) = 0. \end{cases}$$

Finally, for each $V \in \mathcal{V}$, define $\Omega_{\mathcal{U}}^{\mathcal{V}} : \mathcal{V} \to \mathbb{Z}$ by

$$\Omega_{\mathcal{U}}^{\mathcal{V}}(V) = \Delta_{\mathcal{U}}^{\mathcal{V}}(j) + \Theta_{\mathcal{U}}^{\mathcal{V}}(V) \quad \text{where } j \text{ is the integer such that } V \in \mathcal{B}(C_j').$$

The number $\Delta_{\mathcal{U}}^{\mathcal{V}}(j)$ measures the "wrapping position" of $C_j'$ relative to $C_0'$ from the chain $[C_0', \ldots, C_j']$, $\Theta_{\mathcal{U}}^{\mathcal{V}}(V)$ measures the wrapping position of $V$ relative to $C_j'$ from the chain $[C_j', \ldots, p(V), V]$, and $\Omega_{\mathcal{U}}^{\mathcal{V}}(V)$ measures the wrapping position of $V$ relative to $C_0'$ from the chain $[C_0', \ldots, C_j', \ldots, p(V), V]$. (See Fig. 1.)

Suppose $\mathcal{U}$ is a 1-cyclic cover of $X$, $\mathcal{C} \subset \mathcal{U}$ is the circle-chain of $\mathcal{U}$ and $H$ is a tree-like subcontinuum of $X$ such that $H \cap C \neq \emptyset$ for some $C \in \mathcal{C}$. Define $\mathcal{T}(H, \mathcal{U})$ to be some tree-cover of $H$ that refines $\mathcal{U}$ and has minimum cardinality. Although $\mathcal{T}(H, \mathcal{U})$ itself is not uniquely determined, the cardinality of $\mathcal{T}(H, \mathcal{U})$ is uniquely determined and this uniqueness will be used in the main result.

**Proposition 3.** *Suppose that* $1 < |\mathcal{T}(H, \mathcal{U})|$. *If* $U \in \mathcal{U}$ *and* $V \in \mathcal{T}(H, \mathcal{U})$ *such that* $V \subset U$, *then* $V \cap \operatorname{core}(U) \neq \emptyset$.

**Proof.** Suppose that $V \cap \operatorname{core}(U) = \emptyset$, then there exists a $U' \in \mathcal{U}$ distinct from $U$ such that $V \subset U \cap U'$. Let $V' \in \mathcal{T}(H, \mathcal{U})$ such that $V \cap V' \neq \emptyset$. Then $V' \subset U$ or $V' \subset U'$. Hence $V \cup V' \subset U$ or $V \cup V' \subset U'$. Thus $(\mathcal{T}(H, \mathcal{U}) - \{V, V'\}) \cup \{V \cup V'\}$ is a tree cover of $H$ that refines $\mathcal{U}$ with cardinality less than $\mathcal{T}(H, \mathcal{U})$ which contradicts the minimality of $\mathcal{T}(H, \mathcal{U})$. □

Define $\mathcal{Q}(H, \mathcal{U}) = \{Q \in \mathcal{T}(H, \mathcal{U}) \mid Q \subseteq C \text{ for some } C \in \mathcal{C}\}$ as the *trunk* of $\mathcal{T}(H, \mathcal{U})$.

**Proposition 4.** *$\mathcal{Q}(H,\mathcal{U})$ is a connected collection of open sets.*

**Proof.** Suppose that $\mathcal{Q}(H,\mathcal{U})$ is not connected, then there exist nonempty disjoint sets $\mathcal{Q}_a$, $\mathcal{Q}_b$ such that $\mathcal{Q}(H,\mathcal{U}) = \mathcal{Q}_a \cup \mathcal{Q}_b$ and for every $Q_A \in \mathcal{Q}_a$ and $Q_B \in \mathcal{Q}_b$ there is no chain in $\mathcal{Q}(H,\mathcal{U})$ from $Q_A$ to $Q_B$. Let $Q^a \in \mathcal{Q}_a$ and $Q^b \in \mathcal{Q}_b$. Since $\mathcal{T}(H,\mathcal{U})$ is a tree-cover there exists a unique chain $\mathcal{C}_Q = [T_0, T_1, \ldots, T_n] \subset \mathcal{T}(H,\mathcal{U})$ such that $T_0 = Q^a$ and $T_n = Q^b$.

Let $j_a$ be the largest integer in $\{0, \ldots, n-1\}$ such that $T_{j_a} \in \mathcal{Q}_a$ and let $j_b$ be the smallest integer of $\{1, \ldots, n\}$ greater than $j_a$ such that $T_{j_b} \in \mathcal{Q}_b$. If $j_a + 1 = j_b$, then $[T_{j_a}, T_{j_b}]$ is a chain from $\mathcal{Q}_a$ to $\mathcal{Q}_b$ contained in $\mathcal{Q}(H,\mathcal{U})$ which is a contradiction. So suppose that $j_b > j_a + 1$. Then there exists $U_{j_a+1} \in \mathcal{U} - \mathcal{C}$ such that $T_{j_a+1} \subset U_{j_a+1}$. Let $C_{j_a}, C_{j_b} \in \mathcal{C}$ such that $T_{j_a} \subset C_{j_a}$, $T_{j_b} \subset C_{j_b}$, and define $\mathcal{U}_{ab} = \{U \in \mathcal{U} \mid T_i \subset U \text{ for } i \in \{j_a+1, \ldots, j_b-1\}\}$. Since $[T_{j_a+1}, \ldots, T_{j_b-1}]$ is connected, $\mathcal{U}_{ab}$ is connected. Also note that each $T_i \notin \mathcal{Q}_a \cup \mathcal{Q}_b$ and hence, $T_i \subset U_i \notin \mathcal{C}$ for every $i \in \{j_a+1, \ldots, j_b-1\}$. Thus, $\mathcal{U}_{ab} \cap \mathcal{C} = \emptyset$. It follows that $\mathcal{U}_{ab}$ contains no circle-chain and hence must be a tree-cover. Notice that since $T_{j_a+1} \subset U_{j_a+1}$ it follows that $U_{j_a+1} \in \mathcal{U}_{ab}$ and since $T_{j_a+1} \cap T_{j_a} \neq \emptyset$, we may conclude that $U_{j_a+1} \cap C_{j_a} \neq \emptyset$. Thus, $\mathcal{U}_{ab} \subset \mathcal{B}(C_{j_a})$. Likewise, $U_{j_b-1} \cap C_{j_b} \neq \emptyset$ and thus, $\mathcal{U}_{ab} \subset \mathcal{B}(C_{j_b})$. So $\mathcal{B}(C_{j_a}) \cap \mathcal{B}(C_{j_b}) \neq \emptyset$, and it follows that $C_{j_a} = C_{j_b}$. Therefore, $j_a = j_b$ which is a contradiction. □

**Proposition 5.** *Suppose that A, B, and C are distinct elements of $\mathcal{T}(H,\mathcal{U})$ such that $A_1 \cap A_2 \neq \emptyset$ and $A_2 \cap A_3 \neq \emptyset$. If $U_1, U_2, U_3 \in \mathcal{U}$ such that $A_i \subseteq U_i$ for $i \in \{1, 2, 3\}$, then $U_1$, $U_2$ and $U_3$ are all distinct.*

**Proof.** If $U_i = U_j$ for $i \neq j$, then $(\mathcal{T}(H,\mathcal{U}) - \{A_i, A_j\}) \cup \{A_i \cup A_j\}$ would be a tree-cover of $H$ that refines $\mathcal{U}$ but has cardinality less than $\mathcal{T}(H,\mathcal{U})$ which is a contradiction that $\mathcal{T}(H,\mathcal{U})$ is minimal. □

**Proposition 6.** *$\mathcal{Q}(H,\mathcal{U})$ is a chain.*

**Proof.** Since $\mathcal{Q}(H,\mathcal{U})$ is a connected subset of a tree-cover. It must also be a tree-cover. So it suffices to show that $\mathcal{Q}(H,\mathcal{U})$ has no branching. Suppose there exist distinct $C', C'_a, C'_b, C'_c \in \mathcal{Q}(H,\mathcal{U})$ such that $C' \cap C'_a \neq \emptyset$, $C' \cap C'_b \neq \emptyset$ and $C' \cap C'_c \neq \emptyset$. Let $C, C_a, C_b, C_c \in \mathcal{C}$ such that $C' \subset C$, $C'_a \subset C_a$, $C'_b \subset C_b$ and $C'_c \subset C_c$. Then by Proposition 5, $C$, $C_a$, $C_b$ and $C_c$ are all distinct. Furthermore $C \cap C_a$, $C \cap C_b$ and $C \cap C_c$ are all nonempty. Thus, $\mathcal{C}$ is not a circle-chain which is a contradiction. □

**Lemma 7.** *Let $\mathcal{C} = [C_0, \ldots, C_{n-1}]_\circ$ be the circle-chain of $\mathcal{U}$. Then $\mathcal{Q}(H,\mathcal{U})$ is a chain indexed as follows:*

$$\big[C_\alpha(i), C_{\alpha+1}(i), \ldots, C_{n-1}(i), C_0(i+1), \ldots, C_{n-1}(i+1), C_0(i+1), \ldots, C_\beta(j)\big]$$

*where $\alpha, \beta \in \{0, 1, \ldots, n-1\}$ and $C_k(m) \subseteq C_k$ for each $m$. Also $i$ is an arbitrarily chosen integer and $j$ denotes an index which will be given in the proof.*

**Proof.** Pick any integer $i$. Since $\mathcal{Q}(H,\mathcal{U})$ is a chain, it must have 2 endlinks say $U$ and $V$. Let $U'$ and $V'$ be the unique links of $\mathcal{Q}(H,\mathcal{U})$ that intersect $U$ and $V$, respectively. There exist $\alpha, \alpha', \beta, \beta' \in \{0, 1, \ldots, n-1\}$ with the following properties:

(1) either $\alpha' = \alpha + 1$ or $\alpha = n-1$ and $\alpha' = 0$,
(2) either $\beta' = \beta - 1$ or $\beta' = n-1$ and $\beta = 0$,
(3) either $U \subset C_\alpha$, $U' \subset C_{\alpha'}$, $V \subset C_\beta$, and $V' \subset C_{\beta'}$ or $V \subset C_\alpha$, $V' \subset C_{\alpha'}$, $U \subset C_\beta$, and $U' \subset C_{\beta'}$.

Without loss of generality, assume $U \subset C_\alpha$. Then let $C_\alpha(i) = U$ and let $C_{\alpha'}(i) = U'$ if $\alpha' \neq 0$ and $C_{\alpha'}(i+1) = U'$ if $\alpha' = 0$.

Continuing inductively, suppose that $[C_\alpha(i), \ldots, C_\gamma(m)]$ have been found. We have 3 cases to consider:

**Case 1.** $0 < \gamma < n-1$.

Let $Q'$ be the element of $\mathcal{Q}(H,\mathcal{U})$ different from $C_{\gamma-1}(m)$ that intersects $C_\gamma(m)$. It follows from Proposition 5 that $Q' \subset C_{\gamma+1}$. Let $C_{\gamma+1}(m) = Q'$.

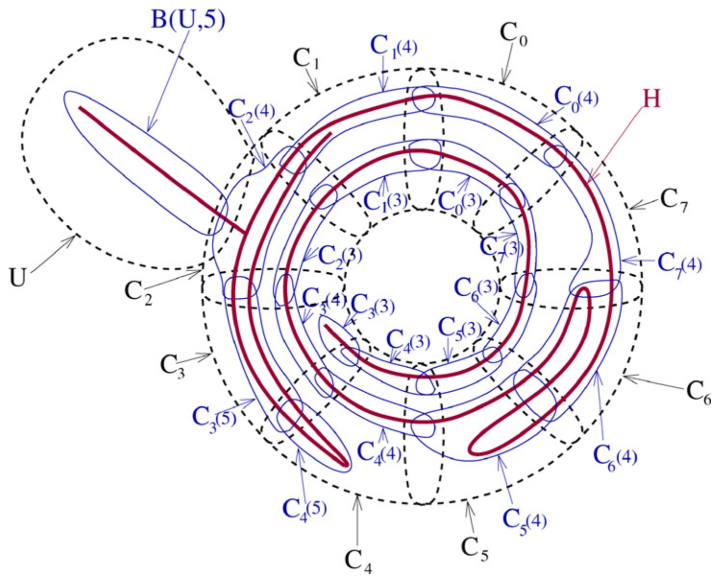


Fig. 2. Here $i$ is arbitrarily chosen to be 3. $\mathcal{T}(H,\mathcal{U})$ is created from $\mathcal{U}$. Notice that $W(H,\mathcal{U}) = 2 = 5 - 3$ since $\beta = 4 \geqslant 3 = \alpha$.

**Case 2.** $\gamma = 0$.
Let $Q'$ be the element of $\mathcal{Q}(H,\mathcal{U})$ different from $C_{n-1}(m-1)$ that intersects $C_\gamma(m)$. It follows again from Proposition 5 that $Q' \subset C_{\gamma+1}$. Let $C_{\gamma+1}(m) = Q'$.

**Case 3.** $\gamma = n-1$.
Let $Q'$ be the element of $\mathcal{Q}(H,\mathcal{U})$ different from $C_{\gamma-1}(m)$ that intersects $C_\gamma(m)$. It also follows from Proposition 5 that $Q' \subset C_0$. Let $C_0(m+1) = Q'$.

Since $\mathcal{Q}(H,\mathcal{U})$ is finite, this process will stop at the endlink $V = C_\beta(j)$. □

Define the *wrapping number* $W(H,\mathcal{U})$ of $H$ on $\mathcal{U}$ by:

$$W(H,\mathcal{U}) = \begin{cases} j-i & \text{if } \alpha \leqslant \beta, \\ j-i-1 & \text{if } \alpha > \beta, \end{cases}$$

where $i$, $j$, $\alpha$ and $\beta$ are as in Lemma 7. Let $B(U,s)$ be the unique element of $\mathcal{T}(H,\mathcal{U})$ such that

(1) $B(U,s) \subset U \in \mathcal{U}$,
(2) there exists a chain $\mathcal{D} \subseteq \mathcal{T}(H,\mathcal{U})$ from $C_k(s)$ to $B(U,s)$ such that $\mathcal{D} \cap \mathcal{Q}(H,\mathcal{U}) = \{C_k(s)\}$ where $U \in \mathcal{B}(C_k)$.

Notice that $B(U,s)$ is at the position where $\mathcal{T}(H,\mathcal{U})$ has "wrapped" $\mathcal{U}$ $s$ times. Also, notice that $B(C_k,s) = C_k(s)$. Here, $U$ is called the *symbol* and $s$ is called the *index* of $B(U,s)$. (See Fig. 2.)

**Proposition 8.** *Suppose $B(U,s_1)$, $B(V,s_2)$ are distinct elements of $\mathcal{T}(H,\mathcal{U})$. Then $B(U,s_1) \cap B(V,s_2) \neq \emptyset$ implies that $U \cap V \neq \emptyset$ and also the following*:

(1) *if $U \cup V \not\subset C_0 \cup C_{n-1}$, then $s_1 = s_2$,*
(2) *if $U = C_0$ and $V = C_{n-1}$, then $s_2 = s_1 - 1$,*
(3) *if $V = C_0$ and $U = C_{n-1}$, then $s_1 = s_2 - 1$.*

**Proof.** Clearly, $B(U,s_1) \cap B(V,s_2) \neq \emptyset$ implies that $U \cap V \neq \emptyset$. Suppose that (1), (2) and (3) are false. Let $p, q$ be the integers such that $U \in \mathcal{B}(C_p)$ and $V \in \mathcal{B}(C_q)$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the respective unique chains in $\mathcal{T}(H,\mathcal{U})$

from $B(U,s_1)$ to $C_p(s_1)$ and from $B(V,s_2)$ to $C_q(s_2)$. Finally, let $\mathcal{C}'$ be the unique chain in $\mathcal{Q}(H,\mathcal{U})$ from $C_p(s_1)$ to $C_q(s_2)$. Then $\mathcal{D}_1 \cup \mathcal{C}' \cup \mathcal{D}_2$ is a circle-subchain in tree-cover $\mathcal{T}(H,\mathcal{U})$ which is a contradiction. □

Next let $\mathcal{U}_0$ and $\mathcal{U}_1$ be 1-cyclic covers of $X$ with such that

(1) $\mathcal{U}_1$ is a degree 1 closure 2-refinement of $\mathcal{U}_0$,
(2) $\mathcal{C}_0 = [C_0^0, \dots, C_{n-1}^0]_\circ$ is the circle chain cover of $\mathcal{U}_0$,
(3) $\mathcal{C}_1 = [C_0^1, \dots, C_{m-1}^1]_\circ$ is the circle chain cover of $\mathcal{U}_1$,
(4) both $C_0^1$ and $C_{m-1}^1$ intersect the core of $C_0^0$.

Define the *wrapping number* $W(\mathcal{U}_1,\mathcal{U}_0)$ of $\mathcal{U}_1$ on $\mathcal{U}_0$ as

$$W(\mathcal{U}_1,\mathcal{U}_0) = \left\lfloor \frac{\max\{\Omega_{\mathcal{U}_0}^{\mathcal{U}_1}(V) \mid V \in \mathcal{U}_1\} - \min\{\Omega_{\mathcal{U}_0}^{\mathcal{U}_1}(V) \mid V \in \mathcal{U}_1\}}{n} \right\rfloor$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$. $W(\mathcal{U}_1,\mathcal{U}_0)$ counts the maximum number of times that subchains of $\mathcal{U}_1$ wrap $\mathcal{U}_0$.

Now suppose that $\mathcal{T}(H,\mathcal{U}_1)$ and $\mathcal{Q}(H,\mathcal{U}_1) =$

$$\left[C_{\alpha_1}^1(i_1), C_{\alpha_1+1}^1(i_1), \dots, C_{n-1}^1(i_1), C_0^1(i_1+1), \dots, C_{n-1}^1(i_1+1), C_0^1(i_1+2), \dots, C_{\beta_1}^1(j_1)\right]$$

have been found as in Lemma 7. Then, for each $U \in \mathcal{U}_0$, let

$$\widehat{B}(U, p+q) = \bigcup \left\{ B(V,p) \in \mathcal{T}(H,\mathcal{U}_1) \,\middle|\, V \subset U \text{ and } q = \left\lfloor \frac{\Omega_{\mathcal{U}_0}^{\mathcal{U}_1}(V)}{n} \right\rfloor \right\}.$$

For example, suppose that $U \in \mathcal{U}_0$ and $W, V \in \mathcal{U}_1$ such that $V, W \subset U$, $\lfloor \frac{\Omega_{\mathcal{U}_0}^{\mathcal{U}_1}(V)}{n} \rfloor = 3$, and $\lfloor \frac{\Omega_{\mathcal{U}_0}^{\mathcal{U}_1}(W)}{n} \rfloor = 5$. Then $B(V,8) \cup B(W,6) \subset \widehat{B}(U,11)$.

Finally, let

$$\mathcal{T}(H,\mathcal{U}_0) = \left\{ \widehat{B}(U,s) \,\middle|\, U \in \mathcal{U}_0 \text{ and } s = p+q \right\}.$$

We first show that under this construction, $\mathcal{T}(H,\mathcal{U}_0)$ is a tree-cover of $H$ that refines $\mathcal{U}_0$ and has minimal cardinality. Then we will compare how $H$ wraps in $\mathcal{U}_1$ to how $H$ wraps in $\mathcal{U}_0$.

**Proposition 9.** *Suppose that $V, V' \in \mathcal{U}_1$ where $V \cap V' \neq \emptyset$. Then*

$$\left|\Omega_{\mathcal{U}_0}^{\mathcal{U}_1}(V) - \Omega_{\mathcal{U}_0}^{\mathcal{U}_1}(V')\right| = 0, 1 \text{ or } n.$$

*Furthermore,* $\lfloor \frac{\Omega_{\mathcal{U}_0}^{\mathcal{U}_1}(C_0^1)}{n} \rfloor = 0$ *and* $\lfloor \frac{\Omega_{\mathcal{U}_0}^{\mathcal{U}_1}(C_{m-1}^1)}{n} \rfloor = 1$.

**Proof.** The proof follows from the fact that $\deg_{\mathcal{U}_0}(\mathcal{U}_1) = 1$. □

The next proposition is similar to Proposition 8 and shows that $\mathcal{T}(H,\mathcal{U}_0)$ and $\mathcal{T}(H,\mathcal{U}_1)$ have similar properties. However, $\mathcal{T}(H,\mathcal{U}_0)$ and $\mathcal{T}(H,\mathcal{U}_1)$ are constructed in different ways, so the proofs of the propositions are different.

**Proposition 10.** *Suppose $\widehat{B}(U_0,s_U)$, $\widehat{B}(V_0,s_V)$ are distinct elements of $\mathcal{T}(H,\mathcal{U}_0)$. Then $\widehat{B}(U_0,s_U) \cap \widehat{B}(V_0,s_V) \neq \emptyset$ implies that $U_0 \cap V_0 \neq \emptyset$ and also the following*:

(1) *If $U \cup V \not\subset C_0^0 \cup C_{n-1}^0$ then $s_U = s_V$.*
(2) *If $U_0 = C_0^0$ then $V_0 = C_{n-1}^0$ and $s_V = s_U - 1$.*
(3) *If $V_0 = C_0^0$ then $U_0 = C_{n-1}^0$ and $s_U = s_V - 1$.*

**Proof.** If $\widehat{B}(U_0, s_U) \cap \widehat{B}(V_0, s_V) \neq \emptyset$, then clearly $U_0 \cap V_0 \neq \emptyset$. It also follows that there exist distinct $B(U_1, p_U)$, $B(V_1, p_V) \in \mathcal{T}(H, \mathcal{U}_1)$ such that $B(U_1, p_U) \subset \widehat{B}(U_0, s_U)$, $B(V_1, p_V) \subset \widehat{B}(V_0, s_V)$ and $B(U_1, p_U) \cap B(V_1, p_V) \neq \emptyset$. By the definition of $\widehat{B}(U_0, s_U)$ and $\widehat{B}(V_0, s_V)$, it follows that $s_U = p_U + \lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(U_1)}{n} \rfloor$ and $s_V = p_V + \lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(V_1)}{n} \rfloor$. Clearly, $U_1 \cap V_1 \neq \emptyset$. We have 9 cases to consider:

**Case 1.** $U_0 \cup V_0 \not\subset C^0_0 \cup C^0_{n-1}$ and $U_1 \cup V_1 \not\subset C^1_0 \cup C^1_{m-1}$.

Then by (1) of Proposition 8, $p_U = p_V$. Also by Proposition 9, $\lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(U_1)}{n} \rfloor = \lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(V_1)}{n} \rfloor$. Hence,

$$s_U = p_U + \left\lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(U_1)}{n} \right\rfloor = p_V + \left\lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(V_0)}{n} \right\rfloor = s_V.$$

Hence, (1) is true.

**Case 2.** $U_0 \cup V_0 \not\subset C^0_0 \cup C^0_{n-1}$, $U_1 = C^1_0$ and $V_1 = C^1_{m-1}$.

Then by (2) of Proposition 8, $p_V = p_U - 1$. Also by Proposition 9, $\lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(U_1)}{n} \rfloor = 0$ and $\lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(V_1)}{n} \rfloor = 1$. Hence,

$$s_U = p_U + \left\lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(U_1)}{n} \right\rfloor = p_U = p_U - 1 + \left\lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(U_1)}{n} \right\rfloor + 1 = p_V + \left\lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(V_1)}{n} \right\rfloor = s_V.$$

Hence, (1) is true.

**Case 3.** $U \cup V \not\subset C^0_0 \cup C^0_{n-1}$, $U_1 = C^1_{m-1}$ and $V_1 = C^1_0$.
This is similar to Case 2.

**Case 4.** $U_0 = C^0_0$, $V_0 = C^0_{n-1}$, and $U_1 \cup V_1 \not\subset C^1_0 \cup C^1_{m-1}$.

Then by (1) of Proposition 8, $p_U = p_V$. Also $U_1 \subset C^0_0$ and $V_1 \subset C^0_{n-1}$. Since $U_1 \cap V_1 \neq \emptyset$ , $\lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(U_1)}{n} \rfloor = \lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(V_1)}{n} \rfloor + 1$ by Proposition 8. Hence,

$$s_U - 1 = p_U + \left\lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(U_1)}{n} \right\rfloor - 1 = p_V + \left\lfloor \frac{\Omega^{\mathcal{U}_1}_{\mathcal{U}_0}(V_1)}{n} \right\rfloor + 1 - 1 = s_V.$$

Hence, (2) is true.

**Case 5.** $U_0 = C^0_0$, $V_0 = C^0_{n-1}$, $U_1 = C^1_0$ and $V_1 = C^1_{m-1}$.
Since $C^1_{m-1}$ intersects the core of $C^0_0$, $V_1$ cannot be contained in $V_0$ which is a contradiction. So this case is impossible.

**Case 6.** $U_0 = C^0_0$, $V_0 = C^0_{n-1}$, $U_1 = C^1_{m-1}$ and $V_1 = C^1_0$.
This is similar to Case 5.

**Cases 7–9.** $V_0 = C^0_0$, and $U_0 = C^0_{n-1}$.
This is similar to Cases 4-6. □

**Lemma 11.** $\mathcal{T}(H, \mathcal{U}_0)$ *is tree-cover of* $H$ *that refines* $\mathcal{U}_0$ *and has minimum cardinality.*

**Proof.** It follows from Proposition 10 that if $\widehat{B}(U, s), \widehat{B}(V, t) \in \mathcal{T}(H, \mathcal{U}_0)$ then $\widehat{B}(U, s) \cap \widehat{B}(V, t) \neq \emptyset$ implies that either $s = t$ and $U, V$ are elements of $\mathcal{U}_0$ intersecting each other, or $s = t - 1$, $\{U, V\} = \{C^0_{n-1}, C^0_0\}$. Hence, a circle-chain in $\mathcal{T}(H, \mathcal{U}_0)$ would imply a circle-chain in $\mathcal{U}_0$ composed of the same symbols. But the only circle-chain in $\mathcal{U}_0$ is $[C^0_0, C^0_2, \ldots, C^0_{n-1}]_\circ$. This would imply that the circle-chain of $\mathcal{T}(H, \mathcal{U}_0)$ must be of the form

$[\widehat{B}(C_0^0,s),\widehat{B}(C_1^0,s),\ldots,\widehat{B}(C_{n-1}^0,s)]_\circ$ for some $s$. However that implies that $\widehat{B}(C_0^0,s)$ and $\widehat{B}(C_{n-1}^0,s)$ intersect which is impossible by Proposition 10. Hence, $\mathcal{T}(H,\mathcal{U}_0)$ is a tree-cover of $H$.

To show that $\mathcal{T}(H,\mathcal{U}_0)$ is minimal, take the union any two distinct elements of $\mathcal{T}(H,\mathcal{U}_0)$ together. It follows from Proposition 3 that if they do not have the same symbol, then their union cannot be contained in any element of $\mathcal{U}_0$ and thus, the new cover is no longer a refinement. On the other hand, if the elements have the same symbol, then they must differ in index. Thus, the new cover will contain a circle-chain and hence, will no longer be a tree-cover. □

The trunk $\mathcal{Q}(H,\mathcal{U}_0)$ of $\mathcal{T}(H,\mathcal{U}_0)$ can now be defined as:

$$\mathcal{Q}(H,\mathcal{U}_0)=\left\{\widehat{B}\left(C_k^0,s\right)\;\middle|\;C_k^0\in\mathcal{C}_0\text{ and }s=p+q\right\}.$$

Then if we let $C_k^0(s)=\widehat{B}(C_k^0,s)$, $\mathcal{Q}(H,\mathcal{U}_0)$ is in the form of Lemma 7.

**Proposition 12.** $W(H,\mathcal{U}_0)\leqslant W(H,\mathcal{U}_1)+W(\mathcal{U}_1,\mathcal{U}_0)+1$.

**Proof.** Notice that $\min\{\lfloor\frac{\Omega_{\mathcal{U}_0}^{\mathcal{U}_1}(V)}{n}\rfloor\mid V\in\mathcal{U}_1\}\leqslant 0$. It follows from this that

$$\begin{aligned}W(H,\mathcal{U}_0)&\leqslant\max\left\{s\;\middle|\;\widehat{B}(U,s)\in\mathcal{T}(H,\mathcal{U}_0)\text{ and }U\in\mathcal{U}_0\right\}-\min\left\{s\;\middle|\;\widehat{B}(U,s)\in\mathcal{T}(H,\mathcal{U}_0)\text{ and }U\in\mathcal{U}_0\right\}\\&\leqslant\max\left\{p\;\middle|\;B(V,p)\in\mathcal{T}(H,\mathcal{U}_1)\text{ and }V\in\mathcal{U}_1\right\}-\min\left\{p\;\middle|\;B(V,p)\in\mathcal{T}(H,\mathcal{U}_1)\text{ and }V\in\mathcal{U}_1\right\}\\&\quad+\max\left\{\left\lfloor\frac{\Omega_{\mathcal{U}_0}^{\mathcal{U}_1}(V)}{n}\right\rfloor\;\middle|\;V\in\mathcal{U}_1\right\}-\min\left\{\left\lfloor\frac{\Omega_{\mathcal{U}_0}^{\mathcal{U}_1}(V)}{n}\right\rfloor\;\middle|\;V\in\mathcal{U}_1\right\}\\&\leqslant W(H,\mathcal{U}_1)+W(\mathcal{U}_1,\mathcal{U}_0)+1.\qquad\square\end{aligned}$$

**Proposition 13.** *Suppose that $H$ is a tree-like subcontinuum of $X$ and $\mathcal{U}_0$, $\mathcal{U}_1$ are finite 1-cyclic open covers of $X$ such that $\mathcal{U}_1$ is a degree 1, 2-refinement of $\mathcal{U}_0$. Then $W(H,\mathcal{U}_1)\leqslant W(H,\mathcal{U}_0)$.*

**Proof.** There exists $C_k^1\in\mathcal{C}_1$ and integers $i_1$, $j_1$ such that

$$C_k^1(i_1),C_k^1(i_1+1),\ldots,C_k^1(j_1)\in\mathcal{T}(H,\mathcal{U}_1)$$

and $W(H,\mathcal{U}_1)=j_1-i_1$. Let $q=\lfloor\frac{\Omega_{\mathcal{U}_0}^{\mathcal{U}_1}(C_k^1)}{n}\rfloor$ and $U\in\mathcal{U}_0$ such that $C_k^1\subset U$. Then

$$\widehat{B}(U,i_1+q),\widehat{B}(U,i_1+q+1),\ldots,\widehat{B}(U,j_1+q)\in\mathcal{T}(H,\mathcal{U}_0).$$

Hence $W(H,\mathcal{U}_1)\leqslant W(H,\mathcal{U}_0)$. □

**Theorem 14.** *Let $h:X\to X$ be a homeomorphism. Suppose that $H$ is a tree-like subcontinuum of $X$ and $\mathcal{U}_0$ and $\mathcal{U}_1$ are finite 1-cyclic open covers of $X$ such that both $\mathcal{U}_1$ and $h(\mathcal{U}_1)$ are degree 1, 2-refinements of $\mathcal{U}_0$. Then*

$$W\left(h^n(H),\mathcal{U}_0\right)\leqslant W(H,\mathcal{U}_0)+nW\left(h(\mathcal{U}_1),\mathcal{U}_0\right)+n.$$

**Proof.** Proof is by induction on $n$. If $n=0$ then the theorem is clearly true. Suppose that

$$W\left(h^{n-1}(H),\mathcal{U}_0\right)\leqslant W(H,\mathcal{U}_0)+(n-1)W\left(h(\mathcal{U}_1),\mathcal{U}_0\right)+n-1.$$

By Proposition 13, $W(h^{n-1}(H),\mathcal{U}_1)\leqslant W(h^{n-1}(H),\mathcal{U}_0)$. Since $W(h^n(H),h(\mathcal{U}_1))=W(h^{n-1}(H),\mathcal{U}_1)$, it follows from the induction hypothesis and Proposition 12 that

$$\begin{aligned}W\left(h^n(H),\mathcal{U}_0\right)&\leqslant W\left(h^n(H),h(\mathcal{U}_1)\right)+W\left(h(\mathcal{U}_1),\mathcal{U}_0\right)+1=W\left(h^{n-1}(H),\mathcal{U}_1\right)+W\left(h(\mathcal{U}_1),\mathcal{U}_0\right)+1\\&\leqslant W(H,\mathcal{U}_0)+nW\left(h(\mathcal{U}_1),\mathcal{U}_0\right)+n.\qquad\square\end{aligned}$$

## 4. Main result

The topological entropy of a map $h$ is a measure of diffusion of points under iterations of $h$. All expansive homeomorphisms of nondegenerate continua have positive entropy. The following definition of entropy is due to Bowen [10]: For a map $h : X \to X$ and a non-negative integer $n$, define

$$d_n^+(x, y) = \max_{0 \leqslant i < n} \mathrm{d}\big(h^i(x), h^i(y)\big).$$

Similarly, if $h$ is a homeomorphism, define

$$d_n^-(x, y) = \max_{-n < i \leqslant 0} \mathrm{d}\big(h^i(x), h^i(y)\big),$$

where again $n \geqslant 0$.

Let $K$ be a compact subset of $X$ and $n$ be a positive integer. A finite subset $E_n$ of $K$ is said to be $(n, \varepsilon)$-*separated* with respect to map $h$ if $x$ and $y$ are distinct elements of $E_n$ implies that $d_n^+(x, y) > \varepsilon$. Let $s_n(\varepsilon, K, h)$ denote the largest cardinality of any $(n, \varepsilon)$-separated subsets of $K$ with respect to $h$. Then

$$s(\varepsilon, K, h) = \limsup_{n \to \infty} \frac{\log s_n(\varepsilon, K, h)}{n}.$$

The *entropy* of $h$ on $X$ is then defined as

$$\mathrm{Ent}(h, X) = \sup\Big\{ \lim_{\varepsilon \to 0} s(\varepsilon, K, h) \;\Big|\; K \text{ is a compact subset of } X \Big\}.$$

A subcontinuum $M$ of $X$ is *stable* under homeomorphism $h : X \to X$ if $\lim_{n\to\infty} \mathrm{diam}(h^n(M)) = 0$. Likewise, $M$ is *unstable* under $h$ if $\lim_{n\to-\infty} \mathrm{diam}(h^n(M)) = 0$.

The following theorems are due to Kato. The second is found in the proof of Theorem 4.1 of [3].

**Theorem 15.** *If $h : X \to X$ is an expansive homeomorphism, then there exists a stable subcontinuum or an unstable subcontinuum.*

**Theorem 16.** *If $h : X \to X$ is an expansive homeomorphism and $M$ is an unstable subcontinuum, then there exists an $\varepsilon > 0$ such that $s(\varepsilon, M, h) > 0$.*

*Likewise, if $M$ is a stable subcontinuum, then there exists an $\varepsilon > 0$ such that $s(\varepsilon, M, h^{-1}) > 0$.*

The proof of the main result now follows in a similar way to the proof that tree-like continua do not admit expansive homeomorphisms [6]. The next proposition is Cantor's original definition of connectedness [2].

**Proposition 17.** *Suppose $X$ is connected and $a, b, \in X$. For every $\varepsilon > 0$ there exists a finite sequence $\{x_i\}_{i=1}^n \subset X$ such that $x_1 = a$, $x_n = b$ and $d(x_i, x_{i+1}) < \varepsilon$.*

The previous sequence is called a *simple chain sequence* from $a$ to $b$ with *mesh* less than $\varepsilon$. For a homeomorphism $h$ and a positive integer $n$, define $\mathcal{L}(h, n, \varepsilon)$ to be a number greater than 0 such that

$$\mathrm{d}(x, y) < \mathcal{L}(h, n, \varepsilon) \quad \text{implies} \quad \mathrm{d}\big(h^i(x), h^i(y)\big) < \varepsilon \quad \text{for all } -n \leqslant i \leqslant n.$$

**Lemma 18.** *(See [6].) Suppose that $h : X \to X$ is a homeomorphism of a continuum $X$ and that $\{x_i\}_{i=1}^m$ is a simple chain sequence of $X$ from $a$ to $b$ with mesh less than $\mathcal{L}(h, n, \varepsilon/6)$. Also, suppose that $\{x_i\}_{i=1}^m$ is contained in some tree-cover $\mathcal{T}$ such that $a$ and $b$ are in the same element $T_1$ of $\mathcal{T}$ and that the mesh of $\{x_i\}_{i=1}^m$ is less than the Lebesgue number of $\mathcal{T}$. If $d_n^-(a, b) \geqslant \varepsilon$, then there exists $x_\alpha, x_\beta \in \{x_i\}_{i=1}^m$ such that $x_\alpha, x_\beta$ are in the same element of $\mathcal{T}$ and $\varepsilon/3 \leqslant d_n^-(x_\alpha, x_\beta) < \varepsilon$.*

**Lemma 19.** *(See [6].) Let $h : X \to X$ be a homeomorphism of a compact space onto itself. Suppose that there are sequences $\{y_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty$ such that $d(h^k(y_n), h^k(z_n)) < \varepsilon$ for all $k \leqslant n$. Then there exists a limit point $y$ of $\{y_i\}_{i=1}^\infty$ and a limit point $z$ of $\{z_i\}_{i=1}^\infty$ such that $d(h^k(y), h^k(z)) < 2\varepsilon$ for all $k$.*

**Theorem 20.** *Degree* 1, 1*-cyclic continua do not admit expansive homeomorphisms.*

**Proof.** Suppose that $h: X \to X$ is an expansive homeomorphism of a 1-cyclic plane continuum with expansive constant $c$. By Theorem 15, there exists a nondegenerate stable or unstable subcontinuum $M$. Without loss of generality, we may assume that $\mathrm{diam}(h^i(M)) < c/3$ for all $i \leqslant 0$. Since $\lim_{i\to-\infty} \mathrm{diam}(h^i(M)) = 0$, it follows that $M$ must be tree-like. By Theorem 16, there exists a $\gamma > 0$ such that $s(\gamma, M, h) > 0$. Let $\varepsilon = \min\{\gamma, c/4\}$ and $\{\delta_k\}_{k=1}^{\infty}$ be a sequence of positive numbers such that each $\delta_k < \mathcal{L}(h, k, \varepsilon/6)$. Let $\{\mathcal{U}_k\}_{k=0}^{\infty}$ be a sequence of 1-cycle covers of $X$ such that

(1) $\mathrm{mesh}(\mathcal{U}_k) < \delta_k$,
(2) both $\mathcal{U}_{k+1}$ and $h(\mathcal{U}_{k+1})$ are 1-degree, closure 2-refinements of $\mathcal{U}_k$.

By Theorem 14,

$$\left|\mathcal{T}\left(h^n(M), \mathcal{U}_k\right)\right| \leqslant |\mathcal{U}_k| W\left(h^n(M), \mathcal{U}_k\right) \leqslant |\mathcal{U}_k|\left(W(M, \mathcal{U}_k) + n\widehat{W}\left(h(\mathcal{U}_{k+1}), \mathcal{U}_k\right) + n\right)$$

which has polynomial growth as $n$ increases. Since $s(\varepsilon, M, h) > 0$, $s_n(\varepsilon, M, h)$ must have exponential growth as $n$ increases. Therefore, for some integer $N_k > k$,

$$s_{N_k}(\varepsilon, M, h) > \left|\mathcal{T}\left(h^{N_k}(M), \mathcal{U}_k\right)\right| + 1.$$

Let $E_k^{N_k}$ be the maximal $(N_k, \varepsilon)$-separated set of $M$. Then by the pigeon-hole principle, there exists $a^k, b^k \in E_k^{N_k}$ such that $h^{N_k}(a^k), h^{N_k}(b^k)$ are in a common element of $\mathcal{T}(h^{N_k}(M), \mathcal{U}_k)$. Since $d_{N_k}^{-}(h^{N_k}(a^k), h^{N_k}(b^k)) \geqslant \varepsilon$, it follows from Lemma 18 that there exists $x_\alpha^k, x_\beta^k \in h^{N_k}(M)$ such that $\varepsilon/3 \leqslant d_{N_k}^{-}(x_\alpha^k, x_\beta^k) < \varepsilon$ and $\mathrm{d}(x_\alpha^k, x_\beta^k) < \delta_k$. Hence, $\mathrm{d}(h^i(x_\alpha^k), h^i(x_\beta^k)) < \varepsilon$ for all $i \leqslant k < N_k$.

Now let $m_k \in \{0, \ldots, N_k - 1\}$ such that $\mathrm{d}(h^{m_k}(x_\alpha^k), h^{m_k}(x_\beta^k)) \geqslant \varepsilon/3$. For simplicity, define $y_k = h^{m_k}(x_\alpha^k)$ and $z_k = h^{m_k}(x_\beta^k)$. Then $\mathrm{d}(h^i(y_k), h^i(z_k)) < \varepsilon$ for all $i < k + m_k$. By Lemma 19, there exist limit points $y$ of $\{y_k\}_{k=1}^{\infty}$ and $z$ of $\{z_k\}_{k=1}^{\infty}$ such that $\mathrm{d}(h^i(y), h^i(z)) \leqslant 2\varepsilon < c$ for all $i$. However, since $\mathrm{d}(y_k, z_k) \geqslant \varepsilon/3$ for all $k > 0$, $y$ and $z$ must be distinct. Therefore, $h$ is not expansive. □

The following interesting results now follow:

**Corollary 21.** 1*-dimensional plane continua that have* 2 *complementary domains do not admit expansive homeomorphisms.*

**Proof.** This follows directly from Theorems 2 and 20. □

**Corollary 22.** *The pseudo-circle does not admit an expansive homeomorphism.*

The following questions remain open.

**Question 1.** *Does there exist a* 1*-dimensional* 3*-separating plane continuum that admits an expansive homeomorphism?*

**Question 2.** *Does there exist a* 2*-dimensional non-separating plane continuum that admits an expansive homeomorphism?*

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