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The variational iteration method for solving a neutral functional-differential equation with proportional delays

Xumei Chen [a,*], Linjun Wang [b]

[a] Faculty of Science, Jiangsu University, Zhenjiang, 212013, PR China
[b] Department of Mathematics, Jilin University, Changchun, 130012, PR China

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ABSTRACT

In this paper, the variational iteration method is applied to neutral functional-differential equations with proportional delays. Illustrative examples are given to show the efficiency of the method. We also compare the performance of the method with that of a particular Runge–Kutta method and a one-leg θ-method.

1. Introduction

The variational iteration method was proposed originally by Ji-Huan He [1]. An elementary introduction to the variational iteration method and some new developments, as well as to new interpretations, can be found in [2–5]. There, the main concepts underlying the variational iteration method, such as the role of general Lagrange multipliers, the restricted variation and correction functionals are explained heuristically. This method has been advantageously employed for solving various kinds of nonlinear problems (see for example [6–14]). It has also been successfully applied to parabolic partial differential equations [6], to nonlinear systems of second-order boundary value problems [7], to multi-pantograph delay equations [9], to heat-like and wave-like equations with variable coefficients [10], to linear and nonlinear Schrödinger equations [12], and to other problems [15].

In this paper, we employ the variational iteration method to study various properties of neutral functional-differential equations with proportional delays. Approximate analytical solutions with high accuracy can be obtained by carrying out only a few steps in the variational iteration method.

Consider the following neutral functional-differential equation with proportional delays,

$$(u(t)+a(t)u(p_m t))^{(m)} = \beta u(t) + \sum_{k=0}^{m-1} b_k(t)u^{(k)}(p_k t) + f(t), \quad t \geq 0, \tag{1.1}$$

with the initial conditions

$$\sum_{k=0}^{m-1} c_{ik}u^{(k)}(0) = \lambda_i, \quad i = 0, 1, \ldots, m-1. \tag{1.2}$$

Here, a and b_k ($k = 0, 1, \ldots, m-1$) are given analytical functions, and $\beta, p_k, c_{ik}, \lambda_i$ denote given constants with $0 < p_k < 1$ ($k = 0, 1, \ldots, m$).

* Corresponding author. Tel.: +86 013604306928.
E-mail address: chenxumei@email.jlu.edu.cn (X. Chen).

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In order to apply the variational iteration method, we rewrite Eq. (1.1) as

$$u^{(m)}(t)=\beta u(t)-(a(t)u(p_m t))^{(m)}+\sum_{k=0}^{m-1}b_k(t)u^{(k)}(p_k t)+f(t),\quad t\geq 0. \tag{1.3}$$

Neutral functional-differential equations with proportional delays represent a particular class of delay differential equation. Such functional-differential equations play an important role in the mathematical modeling of real world phenomena [16]. Obviously, most of these equations cannot be solved exactly. It is therefore necessary to design efficient numerical methods to approximate their solutions. Ishiwata et al. used the rational approximation method [17] and the collocation method [18] to compute numerical solutions of delay differential equations with proportional delays. Hu et al. [19] applied linear multistep methods to compute numerical solutions for neutral delay differential equations. Wang et al. obtained approximate solutions for neutral delay differential equations by continuous Runge–Kutta methods [20] and one-leg θ-methods [21,22].

2. The variational iteration method

In this section, we introduce the basic idea underlying the variational iteration method for solving nonlinear equations. Consider the general nonlinear differential equation

$$Lu+Nu=g(t) \tag{2.1}$$

[1,23], where L is a linear differential operator, N is a nonlinear operator, and g is a given analytical function. The essence of the method is to construct a correction functional of the form

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda(t,s)(Lu_n(s)+N\widetilde{u}_n(s)-g(s))\mathrm{d}s, \tag{2.2}$$

where λ is a Lagrange multiplier which can be identified optimally via the variational theory [1,23], u_n is the approximate solution and $\widetilde{u}_n$ denotes the restricted variation, i.e. $\delta\widetilde{u}_n=0$ [1,23]. After determining the Lagrange multiplier λ and selecting an appropriate initial function u_0, the successive approximations u_n of the solution u can be readily obtained. Consequently, the solution of Eq. (2.1) is given by $u=\lim_{n\to\infty}u_n$ [1,23].

3. Illustrative examples

In this section, we present a selection of examples to illustrate the efficiency of the method proposed in Section 2.

Example 1. Consider the following first-order neutral functional-differential equation with proportional delay:

$$\begin{cases} u'(t)=-u(t)+\dfrac{1}{2}u\left(\dfrac{t}{2}\right)+\dfrac{1}{2}u'\left(\dfrac{t}{2}\right), & 0<t<1,\\ u(0)=1. \end{cases} \tag{3.1}$$

The Lagrange multiplier can be readily identified as $\lambda=-\exp(s-t)$ [3]. As a result, we obtain the iteration formula

$$u_{n+1}(t)=u_n(t)-\int_0^t e^{s-t}\left\{u_n'(s)+u_n(s)-\frac{1}{2}u_n\left(\frac{1}{2}s\right)-\frac{1}{2}u_n'\left(\frac{1}{2}s\right)\right\}\mathrm{d}s. \tag{3.2}$$

Starting with $u_0(t)=1$ and using the iteration formula (3.2), we find

$$u_1(t)=\frac{1}{2}+\frac{1}{2}e^{-t},$$
$$u_2(t)=\frac{1}{4}+\frac{3}{4}e^{-t},$$
$$u_3(t)=\frac{1}{8}+\frac{7}{8}e^{-t},$$
$$\vdots$$
$$u_n(t)=\frac{1}{2^n}+\frac{2^n-1}{2^n}e^{-t}.$$

This yields the exact solution $u(t)=\lim_{n\to\infty}u_n(t)=e^{-t}$. In Table 1 we compare the absolute errors of the variational iteration method (for $n=7,8$) with the ones for the two-stage order-one Runge–Kutta method of [16] and the one-leg θ-method of [21,22] with $\theta=0.8$, using $h=0.01$.

Table 1
Comparison of the absolute errors for Example 1.

t	Two-stage order-one Runge–Kutta method	One-leg θ-method with $\theta = 0.8$	Variational iterative method	
			$n = 7$	$n = 8$
0.1	4.55×10^{-4}	2.57×10^{-3}	7.43×10^{-4}	3.72×10^{-4}
0.2	8.24×10^{-4}	8.86×10^{-3}	1.42×10^{-3}	7.08×10^{-4}
0.3	1.12×10^{-3}	1.72×10^{-2}	2.02×10^{-3}	1.01×10^{-3}
0.4	1.35×10^{-3}	2.66×10^{-2}	2.58×10^{-3}	1.29×10^{-3}
0.5	1.52×10^{-3}	3.63×10^{-2}	3.07×10^{-3}	1.54×10^{-3}
0.6	1.66×10^{-3}	4.58×10^{-2}	3.52×10^{-3}	1.76×10^{-3}
0.7	1.75×10^{-3}	5.47×10^{-2}	3.93×10^{-3}	1.97×10^{-3}
0.8	1.81×10^{-3}	6.29×10^{-2}	4.30×10^{-3}	2.15×10^{-3}
0.9	1.84×10^{-3}	7.02×10^{-2}	4.64×10^{-3}	2.32×10^{-3}
1.0	1.85×10^{-3}	7.66×10^{-2}	4.94×10^{-3}	2.47×10^{-3}

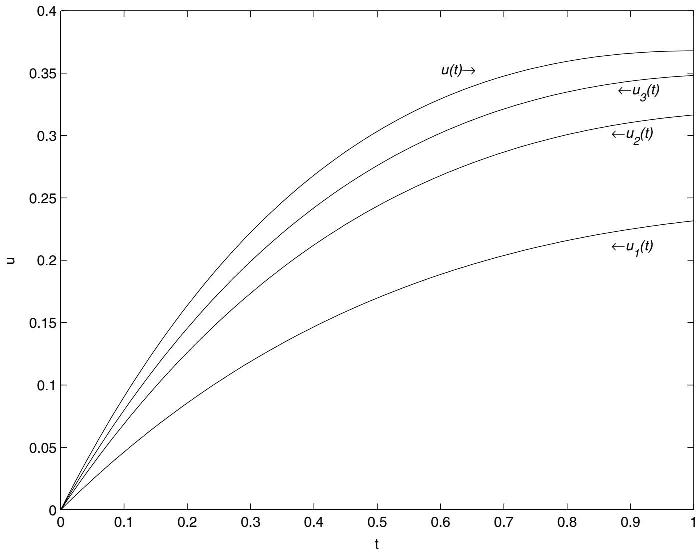

Fig. 1. Comparison of the approximate solutions with the exact solution for Example 2.

Example 2. Consider the first-order neutral functional-differential equation with proportional delay used in [21],

$$\begin{cases} u'(t) = -u(t) + 0.1u(0.8t) + 0.5u'(0.8t) + (0.32t - 0.5)\exp(-0.8t) + \exp(-t), \quad t \geq 0, \\ u(0) = 0, \end{cases} \tag{3.3}$$

which has the exact solution $u(t) = t\exp(-t)$.

The Lagrange multiplier can be readily identified as $\lambda = -\exp(s-t)$ [3]. The corresponding variational iteration formula reads

$$u_{n+1}(t) = u_n(t) - \int_0^t e^{s-t}\{u_n'(s) + u_n(s) - 0.1u_n(0.8s) - 0.5u_n'(0.8s) - \exp(-s) - (0.32s - 0.5)\exp(-0.8s)\}ds, \tag{3.4}$$

and we choose $u_0(t) = 0$ as its starting value. In Fig. 1 we show some of the resulting approximate solutions and the exact solution $u(t) = t\exp(-t)$. In Table 2 we compare the absolute errors of the variational iteration method (for $n = 5, 6$) with those of the two-stage order-one Runge–Kutta method of [16] and the one-leg θ-method [21,22] with $\theta = 0.8$, where $h = 0.01$.

Example 3. Consider the following second-order neutral functional-differential equation with proportional delay,

$$\begin{cases} u''(t) = u'\left(\frac{1}{2}t\right) - \frac{1}{2}tu''\left(\frac{1}{2}t\right) + 2, \quad 0 < t < 1, \\ u(0) = 1, \qquad u'(0) = 0. \end{cases} \tag{3.5}$$

Table 2
Comparison of the absolute errors for Example 2.

t	Two-stage order-one Runge–Kutta method	One-leg θ-method with $\theta = 0.8$	Variational iterative method	
			$n = 5$	$n = 6$
0.1	8.68×10^{-4}	4.65×10^{-3}	2.62×10^{-3}	1.30×10^{-3}
0.2	1.49×10^{-3}	1.45×10^{-2}	4.36×10^{-3}	2.14×10^{-3}
0.3	1.90×10^{-3}	2.57×10^{-2}	5.40×10^{-3}	2.63×10^{-3}
0.4	2.16×10^{-3}	3.60×10^{-2}	5.89×10^{-3}	2.84×10^{-3}
0.5	2.28×10^{-3}	4.43×10^{-2}	5.96×10^{-3}	2.83×10^{-3}
0.6	2.31×10^{-3}	5.03×10^{-2}	5.71×10^{-3}	2.67×10^{-3}
0.7	2.27×10^{-3}	5.37×10^{-2}	5.23×10^{-3}	2.39×10^{-3}
0.8	2.17×10^{-3}	5.47×10^{-2}	4.59×10^{-3}	2.04×10^{-3}
0.9	2.03×10^{-3}	5.35×10^{-2}	3.84×10^{-3}	1.64×10^{-3}
1.0	1.86×10^{-3}	5.03×10^{-2}	3.04×10^{-3}	1.22×10^{-3}

Table 3
Comparison of the absolute errors for Example 4.

t	Two-stage order-one Runge–Kutta method	One-leg θ-method with $\theta = 0.8$	Variational iteration method	
			$n = 5$	$n = 6$
0.1	1.00×10^{-3}	6.10×10^{-3}	3.34×10^{-4}	1.67×10^{-4}
0.2	2.02×10^{-3}	2.58×10^{-2}	1.43×10^{-3}	7.15×10^{-4}
0.3	3.07×10^{-3}	6.47×10^{-2}	3.45×10^{-3}	1.73×10^{-3}
0.4	4.17×10^{-3}	1.37×10^{-1}	6.58×10^{-3}	3.30×10^{-3}
0.5	5.34×10^{-3}	2.81×10^{-1}	1.11×10^{-2}	5.55×10^{-3}

The Lagrange multiplier can be readily identified as $\lambda = s - t$ [3]. We thus generate the approximations u_n by using the iteration formula

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\left\{u_n''(s) - u_n'\left(\frac{1}{2}s\right) + \frac{1}{2}su_n''\left(\frac{1}{2}s\right) - 2\right\} ds. \tag{3.6}$$

Starting with $u_0(t) = 1$ in (3.6), we obtain

$$u_1(t) = t^2 + 1,$$

which coincides with the exact solution.

Example 4. Consider the second-order neutral functional-differential equation with proportional delay,

$$\begin{cases} u''(t) = \frac{3}{4}u(t) + u\left(\frac{t}{2}\right) + u'\left(\frac{t}{2}\right) + \frac{1}{2}u''\left(\frac{t}{2}\right) - t^2 - t + 1, \quad 0 < t < 1, \\ u(0) = u'(0) = 0. \end{cases} \tag{3.7}$$

Here, the Lagrange multiplier is found to be $\lambda = s - t$ [3]. Therefore, the corresponding iteration formula assumes the form

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\left\{u_n''(s) - \frac{3}{4}u_n(s) - u_n\left(\frac{s}{2}\right) - u_n'\left(\frac{s}{2}\right) - \frac{1}{2}u_n''\left(\frac{1}{2}s\right) + s^2 + s - 1\right\} ds. \tag{3.8}$$

Let $u_0(t) = 0$. Then, by the iteration formula (3.8), we obtain

$$u_1(t) = \frac{t^2}{2} - \frac{t^3}{6} - \frac{t^4}{12},$$
$$u_2(t) = \frac{3}{4}t^2 - \frac{1}{8}t^3 - \frac{1}{16}t^4 - \frac{3}{320}t^5 - \frac{13}{5760}t^6,$$
$$u_3(t) = \frac{7}{8}t^2 - \frac{7}{96}t^3 - \frac{7}{192}t^4 - \frac{39}{5120}t^5 - \frac{343}{184\,320}t^6 - \frac{17}{92\,160}t^7 - \frac{91}{2949\,120}t^8.$$

The comparison of these approximate solutions with the exact solution $u(t) = t^2$ is shown in Fig. 2. In Table 3 we compare the absolute errors of the variational iteration method (for $n = 5, 6$) with the two-stage order-one Runge–Kutta method [16] and the one-leg θ-method [21,22] with $\theta = 0.8$, using $h = 0.01$.

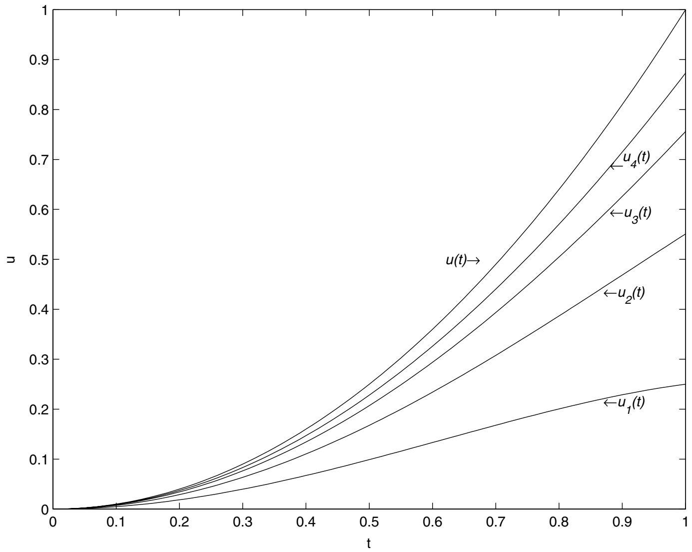

Fig. 2. Comparison of the approximate solutions with the exact solution for Example 4.

Table 4
Comparison of the absolute errors for Example 5.

t	Two-stage order-one Runge–Kutta method	Variational iteration method		
		$n = 4$	$n = 5$	$n = 6$
0.1	4.97×10^{-5}	2.46×10^{-8}	3.07×10^{-9}	9.09×10^{-12}
0.2	4.43×10^{-4}	4.03×10^{-7}	5.04×10^{-8}	2.98×10^{-10}
0.3	1.57×10^{-3}	2.09×10^{-6}	2.62×10^{-7}	2.33×10^{-9}
0.4	3.85×10^{-3}	6.80×10^{-6}	8.49×10^{-7}	1.01×10^{-8}
0.5	7.78×10^{-3}	1.71×10^{-5}	2.13×10^{-6}	3.20×10^{-8}
0.6	1.39×10^{-2}	3.64×10^{-5}	4.55×10^{-6}	8.24×10^{-8}
0.7	2.28×10^{-2}	6.96×10^{-5}	8.69×10^{-6}	1.85×10^{-7}
0.8	3.53×10^{-2}	1.23×10^{-4}	1.53×10^{-5}	3.76×10^{-7}
0.9	5.19×10^{-2}	2.03×10^{-4}	2.54×10^{-5}	7.09×10^{-7}
1.0	7.34×10^{-2}	3.21×10^{-4}	4.01×10^{-5}	1.26×10^{-6}

Example 5. Consider the following third-order neutral functional-differential equation with proportional delays:

$$\begin{cases} u'''(t) = u(t) + u'\left(\frac{t}{2}\right) + u''\left(\frac{t}{3}\right) + \frac{1}{2}u'''\left(\frac{t}{4}\right) - t^4 - \frac{t^3}{2} - \frac{4}{3}t^2 + 21t, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0. \end{cases} \tag{3.9}$$

The Lagrange multiplier is easily identified to be $\lambda = -\frac{(s-t)^2}{2}$ [3]. Hence, the variational iteration formula becomes

$$u_{n+1} = u_n - \int_0^t \frac{(s-t)^2}{2} \left\{ u_n'''(s) - u_n(s) - u_n'\left(\frac{s}{2}\right) - u_n''\left(\frac{s}{3}\right) - \frac{1}{2}u_n'''\left(\frac{s}{4}\right) + s^4 + \frac{s^3}{2} + \frac{4}{3}s^2 - 21s \right\} ds. \tag{3.10}$$

Choosing $u_0(t) = 0$, the above iteration formula yields the following approximate solutions:

$$u_1(t) = \frac{7}{8}t^4 - \frac{1}{45}t^5 - \frac{1}{240}t^6 - \frac{1}{210}t^7,$$
$$u_2(t) = \frac{63}{64}t^4 - \frac{1}{288}t^5 - \frac{1031}{1492\,992}t^6 - \frac{5617}{8709\,120}t^7 - \frac{7144}{104\,509\,440}t^8 - \frac{1}{107\,520}t^9 - \frac{1}{151\,200}t^{10}.$$

The graphs of these approximate solutions and the exact solution $u(t) = t^4$ are contained in Fig. 3. In Table 4 we compare the absolute errors of the variational iteration method ($n = 4, 5, 6$) with the ones for the two-stage order-one Runge–Kutta method of [16], using $h = 0.01$.

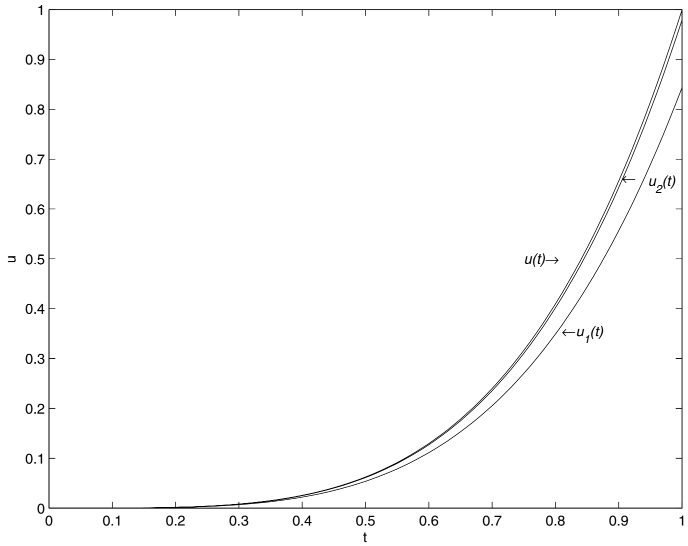

Fig. 3. Comparison of the approximate solutions with the exact solution for Example 5.

4. Conclusions

In this paper, we have demonstrated the feasibility of the variational iteration method for solving neutral functional-differential equations with proportional delays. We obtain high-accuracy approximate solutions, or even the exact solution, after only a few iterations. All the given examples reveal that the results of the variational iteration method are in excellent agreement with those generated by some other methods. The numerical results also show that the variational iteration method yields a very effective and convenient approach to the approximate solution of neutral functional-differential equations with proportional delays.

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