Available online at www.sciencedirect.com
SCIENCE DIRECT®
An International Journal
computers & mathematics with applications
ELSEVIER Computers and Mathematics with Applications 47 (2004) 845–851
www.elsevier.com/locate/camwa

Fixed-Point Theorems in Fuzzy Real Line

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(Received August 2002; revised and accepted September 2003)

Abstract—Some sufficient conditions for the existence of fixed points of increasing operators in fuzzy real line $\mathbb{R}_L$ are given. We also establish some coupled quasi-fixed-point theorems of mixed monotone operators in fuzzy real line $\mathbb{R}_L$. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Fixed point, Fuzzy real line, Condensing operator, Mixed monotone operator, Coupled quasi fixed point.

1. INTRODUCTION

In this paper, by a fuzzy set, we mean a mapping $f : \mathbb{R} \to [0, 1]$, it is also called a fuzzy subset of $\mathbb{R}$.

Let

$$L_0 = \{\alpha \in [0, 1] \mid \alpha > 0\}.$$

DEFINITION 1.1. *(See [1].) By a fuzzy number we mean a mapping $x : \mathbb{R} \to [0, 1]$ of the real line into $[0, 1]$ such that*

(1) *for each $\alpha \in L_0$, the level set $x_\alpha = \{\xi \in \mathbb{R} \mid \alpha \le x(\xi)\}$ is a closed finite interval $[x_{\alpha l}, x_{\alpha r}]$ and*
(2) *$\{\xi \in R \mid 0 < x(\xi)\}$ is bounded.*

The definitions of basic operations and properties for fuzzy numbers also can be found in [1]. For each fuzzy number x and each $\xi \in \mathbb{R}$, we have

$$x(\xi) = \sup\{\alpha \in L_0 \mid \xi \in x_\alpha\}. \tag{1.1}$$

In the following, a characterization of fuzzy numbers is given.

0898-1221/04/$ - see front matter Typeset by $\mathcal{AMS}$-TeX
doi:10.1016/j.camwa.2003.09.024

PROPOSITION 1.2. *(See [1].) For each fuzzy number x, the mappings $\alpha \mapsto x_{\alpha l}$ and $\alpha \mapsto x_{\alpha r}$ of L_0 into $\mathbb{R}$ have the following properties.*

(1) *$\alpha \mapsto x_{\alpha l}$ is isotone, $\alpha \mapsto x_{\alpha r}$ is antitone and $x_{\alpha l} \leq x_{\alpha r}$.*
(2) *$x_{\alpha l} = \sup_{0<\beta<\alpha} x_{\beta l}$ and $x_{\alpha r} = \inf_{0<\beta<\alpha} x_{\beta r}$ for all $\alpha \in L_0$.*
(3) *The infimum $x_{0l} = \inf_{\alpha>0} x_{\alpha l}$ and the supremum $x_{0r} = \sup_{\alpha>0} x_{\alpha r}$ exist in $\mathbb{R}$.*

On the other hand, if $\alpha \mapsto x_{\alpha l}$ and $\alpha \mapsto x_{\alpha r}$ are mappings of L_0 into $\mathbb{R}$ with Properties (1)–(3), then the mapping $x : \mathbb{R} \to L_0$ defined by (1.1) with $x_\alpha = [x_{\alpha l}, x_{\alpha r}]$ for all $\alpha \in L_0$, is the fuzzy number for which $x_\alpha, \alpha \in L_0$, are the level sets.

The set $\mathbb{R}_L$ of all fuzzy numbers is called the fuzzy real line.

DEFINITION 1.3. *The canonical partial ordering $\leq$ on $\mathbb{R}_L$ is defined by*

$$x \leq y \Leftrightarrow x_{\alpha l} \leq y_{\alpha l} \quad \text{and} \quad x_{\alpha r} \leq y_{\alpha r}, \qquad \text{for all } \alpha \in L_0. \tag{1.2}$$

The Hausdorff metric D on $\mathbb{R}_L$ is given by the following definition.

DEFINITION 1.4. *Define $D : \mathbb{R}_L \times \mathbb{R}_L \to \mathbb{R}_+ \cup \{0\}$ by*

$$D(u,v) = \sup_{\alpha \in [0,1]} d(u_\alpha, v_\alpha),$$

where

$$d(A,B) = \inf\{\epsilon > 0 \mid A \subset N(B,\epsilon), B \subset N(A,\epsilon)\},$$

for all A,B in $\wp_k(\mathbb{R}) = \{A \mid A$ is a nonempty compact convex subset of $\mathbb{R}\}$.

Throughout this paper, the metric considered is the Hausdorff metric.

LEMMA 1.5. *Let $x,y,z \in \mathbb{R}_L$. If $x \leq y \leq z$, then $D(z,y) \leq D(z,x)$.*

PROOF. For each $\alpha \in [0,1]$, it follows from $x \leq y \leq z$ that

$$x_{\alpha l} \leq y_{\alpha l} \leq z_{\alpha l} \qquad \text{and} \qquad x_{\alpha r} \leq y_{\alpha r} \leq z_{\alpha r}.$$

Then,

$$|y_{\alpha l} - z_{\alpha l}| \leq |x_{\alpha l} - z_{\alpha l}| \qquad \text{and} \qquad |y_{\alpha r} - z_{\alpha r}| \leq |x_{\alpha r} - z_{\alpha r}|.$$

Thus,

$$\begin{aligned} d([z_{\alpha l}, z_{\alpha r}], [y_{\alpha l}, y_{\alpha r}]) &= \max\{|y_{\alpha l} - z_{\alpha l}|, |y_{\alpha r} - z_{\alpha r}|\} \\ &\leq \max\{|x_{\alpha l} - z_{\alpha l}|, |x_{\alpha r} - z_{\alpha r}|\} \\ &= d([z_{\alpha l}, z_{\alpha r}], [x_{\alpha l}, x_{\alpha r}]), \end{aligned}$$

which implies

$$\begin{aligned} D(z,y) &= \sup_{\alpha \in [0,1]} d([z_{\alpha l}, z_{\alpha r}], [y_{\alpha l}, y_{\alpha r}]) \\ &\leq \sup_{\alpha \in [0,1]} d([z_{\alpha l}, z_{\alpha r}], [x_{\alpha l}, x_{\alpha r}]) = D(x,z). \end{aligned}$$

REMARK 1.6. *(See [2].) $\mathbb{R}_L$ is a complete metric space. ($\mathbb{R}_L$ satisfies the conditions of E^n in [2].)*

In Section 2, we establish some sufficient conditions for the existence of fixed points of increasing operators in fuzzy real line $\mathbb{R}_L$. We also establish some results about quasi-coupled fixed points in fuzzy real line $\mathbb{R}_L$ in Section 3. We hope that our work has some applicability in solving fuzzy equations. For the discussion about characterizing the properties of fuzzy equations, we refer the readers to [3,4].

2. FIXED POINTS OF INCREASING OPERATORS

In order to discuss our main results, we need the following definitions, which are similar to those in crisp results in [5].

Let $[u_0, v_0] = \{x \in \mathbb{R}_L \mid u_0 \le x \le v_0\}$, where $\le$ is the canonical partial ordering.

DEFINITION 2.1. *A set $S \subset \mathbb{R}_L$ is said to be compact if any sequence $\{s_k\}$ in S has a subsequence converging to a point in S in the metric space $\mathbb{R}_L$.*

DEFINITION 2.2. *Let $B \subset [u_0, v_0]$ be a closed set of $\mathbb{R}_L$. A is an operator from B in to B.*

(a) *A is said to be increasing if $x \le y$ implies $Ax \le Ay$ for all $x, y \in B$.*
(b) *A is said to be condensing if it is continuous, bounded and $r(A(S)) < r(S)$ for any bounded set $S \subset B$ with $r(S) > 0$, where $r(S)$ denote the measure of noncompactness of S (cf. [6]).*

THEOREM 2.3. *Let $u_0, v_0 \in \mathbb{R}_L$, $u_0 < v_0$. Let $B \subset [u_0, v_0]$ be a closed set of $\mathbb{R}_L$ such that $u_0, v_0 \in B$. Suppose that $A : B \to B$ is an increasing operator such that*

$$u_0 \le Au_0, \qquad Av_0 \le v_0. \tag{2.1}$$

Suppose A is condensing, then A has a maximal fixed point x^* and a minimal fixed point x_* in B, moreover

$$x^* = \lim_{n\to\infty} v_n, \qquad x_* = \lim_{n\to\infty} u_n, \tag{2.2}$$

where $v_n = Av_{n-1}$ and $u_n = Au_{n-1}$, $n = 1, 2, 3, \dots$ and

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{2.3}$$

PROOF. Since A is increasing, it follows from (2.1) that (2.3) holds. Now, we prove that $\{u_n\}$ converges to some $x_* \in \mathbb{R}_L$ and $Ax_* = x_*$. The set $S = \{u_0, u_1, u_2, \dots\}$ is bounded and $S = A(S) \cup \{u_0\}$, hence, $r(S) = r(A(S))$. It follows from A is condensing that $r(S) = 0$, i.e., S is a relatively compact set. Hence, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $u_{n_k} \to x_*$ for some $x_* \in \mathbb{R}_L$ (note that $\mathbb{R}_L$ is complete). Clearly, $u_n \le x_* \le v_n$ $(n = 1, 2, \dots)$. When $m > n_k$, it follows from Lemma 1.5 that $D(x_*, u_m) \le D(x_*, u_{n_k})$. Thus, $u_m \to x_*$ as $m \to \infty$. Taking limit $n \to \infty$ on both sides of the equality $u_n = Au_{n-1}$, we get $x_* = Ax_*$ since A is continuous and B is closed.

Similarly, we can prove that $\{v_n\}$ converges to some $x^* \in \mathbb{R}_L$ and $Ax^* = x^*$.

Finally, we prove that x^* and x_* are the maximal and minimal fixed point of A in B, respectively. Let $\bar{x} \in B$ and $A\bar{x} = \bar{x}$. Since A is increasing, it follows from $u_0 \le \bar{x} \le v_0$ that $Au_0 \le A\bar{x} \le Av_0$, i.e., $u_1 \le \bar{x} \le v_1$. Using the same argument, we get $u_2 \le \bar{x} \le v_2$, and, in general, $u_n \le \bar{x} \le v_n$ $(n = 1, 2, 3, \dots)$. Now, taking limit $n \to \infty$, we obtain $x_* \le \bar{x} \le x^*$.

Immediately, apply Theorem 2.3, we have the following.

COROLLARY 2.4. *Let the conditions of Theorem 2.3 be satisfied. Suppose that A has only one fixed point $\bar{x}$ in B. Then, for any $x_0 \in B$, the successive iterates*

$$x_n = Ax_{n-1}, \qquad n = 1, 2, 3, \dots$$

converges to $\bar{x}$, i.e., $D(x_n, \bar{x}) \to 0$ as $n \to \infty$.

In the rest of this section, we will apply Theorem 2.3 to solve the following fuzzy equation:

$$Ex^2 + Fx + G = x, \tag{FE}$$

where E, F, G, and x are fuzzy numbers.

First, we characterize the Hausdorff metric in fuzzy number by supnorm in $C[0, 1]$.

DEFINITION 2.5. *For each fuzzy number x, we define two functions x_L and x_R from $[0,1]$ into $\mathbb{R}$ by $x_L(\alpha) = x_{\alpha l}$ and $x_R(\alpha) = x_{\alpha r}$ for each $\alpha \in [0,1]$.*

THEOREM 2.6. *For each fuzzy number x, x_R, and x_L belong to $C[0,1]$, i.e., they are continuous functions on $[0,1]$.*

PROOF. Let $\alpha > 0$. We will compute the right and left limits of $\lim_{\beta\to\alpha} x_L(\beta)$ and show that they are equal to $x_L(\alpha)$.

Let $\alpha_n > \alpha$ and $\alpha_n \to \alpha$ as $n \to \infty$. Suppose that $x_L(\alpha_n)$ does not converge to $x_L(\alpha)$, then there is an $\epsilon > 0$ such that $x_L(\alpha_n) > x_L(\alpha) + \epsilon$ for sufficiently large nature number n. On the other hand, by the definition of fuzzy numbers, we know that $\{x_L(\alpha_n)\}$ is decreasing and bounded. So, $x_L(\alpha_n)$ converges to a number L and $L \geq x_{\alpha l} + \epsilon$. Moreover, by the definition of x_L, we know that $x(L) \leq \alpha_n$ and it implies $x(L) \leq \alpha$. So, $L \leq x_L(\alpha) = x_{\alpha l}$. It follows a contradiction. Hence, $\lim_{\alpha_n\to\alpha} x_L(\alpha_n) = x_L(\alpha)$. So, the right limit of $\lim_{\beta\to\alpha} x_L(\beta)$ is equal to $x_L(\alpha)$. Similarly, we can show the left limit of $\lim_{\beta\to\alpha} x_L(\beta)$ is equal to $x_L(\alpha)$. So, x_L is continuous on $(0,1]$. Using the same argument, we also can show that x_L is continuous at 0. Hence, x_L is continuous on $[0,1]$.

The argument of x_R is continuous on $[0,1]$ is similar.

The following theorem is a characterization of Hausdorff in fuzzy number by supnorm on $C[0,1]$.

THEOREM 2.7. *Suppose that x and y are fuzzy numbers, then $D(x,y) = \max(\|x_L - y_L\|_\infty, \|x_R - y_R\|_\infty)$.*

PROOF. It is obvious.

Let $M > 0$ be a fixed real number. Let $B_M = \{x \in \mathbb{R}_L \mid 0 \leq x \leq 1, |x_L(\alpha) - x_L(\beta)| \leq M|\alpha - \beta|$ and $|x_R(\alpha) - x_R(\beta)| \leq M|\alpha - \beta|$ for each $\alpha, \beta \in [0,1]\}$.

THEOREM 2.8. *B_M is a closed subset of $\mathbb{R}_L$.*

PROOF. Suppose $x_n \in B_M$ and $\lim_{n\to\infty} x_n = x$ in $\mathbb{R}_L$. We show that $x \in B_M$. By Theorem 2.7, we know that $\|(x_n)_L - x_L\|_\infty \to 0$ as $n \to \infty$. It implies that for any $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $\|(x_n)_L - x_L\|_\infty \leq \epsilon$. Hence,

$$\begin{aligned}|x_L(\alpha) - x_L(\beta)| &\leq |x_L(\alpha) - (x_n)_L(\alpha)| + |(x_n)_L(\alpha) - (x_n)_L(\beta)| + |x_L(\beta) - (x_n)_L(\beta)| \\ &\leq 2\epsilon + M|\alpha - \beta|,\end{aligned}$$

for α, $\beta \in [0,1]$. Since ϵ is arbitrary, it implies that $|x_L(\alpha) - x_L(\beta)| \leq M|\alpha - \beta|$ for $\alpha, \beta \in [0,1]$.

Using the same argument, we also can show that $|x_R(\alpha) - x_R(\beta)| \leq M|\alpha - \beta|$ for each $\alpha, \beta \in [0,1]$. So, B_M is a closed set of $\mathbb{R}_L$.

THEOREM 2.9. *Let $M > 0$ be a real number. Suppose that E, F, and G are fuzzy numbers and satisfy the following conditions*

(a) *$E > 0$, $E_L(\alpha), E_R(\alpha) \leq 1/6$ for each $\alpha \in [0,1]$. $|E_L(\alpha) - E_L(\beta)| < (M/6)|\alpha - \beta|$ and $|E_R(\alpha) - E_R(\beta)| < (M/6)|\alpha - \beta|$ for each $\alpha, \beta \in [0,1]$;*
(b) *F and G are defined as the same conditions as in (a).*

Then, (FE) has a solution in B_M.

In order to prove Theorem 2.9, we give the following lemma.

LEMMA 2.10. *Suppose that B is a subset of $\mathbb{R}_L$. Let $B_L = \{x_L \mid x \in B\}$ and let $B_R = \{x_R \mid x \in B\}$. If B_L and B_R are compact in $(C[0,1], \|\cdot\|_\infty)$, then B is a compact set in $\mathbb{R}_L$.*

PROOF. Let $\{x_n\}$ be a sequence of B. We show that $\{x_n\}$ has a convergent subsequence. Since B_L is a compact set of $C[0,1]$, $\{(x_n)_L\}$ has a convergent subsequence $\{(x_{n_k})_L\}$ in $C[0,1]$. Also, B_R is a compact set of $C[0,1]$, $\{(x_{n_k})_R\}$ has a convergent subsequence $\{(x_{n_l})_R\}$ in $C[0,1]$. By Theorem 2.7, $\{(x_{n_l})\}$ is convergent in $\mathbb{R}_L$. So, B is a compact set in $\mathbb{R}_L$.

PROOF OF THEOREM 2.9. Let $y \in B_M$. Let $Y = Ey^2 + Fy + G$. Then,

$$\begin{aligned}
&|Y_L(\alpha) - Y_L(\beta)| \\
&= |E_L(\alpha)y_L^2(\alpha) + F_L(\alpha)y_L(\alpha) + G_L(\alpha) \\
&\quad - E_L(\beta)y_L^2(\beta) - F_L(\beta)y_L(\beta) - G_L(\beta)| \\
&\le y_L^2(\alpha)|E_L(\alpha) - E_L(\beta)| + E_L(\beta)|y_L(\alpha) + y_L(\beta)| \cdot |y_L(\alpha) - y_L(\beta)| \\
&\quad + y_L^2(\alpha)|F_L(\alpha) - F_L(\beta)| + F_L(\beta)|y_L(\alpha) - y_L(\beta)| + |G_L(\alpha) - G_L(\beta)| \\
&\le \frac{M}{6}|\alpha - \beta| + \frac{2M}{6}|\alpha - \beta| + \frac{M}{6}|\alpha - \beta| + \frac{M}{6}|\alpha - \beta| + \frac{M}{6}|\alpha - \beta| \\
&= M|\alpha - \beta|,
\end{aligned} \tag{2.4}$$

for each $\alpha, \beta \in [0,1]$. Similarly, we also can show that

$$|Y_R(\alpha) - Y_R(\beta)| \le M|\alpha - \beta|. \tag{2.5}$$

Moreover, it is easy to see that $0 \le Ex^2 + Fx + G \le 1$ for each $x \in \mathbb{R}_L$. So, we can define operator A from B_M into B_M by $Ax = Ex^2 + Fx + G$. By definition of A and Theorem 2.7, it is easy to see that A is continuous and increasing. Let $S \subset B_M$ be a bounded set with $r(S) > 0$. We show that the closure of $A(S)$ in $\mathbb{R}_L$ is a compact set and then $r(A(S)) < r(S)$. Let B be the closure of $A(S)$ in $\mathbb{R}_L$. By Lemma 2.10, it is sufficient to show that B_L and B_R are compact in $C[0,1]$. It is easy to see that B_L is a bounded set in $C[0,1]$. Let $f \in B_L$. It follows from (2.4) that $|f(\alpha) - f(\beta)| \le M|\alpha - \beta|$ for each $\alpha, \beta \in [0,1]$. So, B_L is equicontinuous. So, B_L is compact by Arzela-Ascoli theorem. Similarly, B_R is compact set. Hence, by Lemma 2.10, we know that B is compact. So, $r(A(S)) < r(S)$. Finally, by Theorem 2.3, there is an $y \in B_M$ such that $Ay = y$, i.e., (FE) has a solution.

REMARK 2.11. It is well known that to solve a fuzzy equation is very difficult, for the details, we refer the readers to [4]. Our result can also be generalized into a system, we will discuss in the following section.

3. COUPLED QUASI FIXED POINTS OF MIXED MONOTONE OPERATORS

Many of the results considered so far for a single equation will be extended to systems. The monotone properties can be used to deal with system of inequalities, and we shall define this property.

DEFINITION 3.1. *Let $u_0, v_0 \in \mathbb{R}_L$. Let $B \subset [u_0, v_0]$ be a closed subset of $\mathbb{R}_L$. A is an operator from $B \times B$ in to B.*

(a) *A is said to be mixed monotone if $A(x,y)$ is increasing in x and decreasing in y, i.e.,*

$$\text{if } x_1 \le x_2,\ x_1, x_2 \in B \text{ implies } A(x_1, y) \le A(x_2, y),\ \text{for each } y \in B;$$
$$\text{if } y_1 \le y_2,\ y_1, y_2 \in B \text{ implies } A(x, y_1) \ge A(x, y_2),\ \text{for each } x \in B.$$

(b) *A is said to be completely continuous if it is continuous and compact. Notice that the compactness means that the set $A(S)$ is relatively compact for any bounded set $S \subset B \times B$.*

(c) *Point $(x^*, y^*) \in B \times B$ is said to be a coupled quasi fixed point of A if $A(x^*, y^*) = x^*$ and $A(y^*, x^*) = y^*$.*

(d) *$x^* \in B$ is called a fixed point of A if $A(x^*, x^*) = x^*$.*

REMARK 3.2. Evidently, if x^* is a fixed point of A, then (x^*, x^*) is a coupled quasi fixed point of A.

THEOREM 3.3. *Let $u_0, v_0 \in \mathbb{R}_L$, $u_0 < v_0$. Let B be a closed subset of $[u_0, v_0]$ such that $u_0, v_0 \in B$. Let $A : B \times B \to B$ be a mixed monotone operator such that*

$$u_0 \le A(u_0, v_0), \qquad A(v_0, u_0) \le v_0. \tag{3.1}$$

Suppose A is completely continuous, then A has a coupled quasi fixed point $(x^*, y^*) \in B \times B$, which is minimal and maximal in the sense that $x^* \le \bar{x} \le y^*$ and $x^* \le \bar{y} \le y^*$ for any coupled quasi fixed point $(\bar{x}, \bar{y}) \in B \times B$ of A; moreover, we have

$$x^* = \lim_{n\to\infty} u_n, \qquad y^* = \lim_{n\to\infty} v_n, \tag{3.2}$$

where $u_n = A(u_{n-1}, v_{n-1})$ and $v_n = A(v_{n-1}, u_{n-1})$, $n = 1, 2, 3, \dots$, which satisfy

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{3.3}$$

PROOF. From (3.1), we know

$$u_0 \le u_1 \le v_1 \le v_0.$$

Suppose $u_{n-1} \le u_n \le v_n \le v_{n-1}$. Since A is mixed monotone,

$$u_n = A(u_{n-1}, v_{n-1}) \le A(u_n, v_{n-1}) \le A(u_n, v_n) = u_{n+1},$$
$$v_n = A(v_{n-1}, u_{n-1}) \ge A(v_n, u_{n-1}) \ge A(v_n, u_n) = v_{n+1},$$

and

$$u_{n+1} = A(u_n, v_n) \le A(v_n, u_n) = v_{n+1}.$$

Hence, by induction, (3.3) holds. Define

$$S = \{u_1, u_2, \dots\}.$$

Then, it follows from A is completely continuous that S is relatively compact. And hence, from the proof of Theorem 2.3, we know $u_n \to x^* \in B$.

Similarly, we can prove that $\{v_n\}$ converges to some $y^* \in B$.

Since A is continuous,

$$u_n = A(u_{n-1}, v_{n-1}) \to A(x^*, y^*)$$

and

$$v_n = A(v_{n-1}, u_{n-1}) \to A(y^*, x^*).$$

Thus, $A(x^*, y^*) = x^*$ and $A(y^*, x^*) = y^*$, i.e., (x^*, y^*) is a coupled quasi fixed point of A.

Finally, we prove that maximal and minimal property of (x^*, y^*). Let $(\bar{x}, \bar{y}) \in B \times B$ is any coupled quasi fixed point of A. Since $u_0 \le \bar{x} \le v_0$ and $u_0 \le \bar{y} \le v_0$,

$$u_1 = A(u_0, v_0) \le A(u_0, \bar{y}) \le A(\bar{x}, \bar{y}) = \bar{x} \le A(v_0, \bar{y}) \le A(v_0, u_0) = v_1$$

and

$$u_1 = A(u_0, v_0) \le A(u_0, \bar{x}) \le A(\bar{y}, \bar{x}) = \bar{y} \le A(v_0, \bar{x}) \le A(v_0, u_0) = v_1.$$

Similarly, $u_2 \le \bar{x} \le v_2$, $u_2 \le \bar{y} \le v_2$ and in general

$$u_n \le \bar{x} \le v_n; \quad u_n \le \bar{y} \le v_n, \qquad n = 0, 1, 2, \dots. \tag{3.4}$$

Now, taking limits in (3.4), we obtain $x^* \le \bar{x} \le y^*$ and $x^* \le \bar{y} \le y^*$.

COROLLARY 3.4. *Let the conditions of Theorem 3.3 be satisfied. Suppose there exists* $0 < \alpha < 1$ *such that*

$$D(A(x,y), A(y,x)) \leq \alpha D(x,y), \qquad (x,y) \in B \times B. \tag{3.5}$$

Then, A has exactly one fixed point $\bar{x}$ in B and, if we successively construct the sequences

$$\begin{aligned} x_n &= A(x_{n-1}, y_{n-1}), \\ y_n &= A(y_{n-1}, x_{n-1}), \end{aligned} \qquad n = 1, 2, 3, \dots, \tag{3.6}$$

for any initial $(x_0, y_0) \in B \times B$, we have

$$D(x_n, \bar{x}) \to 0, \quad D(y_n, \bar{x}) \to 0, \qquad \text{as } n \to \infty. \tag{3.7}$$

PROOF. By (3.5), we know

$$D(v_n, u_n) = D(A(v_{n-1}, u_{n-1}), A(u_{n-1}, v_{n-1})) \leq \alpha D(v_{n-1}, u_{n-1}), \qquad n = 1, 2, \dots,$$

and so

$$D(v_n, u_n) \leq \alpha^n D(v_0, u_0) \to 0 (n \to \infty).$$

Hence, by (3.2), $x^* = y^*$. Let $\bar{x} = x^* = y^*$, then $\bar{x}$ is a fixed point of A. By virtue of the minimal and maximal property (x^*, y^*), we show easily that $\bar{x}$ is the unique fixed point of A in B. Now, let $(x_0, y_0) \in B \times B$ be given and (3.6) be constructed. Similar to the establishment of (3.4), we get

$$u_n \leq x_n \leq v_n, \quad u_n \leq y_n \leq v_n, \qquad n = 0, 1, 2, \dots. \tag{3.8}$$

It follows from (3.2),(3.8) that $\bar{x} = x^* = y^*$ and property of D that (3.7) holds.

REFERENCES

1. W. Gähler and S. Gähler, Contributions to fuzzy analysis, *Fuzzy Sets and Systems* **105**, 201–224, (1999).
2. O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems* **24**, 301–317, (1987).
3. G.J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic, Theory and Applications*, Prentice Hall, Upper Saddle River, NJ, (1995).
4. J.J. Buckley, Solving fuzzy equations, *Fuzzy Sets and Systems* **50**, 1–14, (1992).
5. D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, MA, (1988).
6. V. Lakshmikantham and S. Leela, *Nonlinear Differential Equations in Abstract Spaces*, Pergamon, Oxford, (1981).