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Generalized conjunction/disjunction

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Abstract

The generalized conjunction/disjunction function (GCD) is a continuous logic function of two or more variables that integrates conjunctive and disjunctive properties in a single function. It is used as a mathematical model of simultaneity and replaceability of inputs. Special cases of this function include the full (pure) conjunction, the partial conjunction, the arithmetic mean, the partial disjunction, and the full (pure) disjunction. GCD enables a continuous transition from the full conjunction to the full disjunction, using a parameter that specifies a desired level of conjunction (andness) or disjunction (orness). In this paper, we investigate and compare various versions of GCD and other mathematical models of simultaneity and replaceability that are applicable in the areas of system evaluation, and information retrieval.

Keywords: Preference logic; Andness; Orness; Generalized conjunction/disjunction; Andor; Orand

1. Introduction

GCD is a mapping $\lambda : [0,1]^n \to [0,1], n > 1$, that has properties similar to logic functions of conjunction and disjunction. The level of similarity is adjustable using a parameter $\alpha \in [0,1]$, called the conjunction degree (andness), or its complement $\omega = 1 - \alpha$, that is called the disjunction degree (orness) [5,6]. If $\alpha = 1, \omega = 0$, then the GCD behaves as

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the full (pure) conjunction. Similarly, if $\omega = 1, \alpha = 0$, then the GCD behaves as the full (pure) disjunction.

The main application areas for the GCD function are:

(1) *System evaluation*: preference modeling, system comparison, selection, and optimization.
(2) *Classification* (pattern matching): object recognition, and information retrieval (search).

These two areas have different specific requirements that the GCD must satisfy. In some cases, GCD is interpreted as a logic connective used to aggregate logic variables and compute the resulting degree of truth [5–8,11]. In other cases, GCD is interpreted as the averaging operator [16,17,19–22]. The use of GCD in the area of system evaluation includes decision models for evaluation, comparison, and optimization of computers, search engines, browsers, windowed environments, web sites, e-commerce systems, data management systems, Java IDE's, etc. [11].

In this paper, we focus on applications where GCD is interpreted as a logic connective and used to create compound continuous logic functions, such as partial absorption [8], and other more complex logic functions [9,11]. These functions are frequently used in the area of system evaluation and are suitable for building multiple criteria decision models [2,14].

Continuous logic models based on GCD are a generalization of the binary Boolean logic. The classic Boolean logic is based on binary values $B = \{0, 1\}$ and three basic operations: conjunction $x \wedge y = \min(x, y)$, disjunction $x \vee y = \max(x, y)$, and negation $\bar{x} = 1 - x$. The basic operations are used for making compound functions, such as implication $x \to y = \bar{x} \vee y$, and $x \bar{\wedge} y = \overline{x \wedge y}$, nor $x \bar{\vee} y = \overline{x \vee y}$, exclusive or $x \oplus y = (\bar{x} \wedge y) \vee (x \wedge \bar{y})$, and equivalence $x \sim y = (\bar{x} \wedge \bar{y}) \vee (x \wedge y)$.

The binary set B can be replaced by the unit interval $I = [0, 1]$. If $x \in I$ and $y \in I$ the same logic operations (min, max, and $x \mapsto 1 - x$) can be used to get the traditional continuous logic. The next step in generalization is to replace min and max with GCD, which includes the extreme aggregators min and max, as well as all intermediate aggregators located along the path of continuous transition from min to max. The intermediate aggregators include *partial conjunction* and *partial disjunction*. They are used to model various levels of simultaneity and replaceability (substitutability) of input variables.

In this paper, we investigate mathematical models of simultaneity and replaceability. Our goal is to identify desirable properties of GCD and related aggregators, and to analyze their implementations.

2. Simultaneity and replaceability

Simultaneity and replaceability are two fundamental logic connectives that are building blocks for many decision models. In the area of system evaluation we usually have a set of m requirements that a system is expected to satisfy. There are m input variables $x_1, \ldots, x_m, x_i \in I, i = 1, \ldots, m$ that reflect the level of satisfaction of m specific requirements. Using a logic interpretation, x_i is the degree of truth of the statement asserting that the ith requirement is completely satisfied. We call $x_1, \ldots, x_m$ *elementary preferences*. An evaluation criterion is a compound model that uses elementary preferences to compute the *global preference* $y = L(x_1, \ldots, x_m), y \in I$ that reflects the global satisfaction of all requirements. The global preference is interpreted as the degree of truth of the statement that a complex system completely satisfies all requirements.

The decision model $L : I^m \to I$ can be built using functions that reflect logic relationships between individual requirements. These relationships include various levels of simultaneity and replaceability, as well as more complex relationships that combine mandatory, desired, and optional features. A natural environment for creating system evaluation models is a *continuous preference logic* (CPL) that reflects those aspects of human decision making that include adjustable level of andness, orness, and relative importance (weights) [7,11].

Mathematical models of simultaneity and replaceability are fundamental components of all system evaluation models. Assuming that input variables reflect the level of satisfaction of some criteria, the simultaneity is a requirement for coincident high level of satisfaction of input criteria. All mathematical models of simultaneity reward the concurrence of high inputs, and penalize the lack of simultaneity. The most frequently used models of simultaneity are the logic functions of partial and full conjunction.

Replaceability and simultaneity are symmetric concepts. The replaceability is used in cases where any input can compensate insufficient satisfaction of other inputs. All mathematical models of replaceability penalize cases where inputs are all relatively low, and reward cases where at least one of them is sufficiently high.

A classification of three characteristic families of simultaneity and replaceability models is shown in Fig. 1. These families are t-norms/conorms [15], GCD [11], and AIWA [17]. Without loss in generality let us consider the case of two input variables $x \in I$, $y \in I$, $x < y$, and let $a(x, y)$ be an aggregator that is used to model the simultaneity and/or replaceability. In a special case where x and y are equally important, the neutral point

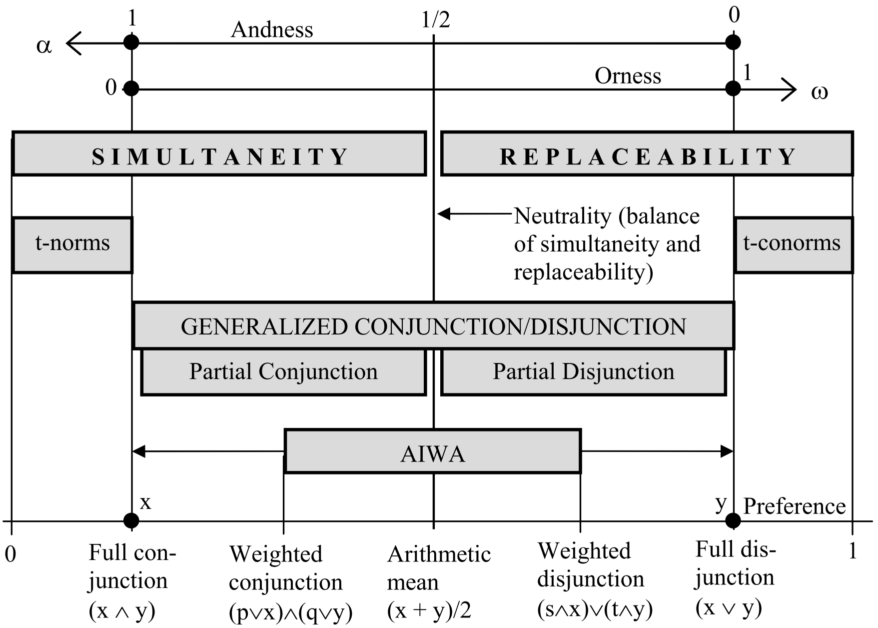

Fig. 1. Families of preference aggregators that model simultaneity and replaceability.

where we have a perfect balance of simultaneity and replaceability is located in the middle of the interval $[x, y]$, and this is the arithmetic mean $(x+y)/2$. If the aggregator $a(x, y)$ satisfies $a(x, y) < (x+y)/2$ then it is a model of simultaneity. Similarly, if $a(x, y) > (x+y)/2$, then $a(x, y)$ is a model of replaceability.

According to Fig. 1, the areas of simultaneity and replaceability are rather wide and include a spectrum of properties. If $0 \leqslant a(x, y) \leqslant x \wedge y$ or $x \vee y \leqslant a(x, y) \leqslant 1$ then these properties are modeled using t-norms and t-conorms [15]. Aggregators that satisfy $x \wedge y \leqslant a(x, y) \leqslant x \vee y$ can be realized using means [13,18,3] and interpreted as continuous preference logic functions. They include GCD and AIWA. If $x \wedge y \leqslant (p \vee x) \wedge (q \vee y) \leqslant a(x, y) \leqslant (s \wedge x) \vee (t \wedge y) \leqslant x \vee y$ then these properties can be modeled using AIWA [17]; in a special case where weighted conjunction has weights $p = q = 0$ the AIWA aggregator can reach conjunction, and in a special case where weighted conjunction has weights $s = t = 1$ the AIWA aggregator can reach disjunction.

GCD is obviously not the only aggregator that can be used to model simultaneity and replaceability. However, GCD is the most suitable family of aggregators for use in the continuous preference logic, and for building compound preference aggregators necessary in system evaluation models. It is important to note that GCD is not a single function, but a family of aggregators that share some basic properties and can be implemented using various means.

3. Basic properties of GCD

The global satisfaction of requirements regularly increases when we increase the satisfaction of any component requirement: if we aggregate m preferences, $y = L(x_1, \ldots, x_m)$, we expect $\partial y / \partial x_i > 0$. This is the main reason why traditional continuous logic functions that are only combinations of min, max, and $x \mapsto 1 - x$ operators cannot be suitable for building decision models for system evaluation and comparison. With these operators, the condition $\partial y / \partial x_i > 0$ holds only for a small subset of input preferences (those that have extreme values in their groups; e.g., in a group of n, if $y = \min(x_1, \ldots, x_n) = x_k$, then $\forall x_i > x_k, \partial y / \partial x_i = 0$). In the majority of cases, this is not acceptable because it implies the inability to improve a system by improving its components. This problem can be solved if instead of pure conjunction (min) and pure disjunction (max), we use appropriate GCD aggregators.

Let $x_1, \ldots, x_n, (x_i \in I, i = 1, \ldots, n)$ be input preferences, and let $\lambda : I^n \to I$, $n > 1$, be a GCD aggregator.

Definition 3.1. An aggregator λ is a *Bounded Quality Range* (BQR) aggregator if it satisfies the condition that the aggregate cannot be better than its best component or worse than its worst component: $x_1 \wedge \cdots \wedge x_n \leqslant \lambda(x_1, \ldots, x_n) \leqslant x_1 \vee \cdots \vee x_n$.

In the area of means BQR is known as the concept of *internality* [3]. It is straightforward to see that BQR aggregators are idempotent: if $x_1 = \cdots = x_n = x \in I$ then $x \wedge \cdots \wedge x = x \leqslant \lambda(x, \ldots, x) \leqslant x = x \vee \cdots \vee x$, and $\lambda(x, \ldots, x) = x$. We assume that all GCD aggregators satisfy the BQR condition, and consequently are idempotent. Of course, at the idempotency line $x_1 = \cdots = x_n = x$ we have that $x_1 \wedge \cdots \wedge x_n = x_1 \vee \cdots \vee x_n = x$ and if there is no difference between conjunction and disjunction, then andness and orness cannot be defined. Therefore, whenever we use andness and orness we assume that at least some of inputs are different, i.e. we assume $x_1 \wedge \cdots \wedge x_n \neq x_1 \vee \cdots \vee x_n$. Under this assumption, for GCD aggregators the andness and orness can be defined as follows.

Definition 3.2. Andness α of a GCD aggregator λ is an indicator of similarity between λ and conjunction $x_1 \wedge \cdots \wedge x_n$. Andness is computed using a similarity metric A that satisfies the following minimum conditions of standard range and monotonicity:

(a) $$\alpha = A(\mathbf{x}, \lambda(x_1,\ldots,x_n)) = \begin{cases} 0, & \lambda(x_1,\ldots,x_n) = x_1 \vee \cdots \vee x_n \\ 1/2, & \lambda(x_1,\ldots,x_n) = (x_1 + \cdots + x_n)/n \\ 1, & \lambda(x_1,\ldots,x_n) = x_1 \wedge \cdots \wedge x_n \end{cases}$$

(b) If $\lambda_1(x_1,\ldots,x_n) < \lambda_2(x_1,\ldots,x_n)$ then $A(\mathbf{x}, \lambda_1(x_1,\ldots,x_n)) > A(\mathbf{x}, \lambda_2(x_1,\ldots,x_n))$, $\mathbf{x} = (x_1,\ldots,x_n)$.

Definition 3.3. Orness ω of a GCD aggregator λ is an indicator of similarity between λ and disjunction $x_1 \vee \cdots \vee x_n$. Orness is computed using a similarity metric Ω that satisfies the following minimum conditions of standard range and monotonicity:

(a) $$\Omega = \Omega(\mathbf{x}, \lambda(x_1,\ldots,x_n)) = \begin{cases} 0, & \lambda(x_1,\ldots,x_n) = x_1 \wedge \cdots \wedge x_n \\ 1/2, & \lambda(x_1,\ldots,x_n) = (x_1 + \cdots + x_n)/n \\ 1, & \lambda(x_1,\ldots,x_n) = x_1 \vee \cdots \vee x_n \end{cases}$$

(b) If $\lambda_1(x_1,\ldots,x_n) < \lambda_2(x_1,\ldots,x_n)$ then $\Omega(\mathbf{x}, \lambda_1(x_1,\ldots,x_n)) < \Omega(\mathbf{x}, \lambda_2(x_1,\ldots,x_n))$.

There are various "metrics of similarity" and various ways to define andness and orness [12]. In all cases we assume that andness and orness are defined as complementary indicators:

Definition 3.4. Andness α and orness ω of a GCD aggregator λ are said to be *complementary* if $A(\mathbf{x}, \lambda(x_1,\ldots,x_n)) + \Omega(\mathbf{x}, \lambda(x_1,\ldots,x_n)) = b = \text{const}$. (We assume $b = 1$ except for symmetric global andness/orness [12] where $b = 0$.)

Definition 3.5. Andness α and orness ω of a GCD aggregator λ are said to be *global* if $A(\mathbf{x}, \lambda(x_1,\ldots,x_n)) = 1 - \Omega(\mathbf{x}, \lambda(x_1,\ldots,x_n)) = \text{const}$. Otherwise, if α and ω are defined so that they are functions of $x_1,\ldots,x_n$, such andness and orness are said to be *local*. In situations where it is necessary to differentiate the type of andness and orness the local andness and orness are denoted α_ℓ, ω_ℓ, and the global andness and orness are denoted α_g, ω_g.

Definition 3.6. A GCD aggregator λ is said to be *parameterized* if it uses a parameter r and satisfies the following conditions of range and monotonicity:

(a) $$\lambda(x_1,\ldots,x_n; r) = \begin{cases} x_1 \vee \cdots \vee x_n, & r = r_d \\ (x_1 + \cdots + x_n)/n, & r = r_a \\ x_1 \wedge \cdots \wedge x_n, & r = r_c \end{cases}$$

(b) There is a path of continuous transition from conjunction to disjunction:
- If $r_c \leqslant r_1 < r_2 \leqslant r_d$ then $\lambda(x_1,\dots,x_n;r_1) \leqslant \lambda(x_1,\dots,x_n;r_2)$
- If $r_d \leqslant r_1 < r_2 \leqslant r_c$ then $\lambda(x_1,\dots,x_n;r_1) \geqslant \lambda(x_1,\dots,x_n;r_2)$
(Assumption: $x_1 \wedge \cdots \wedge x_n \neq x_1 \vee \cdots \vee x_n$)

For each value of r we can find the corresponding andness $\alpha = A(\lambda(x_1,\dots,x_n;r))$ and orness $\omega = \Omega(\lambda(x_1,\dots,x_n;r))$. In cases where r can be computed from the desired value of α or ω the aggregators can be written directly as parameterized functions of orness (or andness, assuming $b = 1$): $\lambda(x_1,\dots,x_n;\omega) = \lambda(x_1,\dots,x_n;1-\alpha)$. So, the basic properties of a parameterized GCD function $\lambda(x_1,\dots,x_n;\omega)$, $\omega \in I$, can be summarized as follows:

(a) Reduction to disjunction: $\lambda(x_1,\dots,x_n;1) = x_1 \vee \dots \vee x_n$.
(b) Reduction to conjunction: $\lambda(x_1,\dots,x_n;0) = x_1 \wedge \dots \wedge x_n$.
(c) Reduction to arithmetic mean: $\lambda(x_1,\dots,x_n;1/2) = (x_1 + \cdots + x_n)/n$.
(d) Internality (BQR): $x_1 \wedge \cdots \wedge x_n \leqslant \lambda(x_1,\dots,x_n;\omega) \leqslant x_1 \vee \cdots \vee x_n$.
(e) Idempotency: $\lambda(x,\dots,x;\omega) = x$.
(f) Continuity: $\lim_{h\to 0} \lambda(x_1,\dots,x_i+h,\dots,x_n;\omega) = \lambda(x_1,\dots,x_i,\dots,x_n;\omega), \quad i = 1,\dots,n$.
(g) Commutativity (symmetry): $\lambda(x_1,\dots,x_n;\omega)$ is not changed if the elements of $(x_1,\dots,x_n)$ are permuted. This property is only a special case, since generally the elements of $(x_1,\dots,x_n)$ can have different importance and in such cases the commutativity is not an acceptable property.
(h) Sensitivity to improvements (a system can be improved if we improve any of its components): $\frac{\partial}{\partial x_i}\lambda(x_1,\dots,x_n;\omega) > 0, 0 < \omega < 1, \quad 0 < x_i < 1, \quad i = 1,\dots,n$.
(i) Sensitivity to orness (higher orness means more replaceability and more chances to satisfy requirements): $\frac{\partial}{\partial \omega}\lambda(x_1,\dots,x_n;\omega) > 0, 0 < \omega < 1, 0 < x_i < 1, i = 1,\dots,n$, $x_1 \wedge \cdots \wedge x_n \neq x_1 \vee \cdots \vee x_n$.
(j) Sensitivity to andness (higher andness means less replaceability and less chances to satisfy requirements): $\frac{\partial}{\partial \alpha}\lambda(x_1,\dots,x_n;\omega) < 0, 0 < \omega < 1, 0 < x_i < 1, i = 1,\dots,n$, $x_1 \wedge \cdots \wedge x_n \neq x_1 \vee \cdots \vee x_n$.

GCD is a mix of conjunctive and disjunctive properties that depend on the value of ω. We use the name *andor*, or *partial conjunction*, and symbol $\triangle$, for $0 < \omega < 0.5$, $(\alpha > \omega)$. The name *orand*, or *partial disjunction*, and symbol ∇, are used for $0.5 < \omega < 1$, $(\alpha < \omega)$. We also use a general symbol $\diamondsuit$ for a general GCD operator that includes the pure conjunction, partial conjunction, arithmetic mean, partial disjunction, and the pure disjunction:

$$y = x_1 \diamondsuit \cdots \diamondsuit x_n = \begin{cases} x_1 \vee \cdots \vee x_n, & \alpha = 0, \omega = 1 \\ x_1 \nabla \cdots \nabla x_n, & 0 < \alpha < 0.5, 0.5 < \omega < 1 \\ (x_1 + \cdots + x_n)/n, & \alpha = \omega = 0.5 \\ x_1 \triangle \cdots \triangle x_n, & 0.5 < \alpha < 1, 0 < \omega < 0.5 \\ x_1 \wedge \cdots \wedge x_n, & \alpha = 1, \omega = 0 \end{cases}$$

We assume that $\diamondsuit$ corresponds to a specific level of orness; so $y = \lambda(x_1,\dots,x_n;\omega)$ and $y = x_1 \diamondsuit \dots \diamondsuit x_n$ are equivalent notations.

Associativity (e.g. $(x_1 \diamondsuit x_2 \diamondsuit x_3) = (x_1 \diamondsuit x_2) \diamondsuit x_3 = x_1 \diamondsuit (x_2 \diamondsuit x_3)$) is a desirable algebraic property (e.g. it helps reducing the errors in weight assessment by grouping input preferences in small groups of up to 5 inputs [10]). Similarly, transfigurations of preference aggregation structures are possible if the partial conjunction and the partial disjunction are mutually distributive (e.g. $x_1 \triangle (x_2 \nabla x_3) = (x_1 \triangle x_2) \nabla (x_1 \triangle x_3)$, $x_1 \nabla (x_2 \triangle x_3) = (x_1 \nabla x_2) \triangle (x_1 \nabla x_3)$). GCD functions based on means do not satisfy algebraic properties of distributivity and associativity. According to [7], the errors (defined as mean values of absolute differences, e.g as the mean of $|x_1 \triangle (x_2 \nabla x_3) - (x_1 \triangle x_2) \nabla (x_1 \triangle x_3)|$ in I^3) are below 2%.This level of errors is not significant because errors in estimating preferences, andness, and weights are regularly significantly larger [10].

4. Means as logic functions

All means satisfy the fundamental GCD property of BQR or internality: $x_1 \wedge \cdots \wedge x_n \leqslant \lambda(x_1, \ldots, x_n; \omega) \leqslant x_1 \vee \cdots \vee x_n$. Therefore, some means can be interpreted as logic functions and the GCD can be organized as a mean. Of course, the theory of means offers a wide spectrum of candidate mathematical models [13,18,3], and the question is which mean is the best material for building the GCD function. Obviously, the most suitable are those means that have weights and adjustable parameters enabling easy adjustment of orness/andness and continuous transition from the pure conjunction to the pure disjunction.

We investigated this problem in [5,6] using a general framework of Bajraktarević means (BM) [1,3]:

$$\lambda(x_1, \ldots, x_n) = F^{-1}\left(\sum_{i=1}^{n} w_i(x_i) F(x_i) / \sum_{i=1}^{n} w_i(x_i)\right)$$

BM uses weight functions $w_i : I \to \{0\} \cup R^+$ and a strictly monotone generator function $F : I \to R$. In a special case of normalized constant weights we have $w_i(x_i) = W_i, W_i > 0, \quad i = 1, \ldots, n, W_1 + \cdots + W_n = 1$, and BM reduces to the weighted quasi-arithmetic mean (QAM):

$$\lambda(x_1, \ldots, x_n) = F^{-1}\left(\sum_{i=1}^{n} W_i F(x_i)\right)$$

Of course, even in this simplified case, there is a spectrum of possible generator functions that yield a variety of GCD properties.

GCD must have an adjustable parameter that enables a well-controlled continuous transition between the pure conjunction and the pure disjunction. This condition simplifies the selection of the generator function F.

The simplest form of the generator function is the power function $F(x) = x^r, r \in R$. This selection yields the weighted power means (WPM) model of GCD [13,3]:

$$M(x_1, \ldots, x_n; r) = \begin{cases} \left(\sum_{i=1}^{n} W_i x_i^r\right)^{1/r}, & 0 < |r| < +\infty \\ \prod_{i=1}^{n} x_i^{W_i}, & r = 0 \\ x_1 \wedge \cdots \wedge x_n, & r = -\infty \\ x_1 \vee \cdots \vee x_n, & r = +\infty \end{cases}$$

$$\frac{\partial}{\partial x_i}M(x_1,\dots,x_n;r) \geqslant 0, \quad i=1,\dots,n$$
$$\frac{\partial}{\partial r}\left(\sum_{i=1}^{n} W_i x_i^r\right)^{1/r} > 0, \quad x_i > 0, \quad i=1,\dots,n, \quad x_1 \wedge \dots \wedge x_n \neq x_1 \vee \dots \vee x_n, \quad |r| < +\infty$$
$$\frac{\partial}{\partial r}\left(\sum_{i=1}^{n} W_i x_i^r\right)^{1/r} = 0, \quad x_1 = \ldots = x_n$$

An advantage of WPM is that their properties are well known in mathematics, because their special cases: harmonic mean ($r=-1$), geometric mean ($r=0$), arithmetic mean ($r=1$), and quadratic mean ($r=2$). In addition, WPM has an important property, $M(x_1,\dots,x_n;r)=0$, $r \leqslant 0$, $x_i=0$, $i \in \{1,\dots,n\}$, that is indispensable for modeling mandatory requirements, the partial absorption function [8], and other compound aggregators [11].

A parameterized GCD model that is related to WPM can be based on counter-harmonic means:

$$H(x_1,\dots,x_n;r) = \begin{cases} x_1 \vee \dots \vee x_n, & r=+\infty \\ \dfrac{\sum_{i=1}^{n} W_i x_i^r}{\sum_{i=1}^{n} W_i x_i^{r-1}}, & 0 \leqslant |r| < +\infty \\ x_1 \wedge \dots \wedge x_n, & r=-\infty \end{cases}$$

Important special cases are: $H(x_1,\dots,x_n;1)=M(x_1,\dots,x_n;1), H(x_1,\dots,x_n;0)=M(x_1,\dots,x_n;-1)$, and (for equal weights and $n=2$) $H(x_1,x_2;1/2)=M(x_1,x_2;0)$.

Another parameterized GCD model can be obtained from QAM using the exponential generator function $F(x)=e^{rx}, r \in R$. This is the weighted exponential mean (WEM):

$$E(x_1,\dots,x_n;r) = \begin{cases} \frac{1}{r}\ln\left(\sum_{i=1}^{n} W_i \mathrm{e}^{rx_i}\right), & 0<|r|<+\infty \\ \sum_{i=1}^{n} W_i x_i, & r=0 \\ x_1 \wedge \dots \wedge x_n, & r=-\infty \\ x_1 \vee \dots \vee x_n, & r=+\infty \end{cases}$$

For the above M, H and E means, the andness and orness are functions of the parameter r, and $\partial\omega/\partial r > 0$, $\partial\alpha/\partial r < 0$.

In the case of two positive variables x and y, a continuous transition from conjunction to disjunction can also be realized using the generalized logarithmic means:

$$L(x,y;r) = \begin{cases} \left(\dfrac{y^{r+1}-x^{r+1}}{(r+1)(y-x)}\right)^{1/r}, & r \neq -1,0,\pm\infty \\ \dfrac{y-x}{\log y - \log x}, & r=-1 \\ \dfrac{1}{e}\left(\dfrac{y^y}{x^x}\right)^{1/(y-x)}, & r=0 \\ x \wedge y, & r=-\infty \\ x \vee y, & r=+\infty \end{cases}$$

By definition, $L(x,x;r)=x$. Easily verifiable special cases are: $L(x,y;1)=M(x,y;1), L(x,y;-2)=M(x,y;0)$, and $L(x,y;-1/2)=M(x,y;1/2)$.

5. Local and global andness/orness

Let us first consider a case of two variables and the GCD function $y=x_1\diamondsuit x_2$. The andness α is a measure of similarity between the GCD function and the full conjunction. Similarly, the orness ω is a measure of similarity between the GCD function and the full disjunction. The *local* orness and andness, introduced in [5], are defined at any given point (x_1,x_2) as follows:

$$\alpha_\ell(x_1,x_2)=\frac{(x_1\vee x_2)-(x_1\diamondsuit x_2)}{(x_1\vee x_2)-(x_1\wedge x_2)},\quad 0\leqslant\alpha_\ell(x_1,x_2)\leqslant 1$$
$$\omega_\ell(x_1,x_2)=\frac{(x_1\diamondsuit x_2)-(x_1\wedge x_2)}{(x_1\vee x_2)-(x_1\wedge x_2)},\quad 0\leqslant\omega_\ell(x_1,x_2)\leqslant 1$$
$$\alpha_\ell(x_1,x_2)+\omega_\ell(x_1,x_2)=1,\quad (x_1\neq x_2)$$

From this definition we have

$$\begin{aligned}(x_1\diamondsuit x_2)&=\alpha_\ell(x_1,x_2)(x_1\wedge x_2)+\omega_\ell(x_1,x_2)(x_1\vee x_2)\\&=\alpha_\ell(x_1,x_2)(x_1\wedge x_2)+[1-\alpha_\ell(x_1,x_2)](x_1\vee x_2)\\&=[1-\omega_\ell(x_1,x_2)](x_1\wedge x_2)+\omega_\ell(x_1,x_2)(x_1\vee x_2)\end{aligned}$$

Therefore, GCD can be interpreted as a combination of conjunction and disjunction.

Local andness and orness are not constant and decision makers cannot adjust nontrivial andness and orness functions for each aggregation block. In system evaluation practice, decision makers can only specify desired global or average levels of andness and orness. The simplest *global* measure of andness and orness is the mean value of local andness and orness $\bar{\alpha}_\ell$ and $\bar{\omega}_\ell$ (in this section $\bar{x}$ denotes the mean value of x; in all other sections $\bar{x}$ denotes the negation of x):

$$\bar{\alpha}_\ell=\int_0^1 \mathrm{d}x_1\int_0^1\alpha_\ell(x_1,x_2)\,\mathrm{d}x_2,\quad \bar{\omega}_\ell=\int_0^1 \mathrm{d}x_1\int_0^1\omega_\ell(x_1,x_2)\,\mathrm{d}x_2$$

For example, if the GCD is realized as a harmonic mean, $y=x_1\diamondsuit x_2=2x_1x_2/(x_1+x_2)$, the mean andness and orness are:

$$\begin{aligned}\bar{\alpha}_\ell&=\int_0^1 \mathrm{d}x_1\int_0^1\frac{(x_1\vee x_2)-2x_1x_2/(x_1+x_2)}{(x_1\vee x_2)-(x_1\wedge x_2)}\mathrm{d}x_2=2\int_0^1 \mathrm{d}x_1\int_{x_1}^1\frac{x_2-2x_1x_2/(x_1+x_2)}{x_2-x_1}\mathrm{d}x_2\\&=2\int_0^1 \mathrm{d}x_1\int_{x_1}^1\frac{x_2}{x_1+x_2}\mathrm{d}x_2=\ln(2)=0.693\\\bar{\omega}_\ell&=1-\bar{\alpha}_\ell=1-\ln(2)=0.307\end{aligned}$$

The cases where $\bar{\alpha}_\ell$ and $\bar{\omega}_\ell$ can be analytically computed in a closed form are very rare. Usually, the values of $\bar{\alpha}_\ell$ and $\bar{\omega}_\ell$ must be obtained, with significant difficulties, using numerical integration.

Instead of mean values of local andness/orness, the global andness/orness can be defined as the andness/orness of means, introduced in [6]:

$$\alpha_g = \frac{\overline{x_1 \vee x_2} - \overline{x_1 \diamondsuit x_2}}{\overline{x_1 \vee x_2} - \overline{x_1 \wedge x_2}}, \quad \omega_g = \frac{\overline{x_1 \diamondsuit x_2} - \overline{x_1 \wedge x_2}}{\overline{x_1 \vee x_2} - \overline{x_1 \wedge x_2}} = 1 - \alpha_g$$

In the case of the harmonic mean we have

$$\overline{x_1 \diamondsuit x_2} = 2\int_0^1 \mathrm{d}x_1 \int_0^1 \frac{x_1 x_2}{x_1 + x_2}\,\mathrm{d}x_2 = \frac{4 - \ln 16}{3}$$
$$\overline{x_1 \wedge x_2} = \int_0^1 \mathrm{d}x_1 \int_0^1 (x_1 \wedge x_2)\,\mathrm{d}x_2 = \int_0^1 \mathrm{d}x_1 \left(\int_0^{x_1} x_2\,\mathrm{d}x_2 + \int_{x_1}^1 x_1\,\mathrm{d}x_2\right) = \frac{1}{3}$$
$$\overline{x_1 \vee x_2} = \int_0^1 \mathrm{d}x_1 \int_0^1 (x_1 \vee x_2)\,\mathrm{d}x_2 = \int_0^1 \mathrm{d}x_1 \left(\int_0^{x_1} x_1\,\mathrm{d}x_2 + \int_{x_1}^1 x_2\,\mathrm{d}x_2\right) = \frac{2}{3}$$
$$\alpha_g = \frac{\overline{x_1 \vee x_2} - \overline{x_1 \diamondsuit x_2}}{\overline{x_1 \vee x_2} - \overline{x_1 \wedge x_2}} = 2 - 3(\overline{x_1 \diamondsuit x_2}) = \ln 16 - 2 = 0.7726$$
$$\omega_g = 1 - \alpha_g = 3(\overline{x_1 \diamondsuit x_2}) - 1 = 3 - \ln 16 = 0.2274$$

Therefore, the global andness/orness of means can be different from the mean value of local andness/orness, but the differences are moderate.

In the important special case of geometric mean the global andness and orness are

$$\overline{x_1 \diamondsuit x_2} = \int_0^1 \mathrm{d}x_1 \int_0^1 \sqrt{x_1 x_2}\,\mathrm{d}x_2 = \int_0^1 \sqrt{x_1}\,\mathrm{d}x_1 \int_0^1 \sqrt{x_2}\,\mathrm{d}x_2 = \frac{4}{9}$$
$$\alpha_g^{(\mathrm{geo})} = 2 - 3(\overline{x_1 \diamondsuit x_2}) = \frac{2}{3}, \quad \omega_g^{(\mathrm{geo})} = 3(\overline{x_1 \diamondsuit x_2}) - 1 = \frac{1}{3}$$

In the case of n variables the global andness and orness are defined as follows [6,7]:

$$\overline{x_1 \wedge \cdots \wedge x_n} = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \ldots \int_0^1 (x_1 \wedge \cdots \wedge x_n)\,\mathrm{d}x_n = \frac{1}{n+1}$$
$$\overline{x_1 \vee \cdots \vee x_n} = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \ldots \int_0^1 (x_1 \vee \cdots \vee x_n)\,\mathrm{d}x_n = \frac{n}{n+1}$$
$$\omega_g = \frac{\overline{x_1 \diamondsuit \cdots \diamondsuit x_n} - \overline{x_1 \wedge \cdots \wedge x_n}}{\overline{x_1 \vee \cdots \vee x_n} - \overline{x_1 \wedge \cdots \wedge x_n}} = \frac{(n+1)(\overline{x_1 \diamondsuit \cdots \diamondsuit x_n}) - 1}{n-1} = 1 - \alpha_g$$
$$\alpha_g = \frac{\overline{x_1 \vee \cdots \vee x_n} - \overline{x_1 \diamondsuit \cdots \diamondsuit x_2}}{\overline{x_1 \vee \cdots \vee x_n} - \overline{x_1 \wedge \cdots \wedge x_n}} = \frac{n - (n+1)(\overline{x_1 \diamondsuit \cdots \diamondsuit x_n})}{n-1} = 1 - \omega_g$$

These definitions show that α_g and ω_g depend on n. For example, in the case of geometric mean we have:

$$\overline{(x_1 x_2 \cdots x_n)^{1/n}} = \int_0^1 x_1^{1/n}\,\mathrm{d}x_1 \cdots \int_0^1 x_n^{1/n}\,\mathrm{d}x_n = \left(\frac{n}{n+1}\right)^n$$
$$\alpha_g^{(\mathrm{geo})}(n) = \frac{n}{n-1}\left[1 - \left(\frac{n}{n+1}\right)^{n-1}\right], \quad \alpha_g^{(\mathrm{geo})}(2) = \frac{2}{3} = 0.667, \quad \lim_{n\to\infty} \alpha_g^{(\mathrm{geo})}(n) = 1 - \frac{1}{e} = 0.632$$
$$\omega_g^{(\mathrm{geo})}(n) = \frac{n}{n-1}\left[\left(\frac{n}{n+1}\right)^{n-1} - \frac{1}{n}\right], \quad \omega_g^{(\mathrm{geo})}(2) = \frac{1}{3} = 0.333, \quad \lim_{n\to\infty} \omega_g^{(\mathrm{geo})}(n) = \frac{1}{e} = 0.368$$

This limit is important because of the following property that holds for WPM: if $\alpha_g(n) \geqslant \alpha_g^{(\text{geo})}(n)$ and $x_i = 0$, $i \in \{1,\ldots,n\}$, then $\lambda(x_1,\ldots,x_n;\alpha_g(n)) = 0$. In other words, if the andness is greater than or equal to the limit andness of the geometric mean $\alpha_g^{(\text{geo})}(n)$, then all inputs become mandatory: a single zero input preference causes the zero output preference.

In the cases of power means and exponential means α_g and ω_g depend on both r and n: $\alpha_g = \phi(n,r)$, $\omega_g = \psi(n,r)$. Similarly, r can be computed from desired values of α_g or ω_g: $r = \rho_n(\alpha_g) = \rho_n(1-\omega_g)$. So, $\phi(n,\rho_n(\alpha_g)) = \alpha_g$, $\psi(n,\rho_n(1-\omega_g)) = \omega_g$.

Numeric values of parameter r for $n = 2$ and nine characteristic levels of global andness and orness, for power and exponential means, are shown in Table 1. In the case of power mean the values of r can be computed from the desired orness using the following numeric approximation: $r = (-0.742 + 3.363\omega_g - 4.729\omega_g^2 + 3.937\omega_g^3)/[(1-\omega_g)\omega_g], \quad 0 \leqslant \omega_g \leqslant 1$.

The exponential mean has a suitable symmetric property $\rho_2(\alpha_g) = -\rho_2(1-\alpha_g)$. Exponential means are an alternative to the power means in cases where the mandatory/sufficient properties (analyzed more precisely in Section 7) are not desirable.

OWA operators [20,22] and ItOWA operators [9,11] are specific forms of GCD that use positional weights and their specific concepts of andness/orness. They are outlined in Section 10.

An important form of global andness/orness is related to quasi-arithmetic means. The logic properties of the quasi-arithmetic mean $x_1 \Diamond \ldots \Diamond x_n = F^{-1}(n^{-1}\sum_{i=1}^{n} F(x_i))$ depend on the convexity/concavity and the increasing/decreasing properties of the generating function F. If $F'(x) > 0$ and $F''(x) > 0$, then function F acts as a nonlinear amplifier that amplifies large values of x and attenuates small values of x. Consequently, the sum $\sum_{i=1}^{n} F(x_i)$ is predominantly affected by the large values of x_i, and this is exactly the desired property of the *orand* function. Increasing the convexity of F increases the orness of the corresponding *orand* function. This gives a possibility to interpret the level of convexity of F as a global orness. According to [12], to realize the *orand* function (partial disjunction) the generating function F must satisfy the condition $F'(x)F''(x) > 0, \quad x \in I$. To realize the *andor* function (partial conjunction) the generating function F must satisfy the condition $F'(x)F''(x) < 0, \quad x \in I$. In both cases the *global generator function andness/orness* are defined as the property of the generator function:

Table 1
Characteristic values $r = \rho_2(\alpha_g)$ for power means and exponential means

Symbol	Global andness		Global orness		r for power means	r for exponential means
	Level	α_g	Level	ω_g		
D	Lowest	0	Highest	1	$+\infty$	$+\infty$
D+	Very low	0.125	Very high	0.875	9.53	14.0
DA	Low	0.25	High	0.75	3.93	5.40
D−	Medium low	0.375	Medium high	0.625	2.02	2.14
A	Medium	0.5	Medium	0.5	1	0
C−	Medium high	0.625	Medium low	0.375	0.26	−2.14
CA	High	0.75	Low	0.25	−0.72	−5.40
C+	Very high	0.875	Very low	0.125	−3.51	−14.0
C	Highest	1	Lowest	0	$-\infty$	$-\infty$

$$\alpha_F = \frac{\int_0^1 F(x)\,dx - F(0)}{F(1) - F(0)}, \quad \omega_F = \frac{F(1) - \int_0^1 F(x)\,dx}{F(1) - F(0)}$$

These formulas hold if additional conditions are satisfied: $F(x)$ must have finite values for $x \in I$, and satisfy boundary conditions related to conjunction, disjunction and the arithmetic mean; these details can be found in [12]. For example, if $F(x) = x^r, r \geqslant 1$, then we have

$$\omega_F(r) = \frac{r}{r+1}, \quad 1 \leqslant r \leqslant +\infty, \quad \omega_F(1) = \frac{1}{2}, \quad \omega_F(+\infty) = 1$$

This relation between the orness ω and the exponent r was first proposed by Larsen in the context of AIWA operators [17]:

$$r = \frac{\omega}{\alpha} = \frac{1-\alpha}{\alpha} = \frac{\omega}{1-\omega}, \quad \omega \geqslant 1/2 \quad (orand)$$

Larsen's orness is close to the global orness of means: $\omega_F(9.53) = 0.905$, $\omega_F(3.93) = 0.797, \omega_F(2.02) = 0.669$ (cf. Table 1).

The global generator function andness/orness formulas can also be used for $F(x) = e^{rx}$ in the whole range $-\infty \leqslant r \leqslant +\infty$:

$$\alpha_F(r) = \frac{\int_0^1 e^{rx}\,dx - 1}{e^r - 1} = \frac{e^r - 1 - r}{r(e^r - 1)}, \quad \omega_F(r) = \frac{(r-1)e^r + 1}{r(e^r - 1)}$$
$$\alpha_F(r) + \omega_F(r) = \alpha_F(r) + \alpha_F(-r) = \omega_F(r) + \omega_F(-r) = 1$$

Generally, andness and orness can be defined in various ways and 9 of them are surveyed in [12]. The precision of decision makers in specifying the desired level of andness/orness is limited (according to [10] the expected andness/orness errors are in the range 6–10%). Practical system evaluators can be trained to use various definitions of andness/orness. Consequently, different definitions of andness/orness might have similar applicability. The traditional global andness/orness of means seems still to be the most suitable, because it is simple and robust, and applicable for all forms of quasi-arithmetic and Bajraktarević means. The mean of local andness is less convenient than the global andness/orness of means, both analytically and numerically. The definitions of andness/orness based on generating functions of quasi-arithmetic means have a practical advantage that they do not depend on n; unfortunately, they are not equally applicable for all forms of the generator function.

6. Weights and relative importance

In the Multicriteria Decision Making (MCDM) area, there is no consensus on the meaning of weights. Choo et al. [4] identify 13 different interpretations of weights in MCDM. These interpretations include weights as degrees of relative importance of component criteria, weights as the level of confidence, the level of evaluator's expertise, etc.

In GCD models for system evaluation and information retrieval, weights are used to express relative importance of input preferences, and this interpretation is one of fundamental observable properties of human reasoning [11]. The relative importance is usually an adjustable constant parameter. In a general case, however, weights can depend on preferences and BM [1,3] provides a convenient mechanism for realizing this property.

In GCD models, we assume that weights and andness/orness are independent parameters. Indeed, in system evaluation practice, evaluators independently think about the relative importance of individual inputs, and then about the desired level of their simultaneity (andness). For example, in the final stage of computer evaluation the global preference y may be a function of preferences that reflect the quality of hardware, software, measured performance, and vendor support (x_h, x_s, x_p, x_v). The evaluator could first investigate the relative importance of these inputs and decide to express the relative importance using weights $W_h = 0.4, W_s = 0.3, W_p = 0.2, W_v = 0.1$ After selecting the weights, the next (independent) step is to select the appropriate level of andness. If this level is selected to be high (aggregator CA in Table 1, $r = -0.72$) then the resulting GCD for preference aggregation is $y = (0.4x_h^{-0.72} + 0.3x_s^{-0.72} + 0.2x_p^{-0.72} + 0.1x_v^{-0.72})^{-1/0.72}$.

In this area, we sometimes encounter inconsistency between a low weight and a high andness. The low weight is interpreted as a low importance. However, the high andness means the requirement for high simultaneity, which indirectly means that all components are necessary and consequently very important. So, a low weight (e.g. less than 5%) and a high andness (e.g. more than 75%) may sometimes represent a contradiction, and should be avoided.

6.1. A contributor–aggregator form of GCD

Let $x_i \in I$ be a preference and let $W_i \in I$ be the corresponding weight (relative importance). The GCD aggregation of n preferences can be expressed in the following *contributor–aggregator* form:

$$y = A(C(x_1, W_1), \ldots, C(x_n, W_n)), \quad C : I^2 \to R, \quad A : R^n \to I$$

The function $C_i = C(x_i, W_i)$ is the contributor function of the ith preference; it defines the relative weighted contribution of the ith input. E.g., in the case of weighted arithmetic mean (WAM), weighted geometric mean (WGM), WPM, WEM, and QAM, we have weights that are normalized to sum 1, and the following CA forms:

$$\begin{aligned}
&C_i = C(x_i, W_i) = W_i x_i, \quad A(C_1, \ldots, C_n) = C_1 + \cdots + C_n \\
&C_i = C(x_i, W_i) = x_i^{W_i}, \quad A(C_1, \ldots, C_n) = C_1 \times \cdots \times C_n \\
&C_i = C(x_i, W_i) = W_i x_i^r, \quad A(C_1, \ldots, C_n) = (C_1 + \cdots + C_n)^{1/r} \\
&C_i = C(x_i, W_i) = W_i e^{rx_i}, \quad A(C_1, \ldots, C_n) = \frac{1}{r} \ln(C_1 + \cdots + C_n) \\
&C_i = C(x_i, W_i) = W_i F(x_i), \quad A(C_1, \ldots, C_n) = F^{-1}(C_1 + \cdots + C_n) \\
&0 < W_i < 1, \quad i = 1, .., n, \quad W_1 + \cdots + W_n = 1
\end{aligned}$$

These examples illustrate some of the many ways to realize contributors and aggregators. The contributor/aggregator approach to GCD organization is close to the intuitive evaluation process, where evaluators first focus on each individual attribute and its relative importance, and then aggregate all contributions to get the global preference for the evaluated system as a whole.

6.2. Multiplicative, exponential, and implicative weights, and their normalization

In decision models weights are usually normalized using one of two basic forms of normalization:

- Sum-normalized weights satisfy the condition

$$0 < W_i < 1, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} W_i = 1$$

- Max-normalized weights satisfy the condition

$$w_i \in I, \quad i = 1, \ldots, n, \quad \max(w_1, \ldots, w_n) = 1$$

The condition $0 < W_i < 1$ assumes that no weight can be zero (because this excludes the corresponding component from consideration). Consequently, sum-normalized weights cannot have the value 1 because that would force all other weights to be zero. As opposed to that, max-normalized weights must include at least one weight that has the value 1.

Two frequently used contributor models are the multiplicative model $C(x_i, W_i) = W_i t(x_i)$, and the exponential model: $C(x_i, W_i) = t(x_i)^{W_i}$, where the function $t: I \to R$ denotes a selected (usually nonlinear) transformation (e.g. the exponential function in WEM, or the power function is WPM).

The multiplicative contributor model is used in all means that can be derived from BM (e.g. in power means and in exponential means), where normalized weights multiply satisfaction degrees. In the case of WAM, $\partial y/\partial x_i = W_i$ and weights denote importance and determine both the level of penalty for a low satisfaction and the level of reward for a high satisfaction of corresponding criteria. High level of penalty/reward for an input can obviously mean only one thing: the input is important. This is a general concept of multiplicative weights that naturally extends to GCD based on the whole family of BM:

$$x_1 \diamondsuit \cdots \diamondsuit x_n = F^{-1}\left(\sum_{i=1}^{n} W_i(x_1, \ldots, x_n) F(x_i)\right)$$

$$W_i(x_1, \ldots, x_n) = w_i(x_i) \Big/ \sum_{j=1}^{n} w_j(x_j)$$

$$0 \leqslant W_i(x_1, \ldots, x_n) \leqslant 1, \quad \sum_{i=1}^{n} W_i(x_1, \ldots, x_n) = 1$$

Similar effect is achieved by the exponential weights in WGM: small weights cause contributors close to one, and such contributors have insignificant effect on the multiplicative aggregator. For high weights the effect of preferences on the value of aggregator is more significant.

The idea of exponential weights is related to logarithmic sensitivity. In the case of QAM we have

$$F(y) = \sum_{i=1}^{n} W_i F(x_i), \quad \frac{\partial F(y)}{\partial x_i} = \frac{\partial F(y)}{\partial y}\frac{\partial y}{\partial x_i} = W_i \frac{\partial F(x_i)}{\partial x_i}$$

$$W_i = \frac{\partial F(y)}{\partial y}\frac{\partial y}{\partial x_i} \Big/ \frac{\partial F(x_i)}{\partial x_i} = \begin{cases} \dfrac{\partial y}{\partial x_i}, & \text{if } F(x) = x \\ \dfrac{\partial \ln y}{\partial \ln x_i} = \dfrac{x_i}{y}\dfrac{\partial y}{\partial x_i} \cong \dfrac{\Delta y}{y} \Big/ \dfrac{\Delta x_i}{x_i}, & \text{if } F(x) = \ln x \end{cases}$$

Therefore, if the QAM generator function is linear then weights represent linear sensitivity coefficients (the ratio between an absolute increment in x and the corresponding absolute increment in y). If the QAM generator function is logarithmic then we have exponential weights that represent logarithmic sensitivity coefficients (the ratio between a relative increment in x and the corresponding relative increment in y).

Another significant form of contributor function is the implicative form that uses max-normalized weights and the transformation function t: $I \to I$, as follows:

$$C(x_i, w_i) = (w_i \Rightarrow t(x_i)) = 1 - [w_i \wedge (1 - t(x_i))]$$
$$= (1 - w_i) \vee t(x_i) = \begin{cases} 1 - w_i, & 0 \leqslant t(x_i) \leqslant 1 - w_i \\ t(x_i), & 1 - w_i \leqslant t(x_i) \leqslant 1 \end{cases}$$
$$w_i \in [0, 1], \quad \max(w_1, \ldots, w_n) = 1, \quad 1 \leqslant \sum_{i=1}^{n} w_i \leqslant n$$

The implicative approach is based on the concept that it is not acceptable that a requirement (criterion) is important and it has low satisfaction. In other words, if a criterion is important, it must be satisfied. In the simplest interpretation, this concept is an implication from weight (relative importance) to preference (degree of satisfaction): $w_i \Rightarrow x_i$, $w_i \in I$, $x_i \in I$.

In the case of multiplicative weights the weights are normalized so that their sum is 1. In the case of implicative weights, however, the weights are normalized so that the maximum weight is 1. Therefore, in the case of equal importance, all multiplicative weights are $1/n$, and all implicative weights are 1.

If $t(x_i) = x_i$, and the aggregator is a pure *and* (cf. [16]), we get the weighted conjunction:

$$y_\wedge = (w_1 \Rightarrow x_1) \wedge \cdots \wedge (w_n \Rightarrow x_n) = (\overline{w_1} \vee x_1) \wedge \cdots \wedge (\overline{w_n} \vee x_n)$$

Using duality $\overline{y_\vee} = y_\wedge(\overline{x_1}, \ldots, \overline{x_n})$ we get the weighted disjunction:

$$y_\vee = \overline{(\overline{w_1} \vee \overline{x_1}) \wedge \cdots \wedge (\overline{w_n} \vee \overline{x_n})} = (w_1 \wedge x_1) \vee \cdots \vee (w_n \wedge x_n)$$

6.3. Implicative importance weighted aggregation between AND and OR

AIWA operators [17] can be interpreted as a form of GCD, and $y_\wedge$ and $y_\vee$ are special cases of AIWA for andness 1 and 0 respectively. In the case of AIWA, conjunction and disjunction in expressions $\overline{w_i} \vee x_i$ and $w_i \wedge x_i$ are implemented as the dual t-conorms and t-norms algebraic sum and product, i.e. the Reichenbach implication, $\overline{w_i} \vee x_i = 1 - w_i(1 - x_i)$, and $w_i \wedge x_i = w_i x_i$. The AIWA operators provide implicative importance weighting (in [17] just called "importance weighting") for all degrees of andness in I:

$$h_\alpha(\mathbf{w}, \mathbf{x}) = \begin{cases} \left(\dfrac{\sum_{i=1}^{n} (w_i x_i)^r}{\sum_{i=1}^{n} w_i^r} \right)^{1/r} & \alpha \leqslant \frac{1}{2} \\ 1 - \left(\dfrac{\sum_{i=1}^{n} (w_i (1 - x_i))^{1/r}}{\sum_{i=1}^{n} w_i^{1/r}} \right)^r & \alpha \geqslant \frac{1}{2} \end{cases}$$

with $r = \frac{1}{\alpha} - 1$. At andness 1/2, the AIWA operator becomes the WAM. At andness 0 and 1, the AIWA operator becomes $h_0(\mathbf{w}, \mathbf{x}) = \max_{i=1}^{n}(w_i x_i)$ and $h_1(\mathbf{w}, \mathbf{x}) = \min_{i=1}^{n}(1 - w_i(1 - x_i))$.

6.4. Illustration of the two weighting forms

To illustrate the difference between the two kinds of weightings, consider the query, for $n = 2$, $Q = ((w_1, C_1), (w_2, C_2))$ where C_i is the ith criterion (constraint). Let $x_i = C_i(X)$

denote the degree of satisfaction of C_i , $i = 1, 2$, by the option (object) X. Let $Q_\alpha(X) = \lambda_{\text{AIWA}}((w_1, x_1), (w_2, x_2); \alpha) = h_\alpha((w_1, w_2), (x_1, x_2))$ denote the overall degree of satisfaction of Q by X, by the AIWA operator (implicative) with andness α. Assume that the implicative importance weights are $(w_1, w_2) = (0.4, 1)$, and the set of options queried is $\{A, B\}$, with $(a_1, a_2) = (0.1, 0.7)$ and $(b_1, b_2) = (0.9, 0.4)$, and $\alpha = 2/3$. By $r = \frac{1}{\alpha} - 1$, we find $r = 1/2$, hence

$$\lambda_{\text{AIWA}}((w_1, x_1), (w_1, x_2); 2/3) = 1 - \left(\frac{(w_1(1 - a_1))^2 + (w_2(1 - a_2))^2}{w_1^2 + w_2^2} \right)^{1/2}.$$

Therefore, the overall degrees of satisfaction of Q by A and B are, respectively

$$Q_{2/3}(A) = \lambda_{\text{AIWA}}((0.4, 0.1), (1, 0.7); 2/3) = 0.565$$
$$Q_{2/3}(B) = \lambda_{\text{AIWA}}((0.4, 0.9), (1, 0.4); 2/3) = 0.442$$

Since $Q_{2/3}(A) > Q_{2/3}(B)$, A is ranked before (better than) B; we denote $A > B$. The reason is that the lower satisfaction of the highly important 2nd criterion ($w_2 = 1$) by B "punishes" B more than the higher satisfaction of the less important 1st criterion ($w_1 = 0.4$) "rewards" it. Let us now consider the similar problem using multiplicative importance weighting, as by the weighted geometric mean (WGM). In the case of two variables, the geometric mean has the global andness 2/3. So, $\lambda_{\text{WGM}}((w_1, x_1), (w_1, x_2)) = x_1^{W_1} x_2^{W_2}$, $W_i = w_i/(w_1 + w_2), i = 1, 2$. In this case,

$$Q_{2/3}(A) = \lambda_{\text{WGM}}((0.4, 0.1), (1, 0.7)) = 0.401$$
$$Q_{2/3}(B) = \lambda_{\text{WGM}}((0.4, 0.9), (1, 0.4)) = 0.504$$

Now, $Q_{2/3}(A) < Q_{2/3}(B)$, yielding $A < B$, i.e. the reverse of the ranking obtained by the implicative importance weighting as represented by the AIWA operator.

Finally, we consider the weighted exponential mean with andness 2/3 as obtained by $r = -3$:

$$\lambda_{\text{WEM}}((w_1, x_1), (w_1, x_2)) = -\frac{1}{3} \ln(W_1 e^{-3x_1} + W_2 e^{-3x_2}) = -\frac{1}{3} \ln\left(\frac{w_1 e^{-3x_1} + w_2 e^{-3x_2}}{w_1 + w_2} \right)$$

In this case:

$$Q_{2/3}(A) = \lambda_{\text{WEM}}((0.4, 0.1), (1, 0.7)) = 0.402$$
$$Q_{2/3}(B) = \lambda_{\text{WEM}}((0.4, 0.9), (1, 0.4)) = 0.484$$

Again we obtain $A < B$, as for the WGM. This could be expected, since the operators are of the same kind, namely multiplicative importance weighted operators at andness 2/3. In fact, the resulting values are rather close to that of the WGM. As these examples illustrate, it is essential to choose the right kind of importance weighting for a decision problem. The difference in the behavior between implicative and multiplicative importance weighting increases as the andness moves away from 1/2. At andness 0 and 1, the importance weights have no effect in the multiplicative case, while they have full effect in the implicative case; at andness 1/2, both the implicative and the multiplicative operators represent the WAM where the weights have full effect.

7. Mandatory and sufficient requirements

In the area of system evaluation, we regularly have the situation where one or more of inputs represent mandatory requirements. In the case of computer evaluation, suppose that the final stage of aggregating preferences includes two components: hardware (x_1) and software (x_2). The global preference of the evaluated computer is computed using the *andor* function $y = x_1 \Delta x_2$. If $x_1 = 0$ (inappropriate hardware) we must reject such a computer (the *andor* function must generate the result $y = 0$). Similarly, if $x_2 = 0$ (e.g. no software), then again $y = 0$. Obviously, both good hardware *and* good software are mandatory requirements that all computers must satisfy. Therefore, we need the *andor* function that satisfies the condition $x_1 \Delta 0 = 0 \Delta x_2 = 0$ (rejection of system that does not satisfy mandatory requirements).

Unfortunately, in this case, the pure conjunction $y = x_1 \wedge x_2$ cannot be used, because such a rigid criterion would not be acceptable in regular cases where $x_1 > 0$ and $x_2 > 0$. Indeed, the majority of evaluators would not accept the equality $0.5 \wedge 0.5 = 1 \wedge 0.5 = 0.5$ that claims that a system with an average hardware and software is equivalent to the system having perfect hardware and an average software. In other words, instead of pure conjunction we need a *partial conjunction (andor)* that satisfies the following *mandatory requirements conditions*:

$$x_1 \Delta 0 = 0 \Delta x_2 = 0$$
$$x_1 \Delta (x_1 + a) > x_1, \quad x_1 > 0, \quad a > 0$$

For example, the geometric mean $\sqrt{x_1 x_2}$ obviously satisfies these conditions, and so do the weighted power means $M(x_1, \ldots, x_n; r)$ for $r \leqslant 0$.

In addition to the use of mandatory requirements in GCD, this property is indispensable for generating the partial absorption function [8], and other more complex logic functions [9,11].

Exponential means $E(x_1, \ldots, x_n; r)$ do not satisfy the mandatory requirements conditions. However, this is a desirable feature in other applications, where the missing satisfaction of one criterion should not eliminate the evaluated object, such as in object recognition and information retrieval.

If we take a function that is dual to a partial conjunction that satisfies the mandatory requirements conditions, we get a partial disjunction (orand) that satisfies the following sufficient requirements conditions:

$$1 \nabla x_2 = x_1 \nabla 1 = 1$$
$$x_1 \nabla (x_1 - a) < x_1, \quad x_1 < 1, \quad a > 0$$

Since the geometric mean $\sqrt{x_1 x_2}$ satisfies the mandatory requirement conditions, it follows that its dual $x_1 \nabla x_2 = 1 - [(1 - x_1)(1 - x_2)]^{1/2}$ satisfies the sufficient requirement conditions. However, such conditions occur seldom in system evaluation practice.

8. De Morgan's GCD functions

De Morgan's laws are a convenient mechanism for creating GCD functions that have various specific properties. Generally, De Morgan's GCD functions are defined as follows:

$$y = \begin{cases} \lambda(x_1, \ldots, x_n; \omega) \\ 1 - \lambda(1 - x_1, \ldots, 1 - x_n; 1 - \omega) \end{cases}; \quad \text{either } \omega \geqslant \frac{1}{2} \text{ or } \omega \leqslant \frac{1}{2}.$$

Of course, such functions always satisfy De Morgan's laws, which for GCD can be written as follows:

$$x_1 \nabla \cdots \nabla x_n = 1 - (1 - x_1) \Delta \cdots \Delta (1 - x_n)$$
$$x_1 \Delta \cdots \Delta x_n = 1 - (1 - x_1) \nabla \cdots \nabla (1 - x_n)$$

These formulas show how to make a partial disjunction if we have a model of partial conjunction and vice versa. In addition, if $x_1 \Delta \ldots \Delta x_n$ is a partial conjunction with andness $\alpha_g = c$, then $1 - (1 - x_1) \Delta \ldots \Delta (1 - x_n)$ is the partial disjunction with orness $\omega_g = c$.

Proof. Let $\bar{x}$ denote the mean value of x, and let $\mu = \overline{x_1 \Delta \ldots \quad \Delta x_n} = \int_0^1 \mathrm{d}x_1 \ldots \int_0^1 (x_1 \Delta \ldots \Delta x_n)\, \mathrm{d}x_n$. Then,

$$\begin{aligned} \overline{1 - (1 - x_1) \Delta \ldots \Delta (1 - x_n)} &= 1 - \int_0^1 \mathrm{d}x_1 \ldots \int_0^1 [(1 - x_1) \Delta \ldots \Delta (1 - x_n)]\, \mathrm{d}x_n \\ &= \begin{vmatrix} 1 - x_i = y_i \\ \mathrm{d}x_i = -\mathrm{d}y_i \\ i = 1, \ldots, n \end{vmatrix} = 1 - \int_0^1 \mathrm{d}y_1 \ldots \int_0^1 (y_1 \Delta \ldots \Delta y_n)\, \mathrm{d}y_n = 1 - \mu \end{aligned}$$

$$\alpha_g = \frac{n - (n+1)(\overline{x_1 \Delta \ldots \Delta x_n})}{n - 1} = \frac{n - (n+1)\mu}{n - 1} = c$$

$$\omega_g = \frac{(n+1)\overline{[1 - (1 - x_1) \Delta \ldots \Delta (1 - x_n)]} - 1}{n - 1} = \frac{(n+1)(1 - \mu) - 1}{n - 1} = \frac{n - (n+1)\mu}{n - 1} = c$$

If the partial conjunction is modeled using the geometric mean then the corresponding (dual) partial disjunction can be modeled as follows:

$$y = x_1 \Delta \ldots \Delta x_n = (x_1 \ldots x_n)^{1/n}, \quad \alpha_g = \alpha_g^{\text{geo}}(n) = c, \quad 1 - 1/e \leqslant c \leqslant 2/3$$
$$z = x_1 \nabla \ldots \nabla x_n = 1 - [(1 - x_1) \ldots (1 - x_n)]^{1/n}, \quad \omega_g = c = \alpha_g^{\text{geo}}(n)$$

The above partial conjunction can be used to model the mandatory requirements ($x_i = 0$ yields y=0, and it is necessary to have $x_i > 0, i = 1, \ldots, n$ to produce $y > 0$). Similarly, the dual partial disjunction can be used to model the sufficient requirements (it is sufficient to have $x_i = 1, \quad i \in \{1, \ldots, n\}$, to produce $z = 1$). Using the harmonic mean, the same effects can be achieved at a higher level of andness/orness, $\alpha_g^{\text{har}}(2) = \ln 16 - 2 = 0.77$ [6]:

$$x_1 \Delta \ldots \Delta x_n = n/(1/x_1 + \cdots + 1/x_n)$$
$$x_1 \nabla \ldots \nabla x_n = 1 - n/[1/(1 - x_1) + \cdots + 1/(1 - x_n)]$$

If we want to avoid mandatory and sufficient requirements we could use the quadratic mean to model the *orand* and *andor*, as follows:

$$x_1 \nabla \ldots \nabla x_n = [(x_1^2 + \cdots + x_n^2)/n]^{1/2}$$
$$x_1 \Delta \ldots \Delta x_n = 1 - \{[(1 - x_1)^2 + \cdots + (1 - x_n)^2]/n\}^{1/2}$$

In this case, neither *orand* can model sufficient requirements, nor can *andor* model mandatory requirements. If $n=2$, then the orness of the quadratic mean is $\omega_g^{\text{quad}}(2) = (\ln(1+\sqrt{2}))/\sqrt{2} = 0.623$ [6]. The andness of the corresponding *andor* is the same, $\alpha = \omega_g^{\text{quad}}(2)$, i.e. it is slightly below the andness of the geometric mean.

Some GCD functions do not satisfy De Morgan's laws. Two such examples of GCD based on the weighted power mean are:

$$M_{M\overline{S}}(x_1,\ldots,x_n;r) = \lim_{s\to r}\left(\sum_{i=1}^{n} W_i x_i^s\right)^{1/s}, \quad r=\rho_n(\alpha_g), \quad 0 \leqslant \alpha_g \leqslant 1$$

$$M_{\overline{M}S}(x_1,\ldots,x_n;r) = 1 - \lim_{s\to 2-r}\left(\sum_{i=1}^{n} W_i (1-x_i)^s\right)^{1/s}, \quad -\infty \leqslant r \leqslant \infty$$

The $M\overline{S}$ version for $\alpha_g \geqslant \alpha_g^{\text{geo}}(n)$ can model the mandatory requirements but cannot model the sufficient requirements. Similarly, the $\overline{M}S$ version cannot model the mandatory requirements but for $r \geqslant 2$ can model the sufficient requirements; e.g. for $r=3$ we have

$$M_{\overline{M}S}(x_1,\ldots,x_n;3) = 1-1/\sum_{i=1}^{n} W_i/(1-x_i); \;\; M_{\overline{M}S}(x_1,\ldots,x_n;3)=1, \;\; x_i=1, \;\; i\in\{1,\ldots,n\}.$$

If $n=2$, the andness $\alpha_g = 2/3$ (and the satisfaction of mandatory requirements) can be achieved using the geometric mean $y=\sqrt{x_1x_2}$, which belongs to the $M\overline{S}$ version of the GCD based on WPM. The same level of andness, but without mandatory requirements, can be achieved using the $\overline{M}S$ type of the GCD function:

$$y = 1-[0.5(1-x_1)^r + 0.5(1-x_2)^r]^{1/r}, \quad r=2.41.$$

The $M\overline{S}$ and $\overline{M}S$ versions of GCD do not satisfy De Morgan's law. However, the corresponding errors are sufficiently small [7], so that the above functions are suitable in many applications.

Two versions of GCD that use weighted power means and satisfy De Morgan's law are:

$$M_{MS}(x_1,\ldots,x_n;r) = \begin{cases} \lim_{s\to r}\left(\sum_{i=1}^{n} W_i x_i^s\right)^{1/s}, & -\infty \leqslant r \leqslant 1 \\ 1-\lim_{s\to 2-r}\left(\sum_{i=1}^{n} W_i (1-x_i)^s\right)^{1/s}, & 1 \leqslant r \leqslant \infty \end{cases}$$

$$M_{\overline{MS}}(x_1,\ldots,x_n;r) = \begin{cases} 1-\lim_{s\to 2-r}\left(\sum_{i=1}^{n} W_i (1-x_i)^s\right)^{1/s}, & -\infty \leqslant r \leqslant 1 \\ \lim_{s\to r}\left(\sum_{i=1}^{n} W_i x_i^s\right)^{1/s}, & 1 \leqslant r \leqslant \infty \end{cases}$$

The MS version of GCD for $r \leqslant 0$ satisfies the mandatory requirements and for $r \geqslant 2$ satisfies the sufficient requirements. In the case of the $\overline{MS}$ version, neither the mandatory conditions nor the sufficient conditions are satisfied for all finite values of r. This function has similar properties as the exponential mean.

The presented four characteristic cases $(MS, \overline{M}S, M\overline{S}, \overline{MS})$ expand the spectrum of properties of the GCD based on weighted power mean.

9. Other models of simultaneity and replaceability

In addition to GCD, there are many other ways to model simultaneity and replaceability. For example, the intersection operators (t-norms) and the union operators (t-conorms) [15], shown in Fig. 1, do not support internality, idempotency, and sensitivity to orness, but have other useful properties, such as associativity and the existence of a neutral (identity) element (namely 1 and 0, respectively); see, for instance, [17].

Some authors propose to combine intersection (denoted $x_1 \triangle_t x_2$) and union operators (denoted $x_1 \nabla_t x_2$) to obtain aggregators with adjustable levels of andness/orness. The idea of such models is either an additive combination $x_1 \diamondsuit_t x_2 = q(x_1 \nabla_t x_2) + (1-q)(x_1 \triangle_t x_2)$, or a multiplicative form $x_1 \diamondsuit_t x_2 = (x_1 \nabla_t x_2)^q (x_1 \triangle_t x_2)^{1-q}, 0 \leqslant q \leqslant 1$ where q is used to adjust the level of orness, and $1-q$ is used to adjust the level of andness. Such operators, proposed by Zimmermann and Zysno [23], are $x_1 \diamondsuit_t x_2 = (1-\bar{x}_1\bar{x}_2)^q (x_1 x_2)^{1-q}$ (based on geometric mean) or $x_1 \diamondsuit_t x_2 = q(1-\bar{x}_1\bar{x}_2) + (1-q)(x_1 x_2)$ (based on the arithmetic mean). Pseudo averaging models of the form $x_1 \diamondsuit x_2 = c_0 + c_1 x_1 + c_2 x_2 + c_{12} x_1 x_2$ are frequently used in the utility theory [2,14].

Means that do not have adjustable parameter can be interpreted as GCD functions with constant andness. For example, for Heronian mean we have

$$h(x,y) = (x + \sqrt{xy} + y)/3, \quad \alpha_g = 0.5556$$

This result can be generalized:

$$h(x,y;p) = p\frac{x+y}{2} + (1-p)\sqrt{xy}, \quad 0 \leqslant p \leqslant 1, \quad \alpha_g = (4-p)/6$$

For the logarithmic mean we have

$$L(x,y) = \frac{y-x}{\log y - \log x}, \quad \alpha_g = 0.613$$

For the centroidal mean we have

$$c(x,y) = \frac{2}{3}\left(\frac{x^2+xy+y^2}{x+y}\right), \quad \alpha_g = 0.4091$$

For each of these functions we can create the corresponding De Morgan dual. For instance, in the case of centroidal mean:

$$d(x,y) = \overline{c(\bar{x},\bar{y})} = 1 - \frac{2}{3}\left(\frac{(1-x)^2 + (1-x)(1-y) + (1-y)^2}{2-x-y}\right),$$
$$\alpha_g = 1 - 0.4091 = 0.5909.$$

These forms of GCD are useful in decision models where the continuous adjustment of andness is not needed.

10. GCD with positional weights

In fuzzy logic, the term "averaging operators" is used for mappings $H : [0,1]^n \rightarrow [0,1], n > 1$, that are monotonic increasing in all its arguments, continuous, symmetric in all its arguments, and idempotent. We notice that GCD functions satisfy these requirements and therefore also are averaging operators.

A particularly interesting family of averaging operators are the OWA (ordered weighted averaging) operators [16,17,19–22]. An OWA operator is characterized by a vector of positional weights $(v_1, \dots, v_n) \in I^n$ that satisfies $v_1 + \cdots + v_n = 1$. The aggregate y of an argument vector $(x_1, \dots, x_n) \in I^n$ is defined by

$$y = v_1 x_{(1)} + \cdots + v_n x_{(n)}$$

where $(\cdot)$ is an index permutation such that $x_{(1)} \geqslant \cdots \geqslant x_{(n)}$. The andness α and orness ω of such an operator are functions of the OWA weighting tuple, namely:

$$\alpha = \frac{v_2 + \cdots + v_{n-1}(n-2) + v_n(n-1)}{n-1},$$
$$\omega = \frac{v_1(n-1) + v_2(n-2) + \cdots + v_{n-1}}{n-1}, \quad \alpha + \omega = 1$$

From this definition, if $v_i = 1/n, \quad i = 1, \dots n$, the OWA operator represents the arithmetic mean, with $\alpha = \omega = 1/2$. If $v_1 = 1$ ($v_i = 0$ for $i > 1$), it represents the pure *or* (the max operator), with orness $\omega = 1$, and if $v_n = 1$ ($v_i = 0$ for $i < n$), it represents the pure *and* (the min operator), with orness $\omega = 0$. If $n = 2$, then $\omega = v_1$ and $\alpha = v_2$ (the weight of the larger argument represents the orness and the weight of the smaller argument represents the andness). The OWA measures of orness and andness can be shown to comply with the measures of global and mean local andness/orness (Section 5).

An advantage of the OWA family is that, for any degree of orness in (0, 1), it allows us to construct multiple averaging operators with different properties. For instance, the following OWA weighing tuples (illustrated with $n = 5$) all have orness $\omega = 0.5$:

$(1/5, 1/5, 1/5, 1/5, 1/5)$	(Arithmetic mean)
$(0, 0, 1, 0, 0)$	(Median)
$(0, 1/3, 1/3, 1/3, 0)$	(Olympic mean)

OWA operators are further characterized by the dispersion of OWA weights. The dispersion is, in its normalized form, defined by:

$$\text{disp}(v_1, \dots, v_n) = -(v_1 \ln v_1 + \cdots + v_n \ln v_n)/\ln n$$

For instance, the dispersion of the above shown three OWA weighting vectors are, respectively, 1, 0, and 0.683.

An iterative OWA operator (ItOWA) was proposed in [9] and analyzed in [11]. We assume $x_i \geqslant x_{i+1},\ i > 0,\ y = v_{n1}x_1 + \cdots + v_{nn}x_n,\ n > 1$, and $(v_{n1}, \dots, v_{nn}) \in I^n$, $v_{n1} + \cdots + v_{nn} = 1, (x_1, \dots, x_n) \in I^n$. The first step in making an ItOWA operator is to select the desired values of ItOWA andness α and ItOWA orness $\omega = 1 - \alpha$. The concept of andness and orness is the same as in other forms of GCD, but its numeric values are specific for ItOWA. The ItOWA weights are computed directly from the desired values of andness/orness. If $n = 2$ the iterative model is the same as OWA:

$$y = x_1 \diamondsuit x_2 = \omega(x_1 \vee x_2) + \alpha(x_1 \wedge x_2) = \omega x_1 + \alpha x_2$$
$$v_{21} = \omega, \quad v_{22} = \alpha$$

In the case of 3 variables, the function $y = x_1 \diamondsuit x_2 \diamondsuit x_3$ is defined by the following iterative procedure:

```
while (x1 − x3 > ε)                //ε = a small error
{
    x12 = ωx1 + αx2;   x13 = ωx1 + αx3;   x23 = ωx2 + αx3;
    x1 = x12;   y = x2 = x13;   x3 = x23;
}
```

This procedure can be expressed in a matrix form as follows:

$$\begin{bmatrix} y \\ y \\ y \end{bmatrix} = \lim_{k\to+\infty} \begin{bmatrix} v_{21} & v_{22} & 0 \\ v_{21} & 0 & v_{22} \\ 0 & v_{21} & v_{22} \end{bmatrix}^k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lim_{k\to+\infty} \begin{bmatrix} \omega & \alpha & 0 \\ \omega & 0 & \alpha \\ 0 & \omega & \alpha \end{bmatrix}^k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The resulting GCD for 3 variables is:

$$y = x_1 \Diamond x_2 \Diamond x_3 = v_{31}x_1 + v_{32}x_2 + v_{33}x_3 = \frac{\omega^2 x_1 + \alpha\omega x_2 + \alpha^2 x_3}{\alpha^2 + \alpha\omega + \omega^2} = \begin{cases} x_1, & \omega = 1 \\ (x_1 + x_2 + x_3)/3, & \omega = 1/2 \\ x_3, & \omega = 0 \end{cases}$$

The ItOWA version of GCD is consistent with OWA averaging operators for $\alpha = 0, 0.5, 1$, and differs for other values of andness. The main difference is that ItOWA computes weights from desired andness/orness, while OWA computes andness/orness from the values of weighs. More details can be found in [11].

The concept of ordered weighted averaging is to model simultaneity by emphasizing the role of low preferences, and to model replaceability by emphasizing the role of high preferences. OWA and ItOWA operators use the arithmetic mean. Consequently, they are not suitable for modeling mandatory and sufficient requirements. However, the same idea can be used in the context of other weighted power means and their combinations with De Morgan's law. Examples of OWA-like operators that satisfy the mandatory and sufficient requirements are WGM (partial conjunction) and its De Morgan dual (partial disjunction):

$$y = \exp(v_{n1}\ln x_1 + \cdots + v_{nn}\ln x_n)$$
$$y = 1 - \exp(v_{n1}\ln(1 - x_1) + \cdots + v_{nn}\ln(1 - x_n))$$
$$x_1 \geqslant \cdots \geqslant x_n, \quad n > 1$$

An extension to (implicative) importance weighted OWA operators is analyzed in [16] and presented in [17] in a form that represents the WAM for orness 0.5. There are several other extensions and properties of OWA operators described in the literature, but the above brief introduction will suffice for the scope of this paper.

11. Conclusions and future work

The GCD operators can be organized, interpreted, and used in a variety of ways. We focused on interpretations in continuous preference logic, and briefly introduced the fuzzy logic averaging operator interpretation. The organization of the GCD operators based on weighted power means is shown to be the most attractive for interpretations in preference logic and applications in system evaluation. We also analyzed a GCD organization based on exponential means. We introduced a distinction between two kinds of importance weighting, namely multiplicative and implicative. These new concepts deserve future research. Comparison of various approaches to definition of andness and orness, as well as new forms of GCD based on Bajraktarević means, also deserve future research.

We have presented and discussed several properties of GCD operators. All operators based on means satisfy the bounded quality range (internality) condition, and are commutative, monotonic, continuous, and idempotent. For the usability of such operators, the level of andness/orness should be easily adjustable. The importance of satisfying component criteria should be easily adjustable through weights. Weighted operators should be a generalization of unweighted operators, so that unweighted operators are obtained when the inputs are evenly weighted. The two kinds of importance weighting generalizations, multiplicative and implicative, should be supported; the choice between these kinds depends on the kind of the decision problem. The multiplicative form is primarily applied for estimating the level of satisfaction of requirements, as applied in, for instance, system evaluation. The implicative form primarily is applied for ranking of options according to their satisfaction of joint criteria (constraints), as applied in selection, classification, and recognition problem solving.

An often-required property in system evaluation is the mandatory requirements property. This says that, regardless of the andness and the (positive) importance weights, the aggregate must evaluate to zero, if at least one of the criteria is not satisfied at all. For other kinds of problems, absence of the mandatory property may be required; for instance, in recognition, where the failure to satisfy a single criterion should not invalidate the option, but just "punish", depending on the importance of satisfying the criterion.

Associativity and distributivity are not properties of GCD operators. However, these properties can be approximated with a tolerable small error. The De Morgan duality applying to GCD operators may also be computationally useful.

Finally, for multiplicative importance weighted GCD operators, we need the sensitivity property $\partial y/\partial x_i > 0$. In addition, assuming that all inputs are not equal, and are positive, the condition $\partial y/\partial \omega > 0$, where ω is the orness, must be satisfied. The property $\partial y/\partial \omega > 0$ does not hold in general for implicative importance weighting.

We have seen that the weighted power means provide a set of useful properties; the mandatory property is obtained for non-positive values of the parameter r, yielding andness degree above about 2/3. The lowest global andness for which the mandatory property is obtained corresponds to the geometric mean ($r = 0$) with andness between 0.667 (at $n = 2$) and 0.632 ($n \gg 1$). If absence of the mandatory property is required with multiplicative weighting for all degrees of andness, then a possible choice is the exponential mean. A nice feature of the exponential mean is its symmetry: for $r = 0$ it generates the arithmetic mean, and the andness for r equals the orness for $-r$.

The implicative importance weighting is provided by the AIWA operators [17] that are based on the power means. These operators do not have the mandatory requirements property.

The GCD function emerges in various forms in system evaluation, classification, recognition, and in other areas. In the continuous logic it is interpreted as a model of adjustable simultaneity and replaceability. In the fuzzy logic it is interpreted as the averaging operator. Various application areas generate a rich spectrum of desired and achieved GCD forms and properties. Our goal was to present various forms and interpretations of GCD in a unifying and comparative way that might cause more convergence in the future research.

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