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LINEAR ALGEBRA AND ITS APPLICATIONS

Eigenvalue location for nonnegative and Z-matrices

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Abstract

Let L_0^k denote the class of $n \times n$ Z-matrices $A = tI - B$ with $B \geqslant 0$ and $\rho_k(B) \leqslant t < \rho_{k+1}(B)$, where $\rho_k(B)$ denotes the maximum spectral radius of $k \times k$ principal submatrices of B. Bounds are determined on the number of eigenvalues with positive real parts for $A \in L_0^k$, where k satisfies, $\lfloor n/2 \rfloor \leqslant k \leqslant n-1$. For these classes, when $k = n-1$ and $n-2$, wedges are identified that contain only the unique negative eigenvalue of A. These results lead to new eigenvalue location regions for nonnegative matrices.

Keywords: Eigenvalue location; Fisher's inequality; M-matrices; Nonnegative matrices

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1. Introduction

Let B be an entry-wise nonnegative $n \times n$ matrix (denoted $B \geqslant 0$), and, for $k = 1, 2, \ldots, n$, let $\rho_k(B)$ denote the maximum spectral radius of the $k \times k$ principal submatrices of B. For completeness, define $\rho_0(B) = -\infty$ and $\rho_{n+1}(B) = \infty$. Observe that $\rho_n(B)$ equals $\rho(B)$, the spectral radius of B. For $k = 0, 1, \ldots, n$, let $L_0^{n,k}$ ($L^{n,k}$) denote the class of Z-matrices $A = tI - B$, in which $B \geqslant 0$ is an $n \times n$ matrix, and $\rho_k(B) \leqslant t < \rho_{k+1}(B)$ ($\rho_k(B) < t < \rho_{k+1}(B)$). Throughout we consider matrices of order n, thus we use the notation L_0^k (L^k) for $L_0^{n,k}$ ($L^{n,k}$). The classes $L_0^0, L_0^1, \ldots, L_0^n$ were introduced in [1] (denoted by $L_0, L_1, \ldots, L_n$), and observed to form a partition of the class of $n \times n$ Z-matrices. Note that L^n is the class of nonsingular M-matrices, L_0^n is the class of (singular and nonsingular) M-matrices, see [2], ch. 6 and that L_0^{n-1} (denoted by N_0) and L_0^{n-2} were first studied in [3]. L_0^{n-2} was denoted by F_0 in [4] and further studied in [5,6]. Note that if $A \in L_0^{n-1}$, then A is irreducible, and nonsingular (see [3]).

From the definitions it follows that $A \in L_0^k$ ($A \in L^k$) if and only if A is a Z-matrix and each $k \times k$ principal submatrix of A is an M-matrix (nonsingular M-matrix), but there is at least one $(k+1)$–by–$(k+1)$ principal submatrix that is not an M-matrix; see [1] Theorem 1.3, for the L_0^k case. In [7] nonsingular matrices in L_0^k were characterized in terms of the principal minors of their inverses.

We are interested in the spectrum of $A \in L_0^k$, and use the following notation. The *characteristic polynomial* of an n–by–n matrix A is $p_A(z) = \det(zI - A)$, and

$$p_A(z) = z^n - c_1 z^{n-1} + c_2 z^{n-2} - c_3 z^{n-3} + \cdots + (-1)^n c_n, \tag{1}$$

where c_k is the sum of the $k \times k$ principal minors of A. In particular, $c_1 = \operatorname{tr} A$ and $c_n = \det A$. The location of the eigenvalues of A can be specified by the *inertia* of A, which is the triple $i_+(A)$, $i_0(A)$, $i_-(A)$, specifying the number of eigenvalues with positive, zero, negative, real parts, respectively. Thus $i_+(A) + i_0(A)$ specifies the number of eigenvalues (counting multiplicities) in the closed right half-plane. For example, if $A \in L_0^n$, then $i_+(A) + i_0(A) = n$, since an M-matrix is positive semi-stable. In Section 3, for $A \in L_0^k$ ($A \in L^k$), where $\lfloor n/2 \rfloor \leqslant k \leqslant n-1$, we determine bounds on $i_+(A) + i_0(A)$ ($i_+(A)$). We also verify Conjecture 2.3 for $A \in L_0^{n-1}$, when $2 \leqslant n \leqslant 6$. In Section 4, for $A \in L_0^{n-1}$ and $A \in L_0^{n-2}$ we determine a wedge containing only one eigenvalue of A, namely the negative eigenvalue.

2. Preliminary results

We begin with the following definition.

Definition 2.1. For $n \geqslant 2$, let

$$g(n) = \begin{cases} n/2+1 & n \equiv 0 (\text{mod } 4), \\ n/2-1/2 & n \equiv 1 (\text{mod } 4), \\ n/2 & n \equiv 2 (\text{mod } 4), \\ n/2+1/2 & n \equiv 3 (\text{mod } 4). \end{cases}$$

Example 2.2. Let $A = -P_n$, where P_n is the basic $n \times n$ circulant (that is, the circulant with first row $(0, 1, 0, \ldots, 0)$). Then $A \in L_0^{n-1}$, where $t = 0 = \rho_{n-1}(P_n)$. It is readily verified that $i_+(A) = g(n)$ if $n \not\equiv 0 \ (\text{mod } 4)$ and $i_+(A) + i_0(A) = g(n)$ if $n \equiv 0 \ (\text{mod } 4)$, since the eigenvalues of P_n have real parts $\cos(2\pi q/n)$ for $q = 0, 1, \ldots, n-1$.

Motivated by this example, by the fact that the Cauchy interlacing inequalities imply that $i_+(A) = n-1$ for symmetric $A \in L_0^{n-1}$, and by numerical results indicating that if $A \in L_0^{n-1}$, it is usually the case that $i_+(A) = n-1$, we make the following conjectures.

Conjecture 2.3. If $n \geqslant 2$ and $A \in L_0^{n-1}$ in which $n \not\equiv 0 \ (\text{mod } 4)$ or $A \in L^{n-1}$ in which $n \equiv 0 \ (\text{mod } 4)$, then

$$n-1 \geqslant i_+(A) \geqslant g(n).$$

It would follow from the above conjecture that if $A \in L_0^{n-1}$ in which $n \equiv 0 \ (\text{mod } 4)$, then

$$n-1 \geqslant i_+(A) + i_0(A) \geqslant g(n).$$

From the definition of L_0^{n-1}, the above conjectures are equivalent to the following conjecture concerning the location of eigenvalues of nonnegative matrices.

Conjecture 2.4. Let $B \geqslant 0$ be an $n \times n$ matrix with $n \geqslant 2$ and let $S = \{\lambda : \lambda$ is an eigenvalue of B with $\operatorname{Re} \lambda \leqslant \rho_{n-1}(B)\}$. Then

$$n-1 \geqslant |S| \geqslant g(n).$$

In considering these conjectures in Section 3, we need the following inequality on the coefficients of the characteristic polynomial of a Z-matrix.

Lemma 2.5. *Let i, j be positive integers satisfying $2 \leqslant i+j \leqslant k \leqslant n$, and let $A \in L_0^k$ be an $n \times n$ matrix with characteristic polynomial (1). Then*

$$c_{i+j} \leqslant c_i \cdot c_j. \tag{2}$$

Moreover, if c_{i+j} is positive, the inequality is strict.

Proof. By the definition of L_0^k each principal minor of order at most k is nonnegative, hence $c_p \geqslant 0$ for $p = 1, 2, \ldots, k$. Thus if $c_{i+j} = 0$, then the inequality holds. Now assume $c_{i+j} > 0$. Consider a principal submatrix of A of order $i + j$. As $A \in L_0^k$, this submatrix is an M-matrix. Without loss of generality (since L_0^k is invariant under permutation similarity) assume that this submatrix has row and column indices $1, 2, \ldots, i + j$, and denote it by $A[1, 2, \ldots, i + j]$. By Fischer's inequality (see, for example, [8], p. 117)

$$\det A[1, 2, \ldots, i + j] \leqslant \det A[1, 2, \ldots, i] \cdot \det A[i + 1, i + 2, \ldots, i + j].$$

Multiply the right-hand side of (2) out. Each summand of c_{i+j} is less than or equal to a distinct product on the right-hand side of (2) by Fischer's inequality. Since there are other positive products on the right-hand side of (2) as well, inequality (2) is strict. □

We make use of Routh's scheme (see [9], Theorem 3.3, p. 142, [10], p. 177) in Sections 3 and 4. A brief description of what we need of this method is as follows. For the polynomial $p_A(z)$ given by (1), construct the Routh array $\{r_{ij}\}$ having the first two rows

$$\{r_{01}, r_{02}, r_{03}, \ldots\} = \{1, c_2, c_4, \ldots\}, \tag{3}$$

$$\{r_{11}, r_{12}, r_{13}, \ldots\} = \{-c_1, -c_3, -c_5, \ldots\}, \tag{4}$$

and ith row defined by

$$r_{ij} = \frac{-\det\begin{bmatrix} r_{i-2,1} & r_{i-2,j+1} \\ r_{i-1,1} & r_{i-1,j+1} \end{bmatrix}}{r_{i-1,1}}, \tag{5}$$

for $i = 2, 3, \ldots, n$. Further, the rows of the Routh's Scheme are then filled with zeros. Associated with each row of Routh's array is a polynomial. Write $z = i\omega$, and let

$$f_1(\omega) = \omega^n - c_2\omega^{n-2} + \cdots \quad \text{and} \quad f_2(\omega) = -c_1\omega^{n-1} + c_3\omega^{n-3} - \cdots$$

For $j = 3, 4, \ldots, n + 1$, inductively define $f_j(\omega) = f_{j-1}(\omega)q_{j-2}(\omega) - f_{j-2}(\omega)$. Thus $-f_j(\omega)$ is the remainder upon dividing $f_{j-2}(\omega)$ by $f_{j-1}(\omega)$. In [10], p. 178 it is shown that $f_j(\omega)$ is the polynomial associated with the jth row of Routh's array. If the array is regular, i.e., $r_{i1} \neq 0$ for all i, then $i_0(A) = 0$ and $i_+(A)$ is equal to the number of variations in sign in the sequence of entries in the first column, namely $\{r_{01}, r_{11}, r_{21}, \ldots\}$. If there is a zero element in the first column, then the scheme (5) cannot be continued; the array is singular, and two types of singularity must be considered. In the type (i) singular case, an entry $r_{p1} = 0$ with $p \geqslant 1$, but there is $q \geqslant 2$ such that $r_{pq} \neq 0$. Then, $i_0(A) = 0$. Replace r_{p1} by a parameter ε (assumed small) and continue the array according to (5). In the type (*ii*) singular case, a row, say the $(j+1)$st, consists entirely of zero entries so that $f_j(\omega)$, the polynomial associated with the jth row, is the last nonzero polynomial obtained by the Euclidean algorithm. In this case the polynomials $f_1(\omega)$ and $f_2(\omega)$ associated with the first two rows of Routh's array, respectively, have a nontrivial GCD and, to continue Routh's algorithm, we replace $f_{j+1}(\omega)$ by $f_j'(\omega)$ (if the roots of $f_j'(\omega)$ are not simple, it will be necessary to repeat this process, see [10], p. 183). In this case $i_0(A)$ may be positive.

The following result, due to Nabben [11], is used in Section 3. We state it here for convenience.

Lemma 2.6 ([11], Theorems 2.8 and 2.10). *Let $A \in L_0^k$ be an $n \times n$ matrix, where $\lfloor n/2 \rfloor \leqslant k \leqslant n-1$. Then A has exactly one negative eigenvalue and* $\det A \leqslant 0$.

Moreover, it follows from the proof of Theorem 2.10 in [11] that, if $A \in L^k$, then $\det A < 0$.

To obtain the wedge results of Section 4, we use the Cauchy index method, see for example [10,12,13], which is described as follows. For any fixed angle $\theta \in (0, \pi)$, write $z = re^{i\theta}$ and

$$p_A(z) = p_A(re^{i\theta}) = U(r) + iV(r), \tag{6}$$

where from (1)

$$U(r) = r^n \cos n\theta - c_1 r^{n-1} \cos(n-1)\theta + \cdots + (-1)^n c_n, \tag{7}$$

$$V(r) = r^n \sin n\theta - c_1 r^{n-1} \sin(n-1)\theta + \cdots + (-1)^{n-1} c_{n-1} r \sin \theta. \tag{8}$$

Let $I_0^\infty(V(r)/U(r))$ denote the Cauchy index of the rational function $V(r)/U(r)$, namely the difference between the number of jumps from $-\infty$ to $+\infty$ and from $+\infty$ to $-\infty$ of the rational function $V(r)/U(r)$ as r increases from 0^+ to ∞. Assuming that $p_A(z)$ has no root z with $\arg z = \theta$, and that $\cos n\theta \neq 0$, the number s of roots of $p_A(z)$ such that $|\arg z| < \theta$ is given by [12], Theorem 1 and Corollary 1:

$$s = I_0^\infty \frac{V(r)}{U(r)} + \frac{n}{\pi}\theta - \frac{1}{\pi}\arctan(\tan n\theta), \tag{9}$$

where $U(r)$ and $V(r)$ are given in (7) and (8), respectively.

3. Inertia results for L_0^k

We now examine the inertia of $A \in L_0^k$, where $\lfloor n/2 \rfloor \leqslant k \leqslant n-1$ and, equivalently, the inertia of nonnegative matrices. For matrices in L_0^{n-1} with zero diagonal, we have the following proposition.

Proposition 3.1. *If $A \in L_0^{n-1}$ and has zero diagonal, then $i_+(A)$, $i_0(A)$, and $i_-(A)$ agree with the number of* nth *roots of $(-1)^n$ that have positive, zero, and negative real parts, respectively.*

Proof. Since $A \in L_0^{n-1}$ has zero diagonal, A is monomial [3], Theorem 2.2. Thus,

$$p_A(z) = z^n + (-1)^n c_n = z^n + (-1)^n \det A,$$

and since $\det A < 0$ [3], Lemma 2.1, the result follows. □

Next we study $i_+(A)$ for $A \in L^k$ in which $\lfloor n/2 \rfloor \leqslant k \leqslant n-1$. If n is odd (even), then $i_+(A)$ is even (odd) since $\det A < 0$ and has exactly one negative (simple) eigenvalue, by Lemma 2.6. Thus

$$n-1 \geqslant i_+(A) + i_0(A) \geqslant i_+(A),$$

giving the upper bound in Conjecture 2.3 for all n. If $n \geqslant 3$ and odd, then $i_-(A) \geqslant 2$, since A is nonsingular and $\operatorname{tr} A \geqslant 0$. Observe that $i_+(A) = 1$ for $n = 2$ and $k = 1$.

Theorem 3.2. *If $A \in L^k$, with $n \geqslant 3$ and $\lfloor n/2 \rfloor \leqslant k \leqslant n-1$, then*

$$n-1 \geqslant i_+(A) \geqslant \begin{cases} 2 & \text{if } n \text{ is odd}, \\ 3 & \text{if } n \text{ is even}. \end{cases}$$

Proof. In view of the remarks preceeding the statement of the theorem, we only need to consider the case in which n is even; so first assume $n \geqslant 6$ (and consequently $k \geqslant 3$). Consider the Routh array for $p_A(z)$ given by (3), (4) and (5) for $A \in L^k$. Since $A \in L^k$, every principal submatrix of A of order $\leqslant$k is a nonsingular M-matrix and thus $c_1, c_2, \ldots, c_k > 0$. The first three entries of the first column are $r_{01} = 1$, $r_{11} = -c_1 < 0$, and $r_{21} = (c_1c_2 - c_3)/c_1 > 0$, by Lemma 2.5. If the array is regular, then (3), (4), and (5) imply that the last nonzero term in the odd rows is $(-1)^n c_n$; in particular, $r_{n1} = (-1)^n c_n < 0$, by Lemma 2.6. Thus, the number of variations in sign in the sequence of entries in the first column of Routh's array is at least 3, giving $i_+(A) \geqslant 3$. If the array is singular, then there is $p \geqslant 3$ such that $r_{p1} = 0$. For type (*i*), $i_0(A) = 0$, replace r_{p1} by ε (small) and continue the scheme. This does not affect the last term in the odd rows, so again $i_+(A) \geqslant 3$. Now assume there is a type (*ii*) singularity and A has no purely imaginary eigenvalue. Suppose $f_{j+1}(\omega) = 0$ and $f_j(\omega)$ is the last nonzero polynomial obtained from the Euclidean algorithm. Then $f_j(\omega)$ must be of even degree with its last nonzero coefficient being $c_n < 0$. Upon replacing $f_{j+1}(\omega)$ by $f_j'(\omega)$ and continuing Routh's algorithm, the last row of Routh's array is identical to the last row in the regular case and so $i_+(A) \geqslant 3$. If A has purely imaginary eigenvalues, then there is $\eta > 0$ such that

$A - \eta I \in L^k$ has no such eigenvalues. By above $i_+(A - \eta I) \geqslant 3$, and so by continuity $i_+(A) \geqslant 3$. If $n = 4$, then $k = 2$ or 3. In either case, c_1, c_2, $c_1 c_2 - c_3 > 0$ and the proof is identical to the proof of the first part. □

Applying a routine continuity argument gives the following corollary.

Corollary 3.3. *If $A \in L_0^k$, with $n \geqslant 3$ and $\lfloor n/2 \rfloor \leqslant k \leqslant n-1$, then*

$$n - 1 \geqslant i_+(A) + i_0(A) \geqslant \begin{cases} 2 & \text{if } n \text{ is odd}, \\ 3 & \text{if } n \text{ is even}. \end{cases}$$

Imposing an additional assumption on the diagonal entries of A, we can extend this result for $A \in L_0^{n-1}$ as follows.

Theorem 3.4. *If $A \in L_0^{n-1}$ with $n \geqslant 3$ and* $\operatorname{tr} A > 0$, *then*

$$n - 1 \geqslant i_+(A) \geqslant \begin{cases} 2 & \text{if } n \text{ is odd}, \\ 3 & \text{if } n \text{ is even}. \end{cases}$$

Proof. If $c_2 > 0$, then a proof as given for Theorem 3.2 establishes the result. So assume $c_2 = 0$ (and therefore from Lemma 2.5, $c_3 = c_4 = \cdots = c_{n-1} = 0$). Thus, $p_A(z) = z^n - c_1 z^{n-1} + c_n$. Following Gantmacher [10], p. 185, let $p_A(z) = F_1(z) + F_2(z)$, where $F_1(z) = z^n + c_n$ and $F_2(z) = -c_1 z^{n-1}$. Since $c_n < 0$ it is easily seen that the GCD of $F_1(z)$ and $F_2(z)$ is a constant. Thus $p_A(z)$ has no root z for which $-z$ is also a root and hence $i_0(A) = 0$. The result follows from Corollary 3.3. □

That it is necessary to state Corollary 3.3 in terms of the closed right half-plane is seen by the following example.

Example 3.5. Consider

$$A = 0I - B = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

It is easily checked that $\rho_{n-1}(B) = \rho_2(B) = 1$, while $\rho_{\lfloor n/2 \rfloor}(B) = \rho_1(B) = 0$, and that the spectrum of B (and A) is $\{-\sqrt{2}, 0, \sqrt{2}\}$. Thus $A \in L_0^{\lfloor n/2 \rfloor}$ has but one eigenvalue in the open right half-plane.

Corollary 3.6. *Let $B \geqslant 0$ be an $n \times n$ matrix with $n \geqslant 3$ and let $S = \{\lambda : \lambda$ is an eigenvalue of B with* $\operatorname{Re} \lambda \leqslant \rho_{\lfloor n/2 \rfloor}(B)\}$. *Then*

$$n-1 \geqslant |S| \geqslant \begin{cases} 2 & \textit{if } n \textit{ is odd}, \\ 3 & \textit{if } n \textit{ is even}. \end{cases}$$

If $B \geqslant 0$ and $n = 3$ or 4, then Corollary 3.6 implies that all eigenvalues λ different from $\rho(B)$ satisfy

$$\operatorname{Re} \lambda \leqslant \rho_{\lfloor n/2 \rfloor}(B).$$

The following example is an illustration.

Example 3.7. Consider,

$$B = \begin{bmatrix} 6 & 2 & 3 \\ 5 & 6 & 0 \\ 1 & 4 & 6 \end{bmatrix}.$$

Since B is row stochastic, it is obvious that $\rho(B) = 11$, and, since $\rho_{\lfloor n/2 \rfloor}(B) = \max\{b_{ii}\} = 6$, all other eigenvalues of B lie in the disk $x^2 + y^2 \leqslant 121$ and on or to the left of the vertical line $x = 6$. In fact, the other two eigenvalues of B are $7/2 \pm i\sqrt{23}/2$.

Using Corollary 3.3 when $k = n-2, n-1$ gives the following.

Corollary 3.8. *If $A \in L^{n-2} \cup L^{n-1}$, where $n = 3$ or 4, then A has a unique negative eigenvalue and the other eigenvalues of A have positive real parts. Furthermore, if $A \in L_0^{n-2} \cup L_0^{n-1}$, where $n = 3$ or 4, then A has a unique negative eigenvalue and the other eigenvalues of A have nonnegative real parts.*

Proposition 3.1 and Theorem 3.4 prove Conjecture 2.3 (and equivalently 2.4) for matrices of orders 2, 3, 4, 5 and 6. The smallest unknown case for $A \in L_0^{n-1}$ is $n = 7$, with $g(7) = 4$, but the possibility of exactly 2 roots in the open right half-plane has not been ruled out. For $n = 9$, $g(9) = 4$, and taking $A = tI - P_9$, in which P_9 is the basic 9×9 circulant and with suitably chosen t: $0 \leqslant t < 1$, shows that $i_+(A)$ can take on the values 4, 6 or 8. Again we do not know whether $i_+(A) = 2$ is possible for $A \in L_0^{n-1}$ when $n = 9$. For general n odd (even), $A = tI - P_n$ and suitable t, $0 \leqslant t < 1$, $A \in L_0^{n-1}$ and $i_+(A)$ can achieve any even (odd) integer in $[g(n), n-1]$.

4. Wedge results for L_0^{n-1} and L_0^{n-2}

Recall that any matrix in $L_0^{n-1} \cup L_0^{n-2}$ has exactly one negative eigenvalue. We show that in fact, if $A \in L_0^{n-1}$, then the negative eigenvalue is the only eigenvalue of A in a sector of semi-angle $\pi/(n-1)$ about the negative real axis.

Similarly, if $A \in L_0^{n-2}$, then this is the only eigenvalue in a sector of semi-angle π/n about the negative real axis. Our proof uses the Cauchy index method as in [12], specifically (9) with $\theta = \pi - \pi/(n-1)$, and $\theta = \pi - \pi/n$, which implies in either case, $\cos n\theta \neq 0$. In the special case, when $n = 2$, it is clear that if $A \in L_0^{n-1}$, then A has one positive and one negative eigenvalue.

Theorem 4.1. *Let A be an $n \times n$ matrix with $n \geqslant 3$. If $A \in L_0^{n-1}$, then all eigenvalues of A distinct from the negative eigenvalue lie in the wedge $|\arg z| \leqslant \pi - \pi/(n-1)$.*

Proof. Consider $A \in L_0^{n-1}$ with $p_A(z)$ given by (6), $U(r)$ and $V(r)$ given by (7) and (8), respectively, with $\theta = \pi - \pi/(n-1)$. Since the theorem is true for $n = 3$ or for $c_1 = 0$ (see Lemma 3.1), we may assume that $n \geqslant 4$ and $c_1 > 0$. Further, assume without loss of generality that A has no eigenvalue on $|\arg z| = \pi - \pi/(n-1)$ (otherwise consider $A + \varepsilon I \in L_0^{n-1}$).

From (9), the number s of roots of $p_A(z)$ such that $|\arg z| < \pi - \pi/(n-1)$ is given by

$$\begin{aligned} s &= I_0^{\infty} \frac{V(r)}{U(r)} + \frac{n}{\pi}(\pi - \pi/(n-1)) - \frac{1}{\pi}\arctan(\tan n(\pi - \pi/(n-1))) \\ &= I_0^{\infty} \frac{V(r)}{U(r)} + n - 1. \end{aligned} \tag{10}$$

Recall that for $A \in L_0^{n-1}$, $c_p \geqslant 0$ for $p = 1, 2, \ldots, n-1$, and $c_n < 0$. Consider first the case when n is even. For $U(r)$ in (7) with $\theta = \pi - \pi/(n-1)$, the coefficient of r^n is $\cos n(\pi - \pi/(n-1)) < 0$. For $k = 1, 2, \ldots, n-1$ the coefficient of r^{n-k} is

$$(-1)^k c_k \cos(n-k)(\pi - \pi/(n-1)).$$

Thus the coefficients of $r^{n-1}, r^{n-2}, \ldots, r^{n/2}$ are $\leqslant 0$, and the coefficients of $r^{n/2-1}, \ldots, r$ are $\geqslant 0$. The constant term in $U(r)$ is $c_n < 0$. Since the number of sign changes in the coefficients of $U(r)$ is either 0 or 2, $U(r)$ has either no positive root or two positive roots, by Descartes' rule of signs. For $V(r)$ in (8) with $\theta = \pi - \pi/(n-1)$, the coefficient of r^n is > 0, the coefficient of r^{n-1} is zero, and for $k = 2, 3, \ldots, n-1$, the coefficient of r^{n-k} is

$$(-1)^k c_k \sin(n-k)(\pi - \pi/(n-1)) \leqslant 0.$$

Hence $V(r)$ has no more than one positive root. If $V(r)$ has no positive root, then $I_0^{\infty}(V(r)/U(r)) = 0$ and $s = n - 1$. So assume $V(r)$ has one positive root, say v. If $U(r)$ has no positive roots, then from (10), $s = n - 1$. Otherwise, $U(r)$ has two positive roots $u_1 \leqslant u_2$. If $c_2 = 0$, then $c_3 = c_4 = \cdots = c_{n-1} = 0$, by Lemma 2.5. Hence $V(r)$ has no positive root, and $s = n - 1$. So assume $c_2 > 0$. Following [13], consider $V(v)/v = 0$. Since the coefficient of r^{n-1} in $V(r)$ is zero, solving for v^{n-1} gives

$$v^{n-1} = \sum_{k=2}^{n-1} (-1)^{k+1} c_k v^{n-k-1} \frac{\sin(n-k)\theta}{\sin n\theta},$$

where $\theta = \pi - \pi/(n-1)$. Substituting for v^{n-1} in $U(v)$, gives

$$U(v) = \sum_{l=2}^{n} (\gamma_l c_l + \delta_l c_1 c_{l-1}) v^{n-l},$$

where $\gamma_l = (-1)^{n-l}(\sin l\theta/\sin\theta)$ and $\delta_l = (-1)^{l-1}\sin(n-l+1)\theta/\sin\theta$. Thus it follows that $\Delta_l = \gamma_l c_l + \delta_l c_1 c_{l-1} \leqslant 0$ for $l = 2, 3, \dots n$, and hence $U(v) \leqslant 0$. Then $\Delta_2 < 0$ as $\delta_2 = 0$, and $\gamma_2 < 0$ hence $U(v) < 0$. Thus $v < u_1$ or $v > u_2$. In either case $I_0^\infty \frac{V(r)}{U(r)} = 0$ and $s = n-1$. In a similar fashion it can also be shown that $s = n-1$ if n is odd. Therefore A has exactly one eigenvalue in the sector $|\arg z| > \pi - \pi/(n-1)$, which must be the negative eigenvalue. The result follows. □

A restatement of Theorem 4.1 for nonnegative matrices follows.

Corollary 4.2. *Let $B \geqslant 0$ be an $n \times n$ matrix with $n \geqslant 2$. Then, the spectral radius, $\rho(B)$, is the only eigenvalue in the wedge*

$$\{z : |\arg(z - \rho_{n-1}(B))| < \pi/(n-1)\}.$$

Consider now $A \in L_0^{n-2}$, with $p_A(z)$ given by (6). Then $c_p \geqslant 0$ for $p = 1, 2, \dots, n-2$, and $c_n \leqslant 0$, with $c_{n-1} < 0$ if $c_n = 0$ [4], Lemma 2.4. In the special case $n = 3$, then $i_+(A) + i_0(A) = 2$ by Corollary 3.3.

Theorem 4.3. *Let A be an $n \times n$ matrix with $n \geqslant 4$. If $A \in L_0^{n-2}$ and A is nonsingular, then all eigenvalues of A distinct from the negative eigenvalue lie in the wedge $|\arg z| \leqslant \pi - \pi/n$. If A is singular, then all eigenvalues of A distinct from the negative eigenvalue lie in the wedge $|\arg z| \leqslant \pi - \pi/(n-1)$.*

Proof. First assume $A \in L_0^{n-2}$ is nonsingular with $p_A(z)$ given by (6), $U(r)$ and $V(r)$ given by (7) and (8), respectively, with $\theta = \pi - \pi/n$. Assume without loss of generality that A has no eigenvalue on $|\arg z| = \pi - \pi/n$ (otherwise take $A + \varepsilon I$). The number s of roots of $p_A(z)$ such that $|\arg z| < \pi - \pi/n$ is given by

$$\begin{aligned} s &= I_0^\infty \frac{V(r)}{U(r)} + \frac{n}{\pi}(\pi - \pi/n) - \frac{1}{\pi}\arctan(\tan n(\pi - \pi/n)) \\ &= I_0^\infty \frac{V(r)}{U(r)} + n - 1. \end{aligned} \tag{11}$$

Recall that for $A \in L_0^{n-2}$, $c_p \geqslant 0$ for $p = 1, 2, \dots, n-2$, and in this case $c_n < 0$. First consider the case when n is even. For $U(r)$ in (7) with $\theta = \pi - \pi/n$, the coefficient of r^n is $\cos n(\pi - \pi/n) < 0$. For $k = 1, 2, \ldots, n-1$, the coefficient of r^{n-k} is

$$(-1)^k c_k \cos(n-k)(\pi - \pi/n).$$

Thus the coefficients of $r^{n-1}, \ldots, r^{n/2+1}$ are $\leqslant 0$, the coefficient of $r^{n/2}$ is zero, and the coefficients of $r^{n/2} - 1, \ldots, r^2$ are $\geqslant 0$. The constant term in $U(r)$ is $c_n < 0$. By Descartes' rule of signs, $U(r)$ has either no positive root or two positive roots. For $V(r)$ in (8) with $\theta = \pi - \pi/n$, the coefficient of r^n is zero, and for $k = 1, 2, \ldots, n-1$ the coefficient of r^{n-k} is

$$(-1)^k c_k \sin(n-k)(\pi - \pi/n),$$

which is $\leqslant 0$, if $k \leqslant n-2$. For $k = n-1$, the coefficient of r^{n-k} (i.e. of r) is opposite in sign from c_{n-1}. So if $c_{n-1} \geqslant 0$, then $V(r)$ has no positive root and hence $I_0^\infty(V(r)/U(r)) = 0$ and (11) implies that $s = n-1$. If $c_{n-1} < 0$, then $V(r)$ has exactly one positive root v. If $U(r)$ has no positive root, then from (11), $s = n-1$. If $U(r)$ has two positive roots, denote them by u_1, u_2 with $u_1 \leqslant u_2$. If $u_1 < v < u_2$, then $I_0^\infty(V(r)/U(r)) = 2$, so $s = n+1$, which is impossible; and if $v = u_1$ or u_2 (or both), then $I_0^\infty(V(r)/U(r)) = 1$, so $s = n$, which is also impossible. This leaves $v < u_1 \leqslant u_2$, or $u_1 \leqslant u_2 < v$ and in either of these cases $I_0^\infty(V(r)/U(r)) = 0$, so from (11), $s = n-1$. Thus the statement is proved for A nonsingular and n even. The case where A is nonsingular and n odd is similar. Now suppose $A \in L_0^{n-2}$ is singular, that is, $c_n = 0$ and $c_{n-1} < 0$. In (1) let $p_A(z) = z \cdot \tilde{p}_A(z)$, where

$$\tilde{p}_A(z) = z^{n-1} - c_1 z^{n-2} + \cdots + (-1)^{n-2} c_{n-2} z + (-1)^{n-1} c_{n-1},$$

where $c_p \geqslant 0$ for $p = 1, 2, \ldots, n-2$, and $c_{n-1} < 0$. If A has no eigenvalue on $|\arg z| = \pi - \pi/(n-1)$, take $\tilde{p}_A(z) = U(r) + iV(r)$, where $z = re^{i(\pi - \pi/(n-1))}$. By the same argument as given in the first part of the proof (for A nonsingular and $c_{n-1} \geqslant 0$), A has exactly one eigenvalue in the sector $|\arg z| > \pi - \pi/(n-1)$. In each case the wedge contains only the negative eigenvalue. The theorem follows. □

A restatement of Theorem 4.3 for nonnegative matrices follows.

Corollary 4.4. *Let $B \geqslant 0$ be an $n \times n$ matrix with $n \geqslant 3$. Then, the spectral radius, $\rho(B)$, is the only eigenvalue in the wedge*

$$\{z : |\arg(z - \rho_{n-2}(B))| < \pi/n\}.$$

Example 4.5. Consider the basic circulant P_n. Then $\rho_{n-1}(P_n) = \rho_{n-2}(P_n) = 0$, and it can easily be verified (see Example 2.2) that $\rho(P_n)$ (which is equal to 1), is the only eigenvalue in the wedge $\{z : |\arg z| < \pi/(n-1)\}$.

Usually Corollaries 4.2 and 4.4 will each provide independent information on localization. However, when $\rho_{n-1}(B) = \rho_{n-2}(B)$ (as was the case in the previous example), the information from Corollary 4.4 is redundant.

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