Existence and Nonexistence of Global Solutions of Some Nonlocal Degenerate Parabolic Equations

WEIBING DENG, YUXIANG LI AND CHUNHONG XIE
Department of Mathematics, Nanjing University
Hankou Road 22, Nanjing 210003, P.R. China
wbdeng@nju.edu.cn
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Abstract—This paper investigates the global existence and nonexistence of positive solutions of the nonlinear degenerate parabolic equation ut = f(u)(Δu + a ∫Ω u dx) with a homogeneous Dirichlet boundary condition. It is proved that there exists no global positive solution if and only if ∫≥ 1/(s f(s)) ds < ∞ and ∫Ω ϕ(x) dx > 1/a, where ϕ(x) is the unique positive solution of the linear elliptic problem -Δϕ(x) = 1, x ∈ Ω; ϕ(x) = 0, x ∈ ∂Ω. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the following degenerate parabolic equation with a nonlocal source:

\[ \begin{align*}
  u_t &= f(u) \left( \Delta u + a \int_{\Omega} u \, dx \right), \quad x \in \Omega, \quad t > 0, \\
  u(x,t) &= 0, \quad x \in \partial \Omega, \quad t > 0, \\
  u(x,0) &= u_0(x), \quad x \in \Omega,
\end{align*} \tag{1.1} \]

where a > 0 and Ω ⊂ R^N is a bounded domain with smooth boundary ∂Ω. In the past several decades, many physical phenomena have been formulated into nonlocal mathematical models (see [1-7] and references therein). The work of this paper is motivated by some recent results of two related problems. On the one hand, the authors of [3,7,8] showed that p = 1 is the blow-up critical exponent of the nonlocal semilinear parabolic equation

\[ u_t = \Delta u + \left( \int_{\Omega} |u|^q \, dx \right)^{p/q}, \quad q \geq 1, \quad p > 0, \tag{1.2} \]
with homogeneous Dirichlet boundary condition. That is to say, if \( p < 1 \), the solutions are global for all initial data while if \( p > 1 \), the solutions blow up for sufficiently large initial data. By a similar discussion as in [7], we can show that \( p - 1 \) is also the critical exponent of the nonlinear equation

\[
 u_t = f(u) \left( \Delta u + \int_{\Omega} u^p \, dx \right) .
\]

(1.3)

However, in the critical case, i.e., \( p - 1 \), when does there exist a global solution? It is not clear. On the other hand, it has been shown that positive solutions of parabolic equations of the form

\[
 u_t = u^p (\Delta u + u) , \quad p > 0,
\]

(1.4)

with homogeneous Dirichlet boundary condition, blow up in finite time if and only if \( X_1 < 1 \) (see [10-14]). Here \( X_1 \) is the first eigenvalue of the Laplacian on \( \Omega \) with zero Dirichlet data on \( \partial \Omega \). Their results show that the first eigenvalue \( X_1 \) plays a crucial role in determining whether or not there exists a global solution. But, for problem (1.1), it seems that \( X_1 \) no longer takes action. Motivated by these results, in this paper, we will establish new criteria for global existence and nonexistence of solutions of problem (1.1).

Throughout this paper, \( u_0(x) \) and \( f(s) \) are assumed to satisfy

\[
 u_0 \in C^1 \left( \bar{\Omega} \right) , \quad u_0 > 0 , \quad \text{in } \Omega ; \quad u_0 = 0 , \quad \frac{\partial u_0}{\partial \nu} < 0 , \quad \text{on } \partial \Omega ;
\]

\[
 f \in C([0, \infty)) \cap C'((0, \infty)) , \quad f > 0 , \quad \text{and } f' > 0 , \quad \text{on } (0, \infty) .
\]

(1.5)

Here \( \nu \) is the unit outward normal vector. Before stating the main results, we give a definition of the solution of problem (1.1).

**DEFINITION.** A positive solution of (1.1) is a function \( u(x, t) \) in \( C([\Omega \times [0, T^*)) \cap C^{2,1}(\Omega \times [0, T^*)) \), positive in \( \Omega \times (0, T^*) \) and satisfying (1.1). If \( T^* = +\infty \), we say the solution is global.

Let \( \varphi(x) \) be the unique positive solution of the following linear elliptic problem:

\[
 -\Delta \varphi(x) = 1 , \quad x \in \Omega ; \quad \varphi(x) = 0 , \quad x \in \partial \Omega .
\]

(1.6)

Denote \( \mu = \int_{\Omega} \varphi(x) \, dx \). Then, let us state our main results.

**THEOREM 1.** If \( \mu > 1/a \) and \( \int_0^\infty 1/(sf(s)) \, ds < \infty \), then there exists no global positive solution of (1.1).

**THEOREM 2.** If \( \mu \leq 1/a \) or \( \int_0^\infty 1/(sf(s)) \, ds = \infty \), then there exists a global positive solution of (1.1).

This paper is organized as follows. In Section 2, we establish the local existence. The proof of the main results will be given in Section 3.

## 2. LOCAL EXISTENCE

Let \( QT = \Omega \times [0, T] \) and \( ST = \partial \Omega \times [0, T] \) for \( 0 < T < \infty \). We first give a maximum principle, which will be used frequently in this paper (see [15, Lemma 2.2] or [8, Lemma 2.1]).

**LEMMA 2.1.** Suppose that \( w(x, t) \in C^{2,1}(QT) \cap C(\bar{Q}_T) \) and satisfies

\[
 w_t - d(x, t) \Delta w \geq c_1(x, t) w + c_2(x, t) \int_{\Omega} c_3(x, t) w(x, t) \, dx , \quad (x, t) \in QT ,
\]

\[
 w(x, t) \geq 0 , \quad (x, t) \in ST ,
\]

\[
 w(x, 0) \geq 0 , \quad x \in \Omega ,
\]

where \( c_1, c_2, c_3 \) are bounded functions and \( c_2, c_3, d \geq 0 \) in \( QT \). Then \( w(x, t) \geq 0 \) on \( \bar{Q}_T \).
To show the local existence of a positive solution of problem (1.1), we consider the following regularized problem:

$$
\begin{align*}
\text{u}_{\varepsilon t} &= f(u_{\varepsilon}) \left( \Delta u_{\varepsilon} + a \int_{\Omega} u_{\varepsilon} \, dx \right), & x \in \Omega, & t > 0, \\
u_{\varepsilon}(x, t) &= \varepsilon, & x \in \partial \Omega, & t \geq 0, \\
u_{\varepsilon}(x, 0) &= u_0 + \varepsilon, & x \in \Omega,
\end{align*}
$$

(2.1)

where $0 < \varepsilon < 1$. By a similar discussion as that of Theorems A.1–A.4 in [7], we know that (2.1) has a unique classical solution $u_\varepsilon(x, t) \geq \varepsilon$, defined on $\Omega \times [0, T^*_\varepsilon)$, where $T^*_\varepsilon$ is the maximal existence time of the solution.

According to Lemma 2.1, we give a comparison principle for problem (2.1).

**Lemma 2.2.** Assume that $w \in C^{2,1}(\Omega_T) \cap C(\bar{\Omega}_T)$ is a nonnegative subsolution (or supersolution) of (2.1). Then $w(x, t) \leq (\geq) u_\varepsilon(x, t)$ on $\bar{\Omega}_T$.

Using Lemma 2.2, we have the following.

**Lemma 2.3.** If $1 > \varepsilon_1 \geq \varepsilon_2 > 0$, then $u_{\varepsilon_1} \geq u_{\varepsilon_2}$ on $(0, T^*_\varepsilon)$ and $T^*_\varepsilon \leq T^*_\varepsilon$.

Then from Lemma 2.3, it follows that $u_\varepsilon$ are monotone with respect to $\varepsilon$. So the limit $T^* = \lim_{\varepsilon \to 0} T^*_\varepsilon$ exists, and as well the point-wise limit

$$
u(x, t) = \lim_{\varepsilon \to 0} u_\varepsilon(x, t) \tag{2.2}$$

exists for $(x, t) \in \bar{\Omega} \times [0, T^*)$. To prove $u(x, t)$ defined by (2.2) is a positive solution of (1.1), we require the following regularity property.

**Lemma 2.4.** For any $\varepsilon \in (0, 1)$ and $T' < T^*_\varepsilon$, we have

$$
\int_0^{T'} \int_{\Omega} \frac{u_{\varepsilon}^2}{f(u_{\varepsilon})} \, dx \, dt + \frac{1}{2} \int_{\Omega} |\nabla u_{\varepsilon}(x, T')|^2 \, dx \leq C_1,
$$

where $C_1$ is a constant independent of $\varepsilon$.

The proof is similar to [14, Lemma 2.3]. We omit it.

Denote by $\lambda_1 > 0$ and $\phi(x)$ the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem:

$$
-\Delta \phi(x) = \lambda \phi(x), \quad x \in \Omega; \quad \phi(x) = 0, \quad x \in \partial \Omega.
$$

It is well known that $\phi(x)$ may be normalized as $\phi(x) > 0$ in $\Omega$ and $\max_{\Omega} \phi(x) = 1$. Thus, from Lemma 2.2, we have the following.

**Lemma 2.5.** Let $h(x, t) = ke^{-\rho t} \phi(x)$, where $k > 0$ is sufficiently small that $k \phi(x) \leq u_0(x)$ and $\rho = \lambda_1 ||f||_{L^\infty(0, k)}$. Then $u_\varepsilon(x, t) \geq h(x, t)$ on $\Omega \times [0, T^*_\varepsilon)$.

Then, by standard arguments (see [9,14]), it follows from Lemmas 2.2–2.5 that $u_\varepsilon \to u$ uniformly with the second derivatives in compact subsets of $\Omega$ and $u$ is a solution of (1.1) on $\Omega \times [0, T^*)$, where $T^*$ is the maximal existence time of $u$. Similarly, we can show that $u(x, t)$ is continuous at any point $(y, t)$, $y \in \partial \Omega$ and $u(y, t) = 0$ (see [9,13]). Thus, we have the following.

**Theorem 2.1.** The function $u(x, t)$ defined by (2.2) is a positive solution of (1.1). Moreover, if $T^* < \infty$, then $\limsup_{t \to T^*} \max_{x \in \Omega} u(x, t) = \infty$. 
3. PROOF OF THE MAIN RESULTS

In this section, we assume \( u(x,t) \) is a positive solution of (1.1) on \( \bar{\Omega} \times [0, T^*) \), where \( T^* \) is the maximal existence time.

**Lemma 3.1.** If \( \mu > 1/a \), then the positive solution \( u \) of (1.1) satisfies \( u(x,t) \geq k \varphi \) for \( (x,t) \in \bar{\Omega} \times [0, T^*) \), where \( k > 0 \) is sufficiently small that \( u_0(x) \geq k \varphi(x) \).

**Proof.** Let \( w = u - k \varphi \); then

\[
\begin{align*}
\partial_t w &= f(u) \left( \Delta w + a \int_{\Omega} w \, dx \right) + kf(u) (a \mu - 1) \\
&\geq f(u) \left( \Delta w + a \int_{\Omega} w \, dx \right).
\end{align*}
\]

Since \( u \) is bounded before its maximal existence time \( T^* \), we have \( u(x,t) \geq k \varphi \) according to Lemma 2.1. \( \blacksquare \)

Denote by \( \varphi_1(x) \) the unique positive solution of the linear elliptic problem

\[
-\Delta \varphi_1(x) = 1, \quad x \in \Omega_1; \quad \varphi_1(x) = 0, \quad x \in \partial \Omega_1.
\]

Here \( \Omega_1 \subset \subset \Omega \). It is obvious that \( \varphi_1(x) \) depends on \( \Omega_1 \) continuously. By the comparison principle for an elliptic equation, we have \( \varphi_1 < \varphi \) on \( \Omega_1 \). Let \( \mu_1 = \int_{\Omega_1} \varphi_1(x) \, dx \); then

\[
\mu_1 = \int_{\Omega_1} \varphi_1 \, dx < \int_{\Omega_1} \varphi \, dx < \int_{\Omega_1} \varphi \, dx = \mu.
\]

**Proof of Theorem 1.** From Lemma 3.1 and \( \mu > 1/a \), it follows that there exist \( \Omega_1 \subset \subset \Omega \) and \( c_0 > 0 \), such that

\[
\mu_1 > \frac{1}{\alpha} \quad \text{and} \quad \mu \geq c_0 > 0, \quad \text{for} \quad x \in \Omega_1, \quad t \in (0, T^*).
\]

Let

\[
\Phi(s) = \int_{c_0}^{s} \frac{ds}{f(s)}, \quad s > c_0.
\]

We see that \( \Phi \) is strictly decreasing and convex on \( (c_0, \infty) \) since \( \Phi''(s) = f'(s)/f^2(s) \geq 0 \). Hence, the inverse function \( \Phi^{-1} \) exists and is also strictly decreasing with

\[
\frac{d}{ds} \Phi^{-1}(s) = -\frac{1}{\Phi'\Phi^{-1}(s)} = -f(\Phi^{-1}(s)).
\]

Let \( \theta(x) = \varphi_1(x)/\mu_1 \) and define \( y : [0, T^*) \rightarrow R \) by

\[
y(t) = \int_{\Omega_1} \Phi(u(x,t)) \theta(x) \, dx.
\]

Taking the derivative of \( y(t) \) with respect to \( t \), we obtain

\[
y'(t) = \int_{\Omega_1} \Phi'(u)u \theta \, dx = -\int_{\Omega_1} \frac{u_1}{f(u)} \theta \, dx \leq \left( \frac{1}{\mu_1} - a \right) \int_{\Omega_1} u \, dx \leq \frac{(1/\mu_1 - a) \int_{\Omega_1} u \theta \, dx}{M},
\]

where \( M \) is a positive constant.
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where \( M = \max_{\theta \in \Omega_1} \{ \theta(x) \} \). By the convexity of \( \Phi \) and Jensen’s inequality, we have
\[
\Phi \left( \int_{\Omega_1} u \theta \, dx \right) \leq \int_{\Omega_1} \Phi(u) \theta \, dx,
\]
which implies, as \( \Phi^{-1} \) decreases,
\[
\int_{\Omega_1} u \theta \, dx \geq \Phi^{-1} \left( \int_{\Omega_1} \Phi(u) \theta \, dx \right) = \Phi^{-1} (y),
\]
which inserted into (3.5) gives
\[
y' (t) \leq \frac{(1/\mu_1 - a) \Phi^{-1}(y(t))}{M}, \tag{3.6}
\]
since \( \mu_1 > 1/a \). Performing the transformation
\[
H(t) = \Phi^{-1} (y(t)), \quad t \in [0, T^*),
\]
we get, from (3.3) and the smoothness of \( y \), that \( H \in C([0, T^*]) \cap C^1([0, T^*)) \) and
\[
H' (t) = \left( \frac{d}{ds} \Phi^{-1} (y) \right) y' = -f (H(t)) y'.
\]
Denote \( b = (a \mu_1 - 1)/(\mu_1 M) \). Then, (3.6) turns into
\[
H' (t) \geq bf (H(t)) H(t), \quad t \in (0, T^*). \tag{3.7}
\]
Furthermore,
\[
H(0) = \Phi^{-1} (y(0)) \geq \Phi^{-1}(0) = c_0. \tag{3.8}
\]
Thus, integrating (3.7) from 0 to \( T^* \), we have
\[
\int_{H(0)}^{H(T^*)} \frac{ds}{s f(s)} \geq bT^*.
\]
That is,
\[
T^* \leq \frac{1}{b} \int_0^\infty \frac{ds}{s f(s)} \triangleq T_s < \infty,
\]
which means \( u(x, t) \) can exist no later than \( t = T_s \), and the proof is completed. \( \square \)

**Proof of Theorem 2.** We set the function \( W(x, t) = K \varphi(x) - u(x, t) \), where \( K \) is sufficiently large that \( K \varphi(x) \geq u_0(x) \) and \( u \) is the positive solution of problem (1.1) defined by (2.2). Suppose \( T^* < +\infty \); then, from \( \mu \leq 1/a \), we have
\[
W_t = f(u) \left( \Delta W + a \int_\Omega W \, dx \right) + K f(u) (1 - a \mu) \geq f(u) \left( \Delta W + a \int_\Omega W \, dx \right).
\]
It follows from Lemma 2.1 that \( u(x, t) \leq K \varphi(x) \) for all \( 0 < t < T^* \). This contradicts Theorem 2.1. Hence, \( u \) exists globally.

Next, we show that if
\[
\int_c^\infty \frac{ds}{s f(s)} = \infty, \quad \forall c > 0, \tag{3.9}
\]
then the positive solution \( u(x, t) \) defined by (2.2) is global.
Choosing $b > u[\Omega]$ and $c > \|u_0\|_{L^\infty(\Omega)}$, we consider the initial problem

$$z' = bf(z)z; \quad z(0) = c. \quad (3.10)$$

Clearly, it follows from (3.9) that $z$ exists for all $0 < t < \infty$ and $z(t) \geq c > 0$. Now, let

$$w(x, t) = z(t) - u(x, t).$$

We see that $w(x, t) \geq 0$ on $\partial \Omega$ and $t = 0$, and

$$w_t = bf(z)z - f(u) \left( \Delta u + a \int_\Omega u \, dx \right) \geq f(u) \left( \Delta w + a \int_\Omega w \, dx \right) + u[\Omega]z \int_0^t f'(\tau z + (1 - \tau) u) \, d\tau w.$$ 

It follows from Lemma 2.1 that $u(x, t) \leq z(t)$ for all $0 < t < T^*$. Hence, $u$ exists globally. This completes the proof of Theorem 2.1.

REFERENCES