Thermal fracture analysis of nonhomogeneous piezoelectric materials using an interaction energy integral method

Fengnan Guo, Licheng Guo *, Hongjun Yu, Li Zhang

Department of Astronautic Science and Mechanics, Harbin Institute of Technology, Harbin 150001, China

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This paper presents a modified interaction energy integral method to analyze the thermal stress intensity factors (TSIFs) and electric displacement intensity factor (EDIF) in nonhomogeneous piezoelectric materials under thermal loading. This modified method is demonstrated to be domain-independent, even when the nonhomogeneous piezoelectric materials contain interfaces with thermo-electro-mechanical properties. As a result, the method is shown to be convenient for determining the TSIFs and EDIF in non-homogeneous piezoelectric materials with interfaces. Several examples are shown, and they successfully verify the domain-independence of the present interaction energy integral. The study results also show that the mismatch of material properties can significantly influence the TSIFs and EDIF, particularly when the crack tip is close to the interface. Crack angles and temperature boundary conditions are also shown to significantly influence the TSIFs and EDIF.

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1. Introduction

Piezoelectric materials, which can be used in sensors, ultrasonic transducers, and piezoelectric motors, have attracted the attention of many researchers (Sosa, 1992; Suo et al., 1992). The piezoelectric effect is identified as the reversible interaction between a mechanical stress and an electric voltage in such a material, i.e., an applied mechanical stress generates a voltage, and an applied voltage changes the shape of the material. In piezoelectric materials, fracture is an important failure mode (Rao and Sunar, 1993). Further understanding of the fracture mechanism in piezoelectrics will contribute to better applications of these materials.

Many researchers have studied the fracture problems in piezoelectric materials under mechanical loading and thermal loading conditions. Sosa (1992) deduced asymptotic expressions for the electromechanical fields in the vicinity of the crack and studied the effects of the electric field on crack arrest and crack skewing in two-dimensional piezoelectric materials. Zhang et al. (2002) and Zhang and Gao (2004) presented the theoretical analyzes and experimental observations of the failure and fracture behavior of piezoelectric materials. Ueda (2007a,b, 2008) obtained the thermal stress intensity factors (TSIFs) and electric displacement intensity factor (EDIF) in a functionally graded piezoelectric strip by solving a series of singular integral equations. Wang and Mai (2002, 2003) and Wang and Noda (2001) studied the fracture problems in piezoelectric materials under steady state thermal loading and thermal shock loading. Kuna (2006) and Kuna and Ricoeur (2008) defined a thermo-electro-mechanical J-integral and studied the thermal crack problems in smart structures. Rao and Kuna (2010) obtained the stress intensity factors (SIFs) and EDIF using interaction integrals in functionally graded piezoelectric materials subjected to thermal loading. However, the authors did not consider a situation in which the piezoelectric materials contain interfaces. Rao (2009) and Rao and Kuna (2008) also studied the fracture problems in functionally graded magneto-electro-elastic materials subjected to mechanical loading using the domain form of interaction integrals. Yu et al. (2012) used interaction integrals to solve the SIFs and EDIF of piezoelectric materials with complex interfaces. The investigators proved that the interaction integral formulation does not involve any derivatives of mechanical and electric properties. However, they did not consider the thermal fracture problems of the piezoelectric materials.

Although many publications present studies of the mechanical and thermal fracture problems of piezoelectric materials, essentially no reports of thermal fracture problems have been given when the integral domain in piezoelectric material contains interfaces. Hence, in this paper, we aim to develop a modified interaction energy integral method that can be used to obtain the fracture parameters in nonhomogeneous piezoelectric materials with interfaces and to study the effects of material discontinuities on the TSIFs and EDIF.
2. The basic equations for piezoelectric materials under thermal loading

In this section, first, the temperature field for the thermal fracture problem in piezoelectric materials is determined. Then, the governing equations for a piezoelectric media subjected to thermal loading in the absence of body forces, electric charges, and heat sources are presented.

The temperature field is assumed to satisfy the 2-D steady Fourier heat conduction equation,

\[ k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} = 0, \]  

where \( k_x \) and \( k_y \) are the thermal conductivities along the x-axis and y-axis. As shown in Fig. 1, the applied boundary conditions are

\[ T_{\mid x=y=W} = T_1, \quad T_{\mid x=0} = T_2, \]

\[ \frac{\partial T}{\partial y} _{\mid x=0} = 0, \quad \frac{\partial T}{\partial y} _{\mid x=W} = 0. \]  

In the Finite Element Method (FEM), the temperature distribution in an element can be expressed as

\[ T^e(x, y) = N^e T = \sum_{i=1}^m N_i T_i, \]  

where \( N \) is the shape function in which the ith component is \( N_i \) and \( T \) is the node temperature in which the ith component is \( T_i \). In this paper, \( m = 4 \) denotes that a 4-node element is adopted. The integration of the Fourier heat conduction equation leads to

\[ \int_A N_i \left( k_x \frac{\partial^2 T^e}{\partial x^2} + k_y \frac{\partial^2 T^e}{\partial y^2} \right) dxdy = 0. \]  

The equation for the temperature of the element can be written as

\[ [K]^e[T^e] = 0, \]  

where \( [K]^e \) is the heat conductance matrix of the element, given as \( [K]^e = \int_A \left[ \frac{\partial N^e_i}{\partial x} \frac{\partial N^e_j}{\partial x} + \frac{\partial N^e_i}{\partial y} \frac{\partial N^e_j}{\partial y} \right] dxdy. \)  

The assembly of the temperature equations of every element leads to the global temperature equations group

\[ [K][T] = [f], \]  

where \([K]\) is the global stiffness matrix, \([T]\) is the vector of nodal temperature, and \([f]\) is the vector of the corresponding element force. Using Eq. (7) and the temperature boundary conditions, the temperature field can be calculated.

The governing equations for a piezoelectric medium subjected to thermal loading in the absence of body forces, electric charges, and heat sources are given below.

The constitutive equations for the piezoelectric material are

\[ \sigma_{ij} = C_{ijkl} e_{kl} - \epsilon_{ij} E_i - \lambda_{ij} \Delta T \]

\[ D_i = e_{ijkl} E_j + \kappa_{ij} E_i - \chi_{ij} \Delta T \]  

(i, j, k, l = 1, 2, 3);

the kinematic equations are

\[ \epsilon_{ij} = \frac{1}{2} \left( u_{ik, j} + u_{jk, i} \right), \quad \delta_{ij} = -\phi_{ij}; \]  

and the equilibrium equations are

\[ \sigma_{ij} = 0, \quad bD_{kl} = 0, \quad h_{ij} = 0. \]  

where \( h_i = -k_i \Delta T \) and \( \Delta T \) is the heat flux and the coefficients of heat conduction, respectively. Additionally, \( \Delta T \) is the difference of the absolute temperature between the temperature and the stress-free reference temperature \( T_0 \). In Eqs. (8)–(10), a comma denotes partial differentiation, and the repeated indices denote summation; \( u_i, \sigma_{ij}, \delta_{ij}, D_i \), and \( E_i \) are the elastic displacements, stresses, the electric potential, electric displacements, and the electric field, respectively; \( C_{ijkl}, \epsilon_{ij}, \kappa_{ij}, \lambda_{ij}, \chi_{ij} \) are the elastic stiffness, piezoelectric constants, and dielectric permittivity, respectively.

In Eq. (8), the temperature stress coefficients \( \lambda_{ij} \) and the pyroelectric displacement constants \( \chi_{ij} \), which are related to the tensors of the thermal expansion coefficients \( \alpha_{ij} \) and the pyroelectric field constants \( g_{ij} \), are shown below:

\[ \lambda_{ij} = C_{ijkl} \alpha_{kl} - \epsilon_{ij} g_{ij}, \]

\[ \chi_{ij} = \epsilon_{ijkl} \alpha_{kl} + \kappa_{ij} g_{ij}. \]  

We define \( e_{ij}^0 \) and \( E_i^0 \), which are the electromechanical parts of the total strain \( e_{ij}^t \) and the total electric field \( E_i^t \), as

\[ e_{ij}^0 = e_{ij}^t - e_{ij}^g = e_{ij}^t - \alpha_{ij} \Delta T, \]

\[ E_i^0 = E_i^t + E_i^g = E_i^t + g_i \Delta T, \]  

where \( e_{ij}^g \) and \( E_i^g \) are the thermo-electro-mechanical components of strain and the electric field, respectively.

According to Eq. (12), the constitutive equations can be simplified as

\[ \sigma_{ij} = C_{ijkl} e_{kl}^0 - \epsilon_{ij} E_i^0, \]

\[ D_i = e_{ijkl} E_j^0 + \kappa_{ij} E_i^0. \]  

According to Hwu (2008), Eq. (13) is equivalent to the following set of equations:

\[ e_{ij}^m = S_{ijkl} \sigma_{kl} + \eta_{ik} D_k, \]

\[ E_i^m = -\eta_{ijkl} \sigma_{kl} + \beta_{ik} D_k. \]  

Using the relationship between the indices \( 11 \to 1, 22 \to 2, 33 \to 3, 23 \to 4, 31 \to 5, 12 \to 6 \), Eq. (13) can be written in Voigt notation as

\[ \sigma_{ij} = C_{ijkl} e_{kl}^m - \epsilon_{ij} E_i^m, \]

\[ D_i = e_{ijkl} E_j^m + \kappa_{ij} E_i^m. \]  

where \( x, \beta = 1, \ldots, 6 \) and \( i, s = 1, 2, 3 \).

For convenience, we define a generalized matrix \( C_{eq} \) for the plane strain state as
Then, Eqs. (13) and (14) can be simplified as
\[
\tilde{\sigma} = C_{ek} \tilde{e}^m,
\]
(17)
where \( \hat{\sigma} \) and \( \hat{e}^m \) are given by
\[
\hat{\sigma} = \left[ \sigma_s \right],
\]
(19)
\[
\hat{e}^m = \left[ \epsilon_{y}^m \right] = \left[ \epsilon_{y}^m - \frac{a_s \Delta T}{C_{y}} \right].
\]
(20)
Using the matrix \( C_{ek} \), we can write
\[
\left[\begin{array}{c} x_i \\ g_i \end{array}\right] = C_{ek}^{-1} \left[\begin{array}{c} \lambda_i \\ Z_i \end{array}\right].
\]
(21)
As a result, the thermal expansion coefficients and the pyroelectric field constants can be expressed by \( \lambda_{il}, Z_{il} \) and the generalized matrix \( C_{ek} \).

3. The interaction energy integral for piezoelectric materials under thermal loading

3.1. Auxiliary fields for piezoelectric materials

In this subsection, the auxiliary fields used in the interaction energy integral method for piezoelectric materials are introduced. As shown in Fig. 2, the polar coordinates \((r, \theta)\), the auxiliary electromagnetic stress and the auxiliary electrical displacement fields can be written as (Rao, 2009; Yu et al., 2012)
\[
\sigma_{0}^{\text{aux}}(r, \theta) = \frac{1}{\sqrt{2\pi r}} \sum_{N} K_{0}^{\text{aux}} P_{l}^{m}(\theta),
\]
(22)
\[
D_{l}^{\text{aux}}(r, \theta) = \frac{1}{\sqrt{2\pi r}} \sum_{N} K_{l}^{\text{aux}} Q_{l}^{m}(\theta).
\]
(23)
The auxiliary mechanical displacements and the electric potential are defined as
\[
u_{l}^{\text{aux}}(r, \theta) = \frac{2}{\sqrt{2\pi r}} \sum_{N} K_{N}^{\text{aux}} d_{l}^{m}(\theta),
\]
(24)
\[
\phi_{l}^{\text{aux}}(r, \theta) = \frac{2}{\sqrt{2\pi r}} \sum_{N} K_{N}^{\text{aux}} e_{l}^{m}(\theta).
\]
(25)

The auxiliary strain fields, the electric fields and their relations to the auxiliary electromechanical stress and auxiliary electrical displacement fields are given by
\[
\epsilon_{y}^{\text{aux}} = \sum_{N} \frac{\eta_{l}}{C_{y}} D_{l}^{\text{aux}},
\]
\[
E_{l}^{\text{aux}} = \eta_{l} D_{l}^{\text{aux}} + \beta_{l} D_{l}^{\text{aux}}.
\]
(26)
where \( \eta_{l} \) are the auxiliary intensity factors, and the subscript \( N \) denotes \( \{I, I, III, IV\} \) which represents different fracture opening modes; \( f^{m}_{l}(\theta) \), \( g^{m}_{l}(\theta) \), \( d^{m}_{l}(\theta) \), and \( e^{m}_{l}(\theta) \) are the angular functions, which are given in the Appendix A. Their values depend only on the material properties at the crack tip.

3.2. Interaction energy integral for piezoelectric materials without interfaces

The J-integral given by Kuna (2006, 2010) for piezoelectric materials is
\[
J = \lim_{\Gamma_{0}^{-} \to \Gamma_{0}^{+}} \int_{\Gamma_{0}^{-}} (H_{0} \delta_{0} - \sigma_{0} u_{0} - D_{0} \phi_{0}) n_{d} d\Gamma,
\]
(27)
where \( H = \sigma_{0} \delta_{0} - D_{0} E_{0} \) is the electric enthalpy density for thermo-electro-mechanical loading and \( \delta_{0} \) is the Kronecker delta. \( \Gamma_{0} \) is the integral path, as shown in Fig. 2.

In this paper, the crack faces are assumed to be mechanical traction-free and electrically impermeable. According to this assumption, it is straightforward to show that the J-integral can be reduced to the following form on one closed contour \( \Gamma_{2} \):
\[
J = \lim_{\Gamma_{0}^{-} \to \Gamma_{2}^{+}} \int_{\Gamma_{0}^{-}} (\sigma_{0} u_{0} + D_{0} \phi_{0} - H_{0} \delta_{0}) n_{d} d\Gamma.
\]
(28)
Here, \( m_{l} \) is the outward normal vector to the contour \( \Gamma_{2} \), \( \Gamma_{2} = \Gamma_{1}^{+} + \Gamma_{2}^{-} - \Gamma_{0} - \Gamma_{1}^{-} \), as shown in Fig. 2, and \( q \) is an arbitrary smooth weight function with values that vary from 1 on \( \Gamma_{0} \) to 0 on \( \Gamma_{1}^{-} \).

Consider two independent equilibrium states of the cracked body. Let state 1 and state 2 correspond to the actual state and an auxiliary state, respectively. The superposition of these two states leads to a new equilibrium state. Using Eq. (28), the J-integral corresponding to the new state, denoted by \( J_{\text{aux}+aux} \), can be expressed as
\[
J_{\text{aux}+aux} = \lim_{\Gamma_{0}^{-} \to \Gamma_{2}^{+}} \int_{\Gamma_{0}^{-}} \left\{ (\sigma_{0}^{\text{aux}} u_{0} + D_{0}^{\text{aux}} \phi_{0} - H_{0}^{\text{aux}} \delta_{0}^{\text{aux}}) + (\sigma_{0}^{\text{aux}} u_{0} + D_{0}^{\text{aux}} \phi_{0} - H_{0}^{\text{aux}} \delta_{0}^{\text{aux}}) \right\} n_{d} d\Gamma.
\]
(29)
The variables with superscript “aux” and the variables without the superscript belong to the auxiliary state 2 and the actual state 1, respectively. The expansion of Eq. (28) leads to
\[
J_{\text{aux}+aux} = f^{\text{aux}} + J^{\text{aux}} + J.
\]
(30)
where the interaction energy integral related to the actual state and auxiliary state can be derived as
\[
I = \lim_{\Gamma_{0}^{-} \to \Gamma_{2}^{+}} \int_{\Gamma_{0}^{-}} \left\{ (\sigma_{0}^{\text{aux}} u_{0} + D_{0}^{\text{aux}} \phi_{0} - H_{0}^{\text{aux}} \delta_{0}^{\text{aux}}) + (\sigma_{0}^{\text{aux}} u_{0} + D_{0}^{\text{aux}} \phi_{0} - H_{0}^{\text{aux}} \delta_{0}^{\text{aux}}) \right\} n_{d} d\Gamma.
\]
(31)
According to the definition of the auxiliary fields and Eqs. (13), (14), and (26), we obtain
\[
\sigma_{0}^{\text{aux}} - D_{0}^{\text{aux}} \delta_{0}^{\text{aux}} = \sigma_{0}^{\text{aux}} - D_{0}^{\text{aux}} E_{0}^{\text{aux}}.
\]
(32)
Thus, the interaction energy integral can be simplified as
\[
I = \lim_{\Gamma_{0}^{-} \to \Gamma_{2}^{+}} \int_{\Gamma_{0}^{-}} \left\{ (\sigma_{0}^{\text{aux}} u_{0} + D_{0}^{\text{aux}} \phi_{0} - H_{0}^{\text{aux}} \delta_{0}^{\text{aux}}) + (\sigma_{0}^{\text{aux}} u_{0} + D_{0}^{\text{aux}} \phi_{0} - H_{0}^{\text{aux}} \delta_{0}^{\text{aux}}) \right\} n_{d} d\Gamma.
\]
(33)
To avoid a potential source of inaccuracy during computation, the contour integral is converted into an equivalent domain integral (Moran and Shih, 1987). By applying the divergence theorem to Eq.
(33), we obtain
\[ I = I_h + I_{nonth}, \]
where
\[ I_h = \int_A \left\{ \sigma_y u_{11} + \sigma_y u_{22} + D_j \phi_{\text{aux}} + D_j \phi_{\text{aux}} \phi_1 \right\} q \, dA, \]
\[ I_{nonth} = \int_A \left\{ \sigma_y u_{11} + \sigma_y u_{22} + D_j \phi_{\text{aux}} + D_j \phi_{\text{aux}} \phi_1 \right\} q \, dA. \]
(35)

Expanding the nonhomogeneous term \( I_{nonth} \), we obtain
\[ I_{nonth} = \int_A \left\{ \sigma_y u_{11} + \sigma_y u_{22} + D_j \phi_{\text{aux}} + D_j \phi_{\text{aux}} \phi_1 \right\} q \, dA. \]
(36)

According to the equilibrium equations, we have
\[ \sigma_{ij} = 0, \quad D_{ij} = 0, \quad \sigma_{yy} = 0, \quad D_{yy} = 0. \]
(38)

Then, the nonhomogeneous term of the interaction energy integral becomes
\[ I_{nonth} = \int_A \left\{ \sigma_y u_{11} + \sigma_y u_{22} + D_j \phi_{\text{aux}} + D_j \phi_{\text{aux}} \phi_1 \right\} q \, dA. \]
(39)

Using the definitions of the strains and the electric field, we can further write the terms \( \sigma_y u_{11} \) and \( D_j \phi_{\text{aux}} \phi_1 \) in Eq.
(39) as
\[ \sigma_y u_{11} = 1/2 \left( \sigma_y u_{11} + \sigma_y u_{22} \right) = \sigma_{yy}, \quad \sigma_{yy} = \sigma_{yy}(u_{11} + u_{22}), \]
(40)
\[ D_j \phi_{\text{aux}} \phi_1 = \sigma_{yy}, \quad D_j \phi_{\text{aux}} \phi_1 = \sigma_{yy}, \quad D_j \phi_{\text{aux}} \phi_1 \]
(41)

Substituting Eqs. (40) and (41) into Eq. (39), we have
\[ I_{nonth} = \int_A \left\{ \sigma_y u_{11} + \sigma_y u_{22} + D_j \phi_{\text{aux}} + D_j \phi_{\text{aux}} \phi_1 \right\} q \, dA. \]
(42)

For further derivation, two extra strain fields, \( \dot{\varepsilon}_{\text{aux}} \) and \( \dot{\varepsilon}_{\text{aux}} \), are introduced as follows
\[ \dot{\varepsilon}_{\text{aux}} = \frac{S_{\text{aux}}}{\eta_{\text{aux}}} \sigma_{\text{aux}} + \eta_{\text{aux}} D_{\text{aux}}, \]
\[ \dot{\varepsilon}_{\text{aux}} = -\eta_{\text{aux}} \sigma_{\text{aux}} + \eta_{\text{aux}} D_{\text{aux}}, \]
(43)

where \( S_{\text{aux}}, \eta_{\text{aux}}, \) and \( \eta_{\text{aux}} \) are the material parameters evaluated at the location of the crack tip. According to the extra strain fields, we can obtain the following formulae:
\[ \dot{\varepsilon}_{\text{aux}} = 1/2 (u_{11} + u_{22}), \]
\[ \dot{\varepsilon}_{\text{aux}} = -\eta_{\text{aux}} \sigma_{\text{aux}} + \eta_{\text{aux}} D_{\text{aux}}. \]
(45)

Using the definitions of the extra strain fields \( \dot{\varepsilon}_{\text{aux}} \) and \( \dot{\varepsilon}_{\text{aux}} \), the first two integrands in Eq.
(42) can be written as
\[ \sigma_y u_{11} = \sigma_y \left( S_{\text{aux}} \sigma_{\text{aux}} + \eta_{\text{aux}} D_{\text{aux}} \right), \]
\[ D_j \phi_{\text{aux}} \phi_1 = D_j \left( \eta_{\text{aux}} \sigma_{\text{aux}} + \beta_{\text{aux}} D_{\text{aux}} \right). \]
(47)

According to Eq. (14), the third and fourth terms in Eq. (42) can be expressed as
\[ \sigma_{\text{aux}} \sigma_{\text{aux}} = \sigma_{\text{aux}} (S_{\text{aux}} \sigma_{\text{aux}} + \eta_{\text{aux}} D_{\text{aux}}), \]
\[ D_j (\eta_{\text{aux}} \sigma_{\text{aux}} + \beta_{\text{aux}} D_{\text{aux}}). \]
(49)

Substituting Eqs. (47)–(50) into Eq. (42), the nonhomogeneous term \( I_{nonth} \) can be determined as
\[ I_{nonth} = \int_A \left\{ \sigma_y (S_{\text{aux}} \sigma_{\text{aux}} - \eta_{\text{aux}} \sigma_{\text{aux}}) + \sigma_y (\eta_{\text{aux}} \sigma_{\text{aux}} - \eta_{\text{aux}} \sigma_{\text{aux}}) D_{\text{aux}} + D_j (\eta_{\text{aux}} \sigma_{\text{aux}} - \eta_{\text{aux}} \sigma_{\text{aux}}) D_{\text{aux}} \right\} q \, dA, \]
(51)

Finally, the interaction energy integral for the thermal fracture problems of piezoelectric materials can be written as
\[ I = \int_A \left\{ \sigma_y u_{11} + \sigma_y u_{22} + D_j \phi_{\text{aux}} + D_j \phi_{\text{aux}} \phi_1 \right\} q \, dA \]
\[ + \int_A \left\{ \sigma_y (S_{\text{aux}} \sigma_{\text{aux}} - \eta_{\text{aux}} \sigma_{\text{aux}}) + \sigma_y (\eta_{\text{aux}} \sigma_{\text{aux}} - \eta_{\text{aux}} \sigma_{\text{aux}}) D_{\text{aux}} + D_j (\eta_{\text{aux}} \sigma_{\text{aux}} - \eta_{\text{aux}} \sigma_{\text{aux}}) D_{\text{aux}} \right\} q \, dA \]
\[ + \int_A \left\{ \sigma_y (\eta_{\text{aux}} \sigma_{\text{aux}} - \eta_{\text{aux}} \sigma_{\text{aux}}) + D_j (\eta_{\text{aux}} \sigma_{\text{aux}} - \eta_{\text{aux}} \sigma_{\text{aux}}) D_{\text{aux}} \right\} q \, dA. \]
(52)

where the last term in Eq. (52) is the thermal contribution to the interaction energy integral, which can be denoted by \( \text{thermal} \) and further written as
\[ \text{thermal} = \int_A \left\{ \sigma_y (\eta_{\text{aux}} \sigma_{\text{aux}} - \eta_{\text{aux}} \sigma_{\text{aux}}) + \sigma_y (\eta_{\text{aux}} \sigma_{\text{aux}} - \eta_{\text{aux}} \sigma_{\text{aux}}) D_{\text{aux}} + D_j (\eta_{\text{aux}} \sigma_{\text{aux}} - \eta_{\text{aux}} \sigma_{\text{aux}}) D_{\text{aux}} \right\} q \, dA, \]
(53)

This is a new term in the interaction energy integral for the thermal problems of piezoelectric materials. If all fields involved in Eq.
(52) can be obtained, then the interaction energy integral for piezoelectric materials without any interfaces can be determined.

3.3. Interaction energy integral for piezoelectric materials with an interface

In this section, we discuss the interaction energy integral method for piezoelectric materials with an interface.

As shown in Fig. 3, the integral contour \( \Gamma_1 \) is divided by an interface \( \Gamma_{\text{interface}} \) that is perfectly bonded. Thus, domain \( A \) is divided into two parts, \( A^+ \) and \( A^- \), which are enclosed by contours...
\( \Gamma_{01} \) and \( \Gamma_{02} \), respectively, and where \( \Gamma_{01} = \Gamma_{11} + \Gamma_{\text{interface}} + \Gamma_{31} + \Gamma_3 \) and \( \Gamma_{02} = \Gamma_{21} + \Gamma_{\text{interface}} \).

When the interface intersects the integral path \( \Gamma_2 \), the form of the interaction energy integral in Eq. (33) can be rewritten as

\[
I = \int_{\Gamma} \left\{ \sigma_{ij}^{\text{mix}} u_{ij} + \sigma_{ij}^{\text{mix}} D_j \phi_1 + D_i \phi_j \right\} q_i dA \\
+ \int_{\Gamma} \left\{ \sigma_0 (a_{ijkl} - s_{ijkl}) \sigma_{ij}^{\text{mix}} + \sigma_0 (a_{ijkl} - s_{ijkl}) D_{jk} \phi_1 + D_{ij} \phi_k \right\} q dA
\]

(54)

where \( \sigma_0 \) thermal and \( \gamma_0 \) thermal are domain integrals as a result of the thermal contributions in domains \( A^I \) and \( A^s \), respectively. The domain integrals can be expressed as

\[
I_A^{\text{thermal}} = \int_{\Gamma} \left\{ \sigma_0^{\text{mix}} (\xi_0) \Delta T + \xi_0 \Delta T \right\} q dA,
\]

(55)

\[
I_A^{\text{thermal}} = \int_{\Gamma} \left\{ \sigma_0^{\text{mix}} (\xi_0) + \xi_0 \Delta T \right\} q dA.
\]

(56)

Additionally, \( I_{\text{interface}} \) is a line integral along the interface, determined as the expression shown below

\[
I_{\text{interface}} = \int_{\Gamma_{\text{interface}}} \left\{ \sigma_0^{\text{mix}} (a_{ijkl}^{\text{th}} / 2) e_{ij} - \sigma_0^{\text{mix}} u_{ij} - \sigma_0^{\text{mix}} D_j \phi_1 + D_{ij} \phi_k \right\} q dA
\]

(57)

The temperature fields on both sides of the interface \( \Gamma_{\text{interface}} \) are usually continuous, we have

\[
T^I = T^s = T.
\]

(64)

Because the material interface is in equilibrium, the tractions and the surface charges on both sides of the interface \( \Gamma_{\text{interface}} \) are equal. Thus,

\[
m_0 \sigma_0^i = m_0 \sigma_0^i, \quad m_0 D_0^i = m_0 D_0^i.
\]

(65)

Because the interface is assumed to be perfectly bonded, the derivatives of the mechanical displacements and the actual electric potential with respect to the curvilinear coordinate \( \xi_2 \) are equal on both sides of the interface, \( I_{\text{interface}} \). This yields

\[
\left( \frac{\partial u_i}{\partial \xi_2} \right)^I = \left( \frac{\partial u_i}{\partial \xi_2} \right)^s, \quad \left( \frac{\partial \phi}{\partial \xi_2} \right)^I = \left( \frac{\partial \phi}{\partial \xi_2} \right)^s.
\]

(66)

Applying the strain–displacement relations and using the symmetry of \( \sigma_0^{\text{mix}} \), the first integrand in Eq. (63) can be obtained as

\[
\sigma_0^{\text{mix}} \left( e_{ij}^{\text{th}} - e_{ij}^{\text{th}} \right) m_0 = \sigma_0^{\text{mix}} \left[ \left( \frac{\partial u_i}{\partial \xi_2} \right)^I - \left( \frac{\partial u_i}{\partial \xi_2} \right)^s \right] m_1,
\]

(67)

Applying the chain rule to Eq. (67), we obtain

\[
\sigma_0^{\text{mix}} \left( e_{ij}^{\text{th}} - e_{ij}^{\text{th}} \right) m_0 = \sigma_0^{\text{mix}} \left[ \left( \frac{\partial u_i}{\partial \xi_2} \right)^I - \left( \frac{\partial u_i}{\partial \xi_2} \right)^s \right] \frac{\partial \xi_{ik}}{\partial \xi_2} m_1,
\]

(68)

Using Eq. (63) and the expression \( \frac{\partial \xi_{ik}}{\partial \xi_2} = m_i \), Eq. (68) can be simplified as

\[
\sigma_0^{\text{mix}} \left( e_{ij}^{\text{th}} - e_{ij}^{\text{th}} \right) m_0 = m_0 \sigma_0^{\text{mix}} \left[ \left( \frac{\partial u_i}{\partial \xi_2} \right)^I - \left( \frac{\partial u_i}{\partial \xi_2} \right)^s \right] m_1,
\]

(69)

Using the chain rule and substituting Eq. (66) into the second integrand in Eq. (63), we have

\[
\sigma_0^{\text{mix}} \left[ (u_i)_1^I - (u_i)_1^s \right] m_0 = m_0 \sigma_0^{\text{mix}} \left[ \left( \frac{\partial u_i}{\partial \xi_2} \right)^I - \left( \frac{\partial u_i}{\partial \xi_2} \right)^s \right] m_1.
\]

(70)

Fig. 4. A schematic illustration of a curvilinear coordinate system.
According to Eq. (65), both the third integrand and the fourth term in Eq. (63) are equal to zero. That is,

\[ m_1\left(\sigma_0^{\text{th}} - \sigma_0^{\text{th}}\right)w_{aux} = 0, \quad (71) \]

\[ m_1\left(D_0^{\text{th}} - D_0^{\text{th}}\right)\phi_{aux} = 0. \quad (72) \]

Similarly, using the definition of the electric field and substituting Eq. (66) and \( \frac{\partial}{\partial x} = m_1 \) into the fifth integrand in Eq. (63), we obtain

\[ D_{ij}^{\text{aux}}\left(E_{i}^{\text{aux}} - E_{i}^{\text{aux}}\right)m_1 = -m_1D_{ij}^{\text{aux}}\left(\frac{\partial\phi}{\partial x}\right)^{\text{aux}} - \frac{\partial\phi}{\partial x}\right)^{\text{aux}}m_1. \]

\[ D_{ij}^{\text{aux}}\left(E_{i}^{\text{aux}} - E_{i}^{\text{aux}}\right)m_1 = -m_1D_{ij}^{\text{aux}}\left(\frac{\partial\phi}{\partial x}\right)^{\text{aux}} - \frac{\partial\phi}{\partial x}\right)^{\text{aux}}m_1. \]

Similarly, the sixth integrand in Eq. (63) becomes

\[ D_{ij}^{\text{aux}}\left(\phi_{i}^{\text{aux}} - \phi_{i}^{\text{aux}}\right)m_1 = m_1D_{ij}^{\text{aux}}\left(\frac{\partial\phi}{\partial x}\right)^{\text{aux}} - \frac{\partial\phi}{\partial x}\right)^{\text{aux}}m_1. \]

Substituting Eqs. (69)–(74) into Eq. (63), we obtain

\[ I_{\text{interface}} = \int_{\Gamma_{\text{interface}}} \left\{ \sigma_0^{\text{th}}\left(\sigma_0^{\text{th}} - \sigma_0^{\text{th}}\right)^{\text{aux}} + D_0^{\text{aux}}\left(E_0^{\text{aux}} - E_0^{\text{aux}}\right) \right\} \Delta T m_1 q d\Gamma. \]

Finally, the interaction energy integral in Eq. (54) for piezoelectric materials with an interface can be determined.

Now, the applicability of the present interaction energy integral is considered. It is evident that the present interaction energy integral is applicable to the study of thermal fracture problems in homogeneous piezoelectric materials in the absence of interfaces. In the following discussion, we assume that three types of interfaces in the piezoelectric materials, including (1) a mechanical interface, (2) an electrical interface, and (3) a thermal interface.

For the first two types of interfaces, there are no derivatives of the mechanical and electrical properties in Eq. (54), thus the interaction energy integral is applicable in these two cases.

Next, we focus on whether the present interaction energy integral is applicable to thermal fracture problems in piezoelectric materials with a thermal interface. For convenience, we discuss two types of thermal interfaces, a weak thermal interface and a strong thermal interface. A weak thermal interface means that the thermal properties are continuous but that their derivatives are not continuous on both sides of the interface. Alternatively, a strong thermal interface means that both the thermal properties and their derivatives are discontinuous on either side of the interface.

To state the problem clearly, we discuss individually whether the terms in Eq. (54) are applicable to the study of problems with a thermal interface.

(i) There are no thermal properties in the first two terms of Eq. (54). Hence, the first two terms are applicable to the thermal interface.

(ii) The values of the thermal properties and their derivatives are continuous in either \( A' \) or \( A \). As a result, the two terms \( i_{\text{thermal}}^{\text{th}} \) and \( i_{\text{thermal}}^{\text{th}} \) are applicable to both weak and strong thermal interfaces.

(iii) For the last term, \( i_{\text{interface}} \), if the thermal interface is weak, the thermal properties on both sides of interface are continuous, so the last term in Eq. (54) vanishes, i.e., \( I_{\text{interface}} = 0 \). Thus, \( I_{\text{interface}} \) is applicable to the case with a weak thermal interface. For a strong thermal interface, although their derivatives are either discontinuous or do not exist on both sides of the interface, the interface integral \( I_{\text{interface}} \) does not contain any derivatives of thermal properties. Thus, \( I_{\text{interface}} \) is also applicable to this case with a strong thermal interface.

Therefore, the interaction energy integral is applicable to all cases discussed above, regardless of whether the material properties on either side of the interface are continuous or not.

Due to the arbitrarily chosen domain size in the above derivation, the present interaction energy integral is domain-independent. In the following subsection, we discuss how to obtain the TSIFs and EDIF using the domain-independent interaction energy integral developed above.

3.4. Extraction of the TSIFs and EDIF from the interaction energy integral

For piezoelectric materials, the relationship between interaction energy integral and the intensity factors are given as (Rao and Kuna, 2008)

\[ I = K_1Y_{11} + K_2Y_{12} + K_3Y_{14} \]

where \( Y_{11} \) and \( Y_{12} \) are components of the \((4 \times 4)\) generalized Irwin matrix \( Y \) given in Appendix B.

If the auxiliary state is the mode-I fracture state, i.e., \( K_1^{aux} = 1 \), \( K_2^{aux} = K_3^{aux} = 0 \), then Eq. (77) can be reduced to

\[ I_1^{th} = K_1Y_{12} + K_2Y_{12} + K_3Y_{14}. \]

Similarly, if the auxiliary state is the mode-II or mode-IV fracture state, Eq. (77) can be reduced to

\[ I_2^{th} = K_1Y_{11} + K_2Y_{12} + K_3Y_{14}. \]

or

\[ I_3^{th} = K_1Y_{14} + K_2Y_{14} + K_3Y_{14}. \]

respectively. By solving the set of Eqs. (78)–(80), the TSIFs and EDIF then can be obtained if the interaction energy integrals \( I_1^{th}, I_2^{th}, \) and \( I_3^{th} \) are determined.

4. Examples and discussion

In this section, first, a thermal crack problem in a piezoelectric plane is studied to verify the validity of the present method. Then, the domain-independence of the interaction energy integral is checked in a nonhomogeneous piezoelectric material containing an interface, as well as lacking an interface. Next, the influence of the material discontinuity on the TSIFs and EDIF is investigated. Finally, the inclined crack problems in piezoelectric materials subjected to different thermal loadings are considered.

4.1. Verification of the present method

To verify the applicability of the present method, a thermal crack problem in a piezoelectric plane is studied. As shown in Fig. 5, a central crack is present in a homogeneous piezoelectric plane. The crack length is \( 2a \) and the thickness is \( 2h \). The length is assumed to be \( 2L = 8h \) relative to an infinitely long plane. It is as-
assumed that uniform temperatures $T_0$ and $-T_0$ are maintained at the top and the bottom planes, respectively. The crack faces are assumed to be thermally and electrically insulated. In 2007, this problem was studied by Ueda (2007b). The material properties and the boundary conditions are the same as those used by Ueda (2007b).

Fig. 6 shows a comparison of the present mode II TSIFs with the analytical values given by Ueda (2007b). It can be seen that the present results agree well with those of Ueda.

4.2. Verification of the domain-independence of the interaction energy integral

Note that in the works by Ueda (2007a, b) and Rao and Kuna (2010), the crack faces are treated as thermally insulated. However, in our work, the crack surfaces are assumed to be permeable with respect to temperature fields.

Three numerical simulations are run to assess the domain-independence of the present interaction energy integral. A piezoelectric plate that contains an edge crack of length $a$ is shown in Fig. 1, where $W$ denotes the width, and $L$, the length. The material properties vary along the $x$-axis and can be expressed as

$$\left( C_i, e_{ij}, k_{ij} \right) = \left( C_{ij0}, e_{ij0}, k_{ij0} \right) \exp(\beta x/W),$$

$$(k_1, k_2) = \left( k_{10}, k_{20} \right) \exp(\omega x/W),$$

$$(\lambda_{ij}, \chi_{ij}) = \left( \lambda_{ij0}, \chi_{ij0} \right) \exp(\delta x/W),$$

where the nonhomogeneous parameters $\beta, \omega, \lambda$, and $\delta$ are positive or negative constants; $(k_1, k_2)$ are the orthotropic coefficients of heat conduction; the subscript 0 indicates that the material parameters whose values are given below are evaluated at the plane $x = 0$.

$C_{110} = 126 \text{ GPa}$, $C_{220} = 117 \text{ GPa}$, $C_{330} = 53 \text{ GPa}$, $C_{130} = 55 \text{ GPa}$, $C_{440} = 353 \text{ GPa}$, $e_{220} = -6.5 \text{ C/m}^2$, $e_{220} = 23.3 \text{ C/m}^2$, $\epsilon_{110} = 17 \text{ C/m}^2$, $\kappa_{110} = 15.1 \times 10^{-8} \text{ C/Vm}$, $\kappa_{220} = 13 \times 10^{-8} \text{ C/Vm}$, $K_{10} = 50 \text{ W/km}$, $k_{20} = 75 \text{ W/km}$, $\lambda_{110} = 1.9738 \times 10^8 \text{ Pa/K}$, $\lambda_{220} = 1.4165 \times 10^8 \text{ Pa/K}$, $\lambda_{120} = 0 \text{ Pa/K}$, $\lambda_{130} = 0 \text{ N/(Vm K)}$, $\lambda_{230} = 0 \text{ N/(Vm K)}$.

The geometry of the plate is $W = L = 20$, and the crack length is given as $a = 9$. The left and right boundary conditions for the temperature field are assumed to be $T_1 = T_2 = 10^\circ \text{C}$. The top and bottom boundaries are established as adiabatic. The initial temperature $T_0$ is taken to be 0 °C. The displacement boundaries are chosen as $u_y = 0$ at the lower left and lower right points, and the other boundaries are free edges. The normalized factors for TSIFs and EDIF are $K = \lambda_{220} \Delta T / \sqrt{\pi a}$ and $K = -\lambda_{220} \Delta T / \sqrt{\pi a}$, respectively, specifying the conditions of generalized plane strain.

First, we set the nonhomogeneous parameters to be same, i.e., $\beta = \omega = \lambda = \delta$. To verify the domain-independence of the TSIFs and EDIF, in this example, five different integral domains with $R_c/R_{tip} = (4, 8, 16, 24, 32)$, as shown in Fig. 7, are chosen. Here, $R_{tip}$ is the edge length of the element that contains the crack tip, and $R_c$ is the radius of the reference circle whose center is located at the crack tip. Table 1 presents the normalized TSIFs and EDIF for different integral domains. It can be found that the relative error is less than 0.1%.

According to Eq. (21), the thermal expansion coefficients $\sigma_{hi}$ and the piezoelectric field constants $g_i$ are homogeneous in the piezoelectric plate when $\beta = \omega = \lambda = \delta$. Next, we discuss whether the domain-independence of the interaction energy integral still holds for nonhomogeneous $\sigma_{hi}$ and $g_i$. The nonhomogeneous parameters are set to be $\beta = 0$, $\omega = 0$, $\lambda = 0$, or $\delta = 0$. Table 2 presents the normalized TSIFs and EDIF in a nonhomogeneous piezoelectric material for different integral domains. It can be seen that the relative error is less than 0.2%.

In the above two examples, the interface integral $I_{interface}$ in Eq. (76) is zero because there is no interface in the plate. At last, we verify the domain-independence of the present method for a piezoelectric plate with an interface. A perfect bonded interface is present in the middle ($x = 0.5W$) of the nonhomogeneous piezoelectric plate. On the left side of the plate, the material is assumed to be homogeneous, that is, the nonhomogeneous parameters $\beta = \omega = \lambda = \delta = 0$. On the right side of the plate, the material properties are also homogeneous, but the properties are expressed as

$$(C_i, e_{ij}, k_{ij}) = (C_{ij0}, e_{ij0}, k_{ij0}),$$

$$(k_1, k_2) = (k_{10}, k_{20}),$$

$$(\lambda_{ij}, \chi_{ij}) = 10 \times (\lambda_{ij0}, \chi_{ij0}).$$

In this case, the interface integral $I_{interface}$ in Eq. (76) is not zero because the thermal expansion coefficients and the piezoelectric field constants on both sides of the plate are discontinuous. Table 3 shows the normalized TSIFs and EDIF for different integral domains. It can be seen that the relative error is less than 0.2%, regardless of whether the interface intersects the integral domain $(R_c/R_{tip} \geq 8)$ or not $(R_c/R_{tip} < 8)$.

Thus, the above three examples successfully verify the domain-independence of the present interaction energy integral method.

4.3. Influence of the thermo-electro-mechanical properties on the TSIFs and EDIF

In this subsection, the influence of the thermo-electro-mechanical properties on the TSIFs and EDIF is investigated. As shown in Fig. 8, an inner crack is present in a nonhomogeneous piezoelectric
plate with an interface. The symbols $A$ and $B$ denote two crack tips. The crack length is $2a$, and the $x$-coordinate of the crack center is $c$. The interface is perfectly bonded and located at the center of the plate ($x = 0.5W$). In this example, the width $W$ and the length $L$ are set to be 20, and the crack length is $a = 1$. The boundary and initial conditions for the temperature field are the same as those used in previous examples.

Table 1
The mode-I TSIFs and EDIF for different integral domains with the same nonhomogeneous parameters.

<table>
<thead>
<tr>
<th>$K_R/K_{tip}$</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>24</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = \omega = \delta = 0$</td>
<td>$K_R/K_{ip}$</td>
<td>0.0386</td>
<td>0.0386</td>
<td>0.0386</td>
<td>0.0386</td>
</tr>
<tr>
<td></td>
<td>$K_{ip}/K_0$</td>
<td>2.4322</td>
<td>2.4321</td>
<td>2.4322</td>
<td>2.4323</td>
</tr>
<tr>
<td>$\omega = \delta = \beta = 0.25$</td>
<td>$K_R/K_{ip}$</td>
<td>0.3417</td>
<td>0.3416</td>
<td>0.3416</td>
<td>0.3416</td>
</tr>
<tr>
<td></td>
<td>$K_{ip}/K_0$</td>
<td>78.8734</td>
<td>78.8613</td>
<td>78.8586</td>
<td>78.8575</td>
</tr>
<tr>
<td>$\beta = \omega = \delta = -0.25$</td>
<td>$K_R/K_{ip}$</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
</tr>
<tr>
<td></td>
<td>$K_{ip}/K_0$</td>
<td>-0.3465</td>
<td>-0.3465</td>
<td>-0.3465</td>
<td>-0.3465</td>
</tr>
</tbody>
</table>

Table 2
The mode-I TSIFs and EDIF for different integral domains with different nonhomogeneous parameters.

<table>
<thead>
<tr>
<th>$K_R/K_{tip}$</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>24</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = \omega = 0$</td>
<td>$K_R/K_{ip}$</td>
<td>-0.0106</td>
<td>-0.0106</td>
<td>-0.0106</td>
<td>-0.0106</td>
</tr>
<tr>
<td>$\delta = -0.25$</td>
<td>$K_{ip}/K_0$</td>
<td>0.2376</td>
<td>0.2376</td>
<td>0.2377</td>
<td>0.2378</td>
</tr>
<tr>
<td>$\omega = \delta = 0$</td>
<td>$K_R/K_{ip}$</td>
<td>-0.2901</td>
<td>-0.29</td>
<td>-0.29</td>
<td>-0.29</td>
</tr>
<tr>
<td>$\beta = -0.25$</td>
<td>$K_{ip}/K_0$</td>
<td>-46.1023</td>
<td>-46.1046</td>
<td>-46.1076</td>
<td>-46.1083</td>
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</tbody>
</table>

Table 3
The TSIFs and EDIF for different integral domains in a piezoelectric plate with an interface.

<table>
<thead>
<tr>
<th>$K_R/K_{tip}$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>16</th>
<th>24</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_R/K_{ip}$</td>
<td>1.2535</td>
<td>1.2535</td>
<td>1.2537</td>
<td>1.2537</td>
<td>1.2537</td>
<td>1.2537</td>
</tr>
<tr>
<td>$K_{ip}/K_0$</td>
<td>0.1032</td>
<td>0.1032</td>
<td>0.1032</td>
<td>0.1032</td>
<td>0.1032</td>
<td>0.1032</td>
</tr>
</tbody>
</table>

Fig. 7. Different integral domains around the crack tip.

Fig. 8. An inner crack in a nonhomogeneous piezoelectric plate containing an interface under thermal loading.
To study the influence of the mechanical properties on the TSIFs and EDIF, the material properties are defined as follows. On the left side of the plate \((x \leq 0.5W)\), the material properties are given by

\[
(C_{ij}, e_{ij}, K_{ij}) = (C_{ij0}, e_{ij0}, K_{ij0}) \exp(\beta x/W) \quad (x \leq 0.5W),
\]

\[
(k_1, k_2) = (k_{10}, k_{20}) \exp(\alpha x/W) \quad (x \leq 0.5W),
\]

\[
(\lambda_{ij}, \chi_{ij}) = (\lambda_{ij0}, \chi_{ij0}) \exp(\delta x/W) \quad (x \leq 0.5W).
\]

On the right side of the plate \((x > 0.5W)\), the thermal conductivity can be expressed as

\[
(k_1, k_2) = (k_{10}, k_{20}) \exp(\alpha x/W) \quad (x > 0.5W).
\]

Three cases for the other material properties are chosen as

\[
\begin{aligned}
\text{Case 1:} & \quad (C_{ij}, e_{ij}, K_{ij}) = (C_{ij0}, e_{ij0}, K_{ij0}) \exp(\beta x/W) \quad (x > 0.5W), \\
\text{Case 2:} & \quad (C_{ij}, e_{ij}, K_{ij}) = (C_{ij0}, e_{ij0}, K_{ij0}) \exp(\beta x/W) \quad (x > 0.5W), \\
\text{Case 3:} & \quad (C_{ij}, e_{ij}, K_{ij}) = (C_{ij0}, e_{ij0}, K_{ij0}) \exp(\beta x/W) \quad (x > 0.5W),
\end{aligned}
\]

where \(\beta = \alpha = \delta = 0.25\). In Case 1, the mechanical and electrical properties and their derivatives are continuous. In Case 2, the mechanical and electrical properties are continuous but their derivatives are discontinuous. In Case 3, both the mechanical and electrical properties and their derivatives are discontinuous. In the above three cases, the interface integral is zero because the thermal expansion coefficient and the pyroelectric field constants are continuous on both sides of the interface.

Note that when the crack tip is on the interface or very close to it, the stresses lose the inverse square root singularity and instead they abide by (Hutchinson and Suo, 1992)

\[
\sigma_{ij} \sim R^{-1}f_{ij}(\theta),
\]

where the factor \(R\) plays a role analogous to that in the regular SIF, and \(f_{ij}(\theta)\) are dimensionless angular distributions. The singularity exponent \(s(0 < s < 1)\) depends on the mismatch of the material elastic properties. If the ratio of the distance between the crack tip and the interface to the crack length is more than 0.03, the inverse square root is still valid (Erdogan et al., 1974; Wang and Chau, 2001; Yu et al., 2009). In this work, a similar assumption as that of Erdogan et al. (1974), Wang and Chau (2001), and Yu et al. (2009) is adopted, namely, the distance from the crack tip to the interface is restricted to be no less than 3% of the crack length. The stress singularity in the computation process should remain \(-0.5\).

Figs. 9–11 show the normalized mode-I TSIFs and EDIF computed by the present method for the three cases. It can be observed that when the crack passes through the interface, the mode-I TSIFs and EDIF for Case 1 vary smoothly. However, a sudden change in the values of the mode-I TSIFs and EDIF can be observed when the crack is close to the interface \((c/W \approx 0.45\) or \(c/W \approx 0.55)\), as in Case 2 and Case 3.

Next, we study two more cases in which the interface integrals are not zero. For these two cases, the material properties on the right side of the plate are set to be

\[
\begin{aligned}
\text{Case 4:} & \quad (C_{ij}, e_{ij}, K_{ij}) = (C_{ij0}, e_{ij0}, K_{ij0}) \exp(\beta x/W) \quad (x > 0.5W), \\
\text{Case 5:} & \quad (C_{ij}, e_{ij}, K_{ij}) = (C_{ij0}, e_{ij0}, K_{ij0}) \exp(\beta x/W) \quad (x > 0.5W),
\end{aligned}
\]

From Eqs. (95) and (96), it can be seen that the interface integrals of Case 4 and Case 5 are not zero. Figs. 12 and 13 show the variation of the normalized mode-I TSIFs and EDIF with the crack position \(c/W\) for Case 4 and Case 5. It can be seen that the variations are not smooth when the crack passes the interface. It can be concluded that the mismatch of material properties at the interface can significantly influence the mode-I TSIFs and EDIF.

4.4. Inclined crack problems in piezoelectric materials under thermal loading

Fig. 14 shows an inclined central crack in a nonhomogeneous piezoelectric plate. The crack length is \(2a\), and the crack angle is \(\theta\). The properties of the piezoelectric plate are adopted as

\[
(C_{ij}, e_{ij}, K_{ij}) = (C_{ij0}, e_{ij0}, K_{ij0}),
\]

\[
(k_1, k_2) = (k_{10}, k_{20}) \exp(\alpha x/W),
\]

\[
(\lambda_{ij}, \chi_{ij}) = (\lambda_{ij0}, \chi_{ij0}) \exp(\delta x/W).
\]
Fig. 10. The normalized mode-I TSIFs and EDIF vary with the crack position for Case 2.

Fig. 11. The normalized mode-I TSIFs and EDIF vary with the crack position for Case 3.

Fig. 12. The normalized mode-I TSIFs and EDIF vary with the crack position for Case 4.
In this example, the nonhomogeneous parameters are adopted as $\alpha = \delta = 0.125$. The crack length is $a = 0.05W = 0.05L$, and the crack angles range from 0° to 150°. Two cases are chosen for the temperature boundary condition on the left and right of the piezoelectric plate, i.e., $T_1 = T_2 = 10{\degree}C$ and $T_1 = 10{\degree}C$, $T_2 = 100{\degree}C$. The boundary conditions on the top and bottom and the initial condition for the temperature field are the same as those given in previous examples. Table 4 and Table 5 show the normalized TSIFs and EDIF for different crack angles and temperature boundaries. It can be seen that the crack angle and the temperature boundary can also significantly influence the TSIFs and EDIF.

### 5. Conclusions

A modified interaction energy integral method is developed to solve the crack problems of nonhomogeneous piezoelectric materials under thermal loading. This modified method is shown to be domain-independent. Thus, it provides the possibility to investigate crack problems in nonhomogeneous piezoelectric materials containing mechanical, electrical, and thermal interfaces. The domain-independence of the approach is verified by several examples. The present method can be used to obtain the TSIFs and EDIF with high efficiency. The influence of the material properties on the TSIFs and EDIF is investigated. It is found that mechanical, electrical, and thermal property mismatch at the interface can

### Table 4

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>120°</th>
<th>135°</th>
<th>150°</th>
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<tbody>
<tr>
<td>$K_I(A)/K_0^I$</td>
<td>0.0965</td>
<td>0.0468</td>
<td>-0.0082</td>
<td>-0.065</td>
<td>-0.1138</td>
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<tr>
<td>$K_II(A)/K_0^I$</td>
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<td>-1.1002</td>
<td>-0.1228</td>
<td>-0.1159</td>
<td>0.0853</td>
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<td>0.1132</td>
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<tr>
<td>$K_I(B)/K_0^I$</td>
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<td>-0.0184</td>
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<td>-0.1841</td>
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<td>-4.7103</td>
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<td>12.6191</td>
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### Table 5

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<th>60°</th>
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</thead>
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<tr>
<td>$K_I(A)/K_0^I$</td>
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<td>1.0474</td>
<td>0.4572</td>
<td>-0.153</td>
<td>-0.6794</td>
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<td>$K_II(A)/K_0^I$</td>
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<td>1.304</td>
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<tr>
<td>$K_IV(A)/K_0^I$</td>
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<td>35.3102</td>
<td>14.7498</td>
<td>-5.7854</td>
<td>-66.2756</td>
<td>-70.2602</td>
<td>-68.4241</td>
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<tr>
<td>$K_I(B)/K_0^I$</td>
<td>1.6618</td>
<td>1.3452</td>
<td>0.4739</td>
<td>-0.6471</td>
<td>-1.4229</td>
<td>-0.4149</td>
<td>0.577</td>
</tr>
<tr>
<td>$K_II(B)/K_0^I$</td>
<td>0.5243</td>
<td>-1.2281</td>
<td>-1.8434</td>
<td>-1.9394</td>
<td>1.5183</td>
<td>1.8567</td>
<td>1.6436</td>
</tr>
<tr>
<td>$K_IV(B)/K_0^I$</td>
<td>-46.7645</td>
<td>-3.1752</td>
<td>23.3218</td>
<td>49.7005</td>
<td>107.9051</td>
<td>106.2828</td>
<td>99.3089</td>
</tr>
</tbody>
</table>
significantly influence the TSIFs and EDIF. Significant influences of the crack angle and the temperature boundary condition on the TSIFs and EDIF also are found.

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Appendix A

The angular functions $f_i^N(\theta), g_i^N(\theta), d_i^N(\theta)$, and $\nu^N(\theta)$ are shown below (Park and Sun, 1995)

$$f_i^N(\theta) = -\sum_{p=1}^{4} Re\left\{ \frac{M_{i2N}N_{SN}p_{z}}{\cos \theta + p_{z} \sin \theta} \right\},$$

$$g_i^N(\theta) = -\sum_{p=1}^{4} Re\left\{ \frac{M_{i2N}N_{SN}p_{z}}{\cos \theta + p_{z} \sin \theta} \right\},$$

$$d_i^N(\theta) = \sum_{p=1}^{4} Re\left\{ A_{i2N}N_{SN} \cos \theta + p_{z} \sin \theta \right\},$$

$$\nu^N(\theta) = \sum_{p=1}^{4} Re\left\{ A_{i2N}N_{SN} \cos \theta + p_{z} \sin \theta \right\}.$$ (A1)–(A4)

Here, $Re\{\cdot\}$ and $Im\{\cdot\}$ denote the real part and the imaginary part, respectively, where $p_{z}$ are conjugate pairs of eigenvalues, and $A_{i2N}$ are the $(4 \times 4)$ eigenvectors, which can be solved by a quadratic eigenvalue problem.

$$\begin{bmatrix} C_{i1i1}^{\text{tip}} & C_{i1i2}^{\text{tip}} \\ C_{i2i1}^{\text{tip}} & C_{i2i2}^{\text{tip}} \end{bmatrix} + \begin{bmatrix} C_{i11}^{\text{tip}} & C_{i12}^{\text{tip}} - \kappa_{i1}^{\text{tip}} \theta \\ C_{i21}^{\text{tip}} & C_{i22}^{\text{tip}} - \kappa_{i2}^{\text{tip}} \theta \end{bmatrix} \begin{bmatrix} \theta^4 \\ \theta^2 \end{bmatrix} = \begin{bmatrix} A_{i1} \\ A_{i2} \end{bmatrix} = 0.$$ (A5)

where $C_{ijkl}^{\text{tip}}$, $\kappa_{i1}^{\text{tip}}$, and $\kappa_{i2}^{\text{tip}}$ are the elastic stiffness, piezoelectric coefficient and dielectric permittivity tensors evaluated at the crack tip location, respectively. Only the four eigenvalues $p_{z}$ having a positive imaginary part and the corresponding eigenvectors are used in Eqs. (A1)–(A4).

The $(4 \times 4)$ matrices $M_{i2N}$ and $N_{SN}$ can be calculated by (Park and Sun, 1995)

$$N_{SN}^{-1} = M_{i2N} = \begin{bmatrix} (C_{i1i1}^{\text{tip}} + C_{i1i2}^{\text{tip}})A_{i2N} + (\kappa_{i1}^{\text{tip}} + \kappa_{i2}^{\text{tip}})p_{z}A_{i2N} \\ (C_{i2i1}^{\text{tip}} + C_{i2i2}^{\text{tip}})A_{i2N} + (\kappa_{i1}^{\text{tip}} + \kappa_{i2}^{\text{tip}})p_{z}A_{i2N} \end{bmatrix}.$$ (A6)

The fracture parameters are functions of the material properties, external loadings, external electrical displacement and geometry.

Appendix B

The Irwin matrix $Y$ can be defined as (Ricoeur and Kuna, 2003)

$$Y = Y_{MN} = \sum_{2}^{4} Im\{A_{i2N}N_{SN}\},$$ (B1)

where $M, N = \{II, I, III, IV\}$ denote the crack opening modes.

References


