Ordinary $p$-Laplacian systems with nonlinear boundary conditions

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Abstract

This paper is concerned with the existence of solutions for the boundary value problem

\[
\begin{align*}
-|u'|^{p-2}u' &+ \varepsilon |u|^{p-2}u = \nabla F(t,u), \quad \text{in } (0,T), \\
((|u'|^{p-2}u')(0), \quad -(|u'|^{p-2}u')(T)) &\in \partial j(u(0), u(T)),
\end{align*}
\]

where $\varepsilon \geq 0$, $p \in (1, \infty)$ are fixed, $j : \mathbb{R}^N \times \mathbb{R}^N \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function and $F : (0,T) \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory mapping, continuously differentiable with respect to the second variable and satisfies some usual growth conditions. Our approach is a variational one and relies on Szulkin’s critical point theory [A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, Ann. Inst. H. Poincaré Anal. Non Linéaire 3 (1986) 77–109]. We obtain the existence of solutions in a coercive case as well as the existence of nontrivial solutions when the corresponding Euler–Lagrange functional has a “mountain pass” geometry.

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1. Introduction

Let \( h_p : \mathbb{R}^N \to \mathbb{R}^N \) be the homeomorphism defined by \( h_p(x) = |x|^{p-2}x, \forall x \in \mathbb{R}^N \), where \( p \in (1, \infty) \) is fixed. For a given function \( j : \mathbb{R}^N \times \mathbb{R}^N \to (-\infty, +\infty] \) proper (i.e., \( D(j) := \{ z \in \mathbb{R}^N \times \mathbb{R}^N : j(z) < +\infty \} \neq \emptyset \)), convex and lower semicontinuous (in short, l.s.c.), we consider the boundary value problem

\[
\begin{align*}
-\left[ h_p(u') \right]' + \varepsilon h_p(u) &= \nabla F(t, u), \quad \text{in } (0, T), \\
(h_p(u')(0), -h_p(u')(T)) &\in \partial j(u(0), u(T)),
\end{align*}
\]

where \( \varepsilon \geq 0 \) is a constant and \( F : (0, T) \times \mathbb{R}^N \to \mathbb{R} \) is a Carathéodory mapping, such that:

\( (H_1) \) \( F(t, \cdot) \) is continuously differentiable, for a.e. \( t \in (0, T) \).

Here \( \nabla F(t, x) \) stands for the gradient of \( F(t, \cdot) \) at \( x \in \mathbb{R}^N \) and \( \partial j \) denotes the subdifferential of \( j \) in the sense of convex analysis.

In this paper we are concerned with the existence of solutions for problem (1.1), (1.2) under the following additional hypotheses on \( F \):

\( (H_2) \) \( F(\cdot, 0) \in L^1(0, T); \)

\( (H_3) \) for each \( \rho > 0 \) there is some \( \alpha_\rho \in L^1(0, T) \) such that

\[
|\nabla F(t, x)| \leq \alpha_\rho(t), \quad \text{for a.e. } t \in (0, T), \forall x \in \mathbb{R}^N \text{ with } |x| \leq \rho,
\]

where \(|\cdot|\) stands for the Euclidean norm on \( \mathbb{R}^N \).

By a solution of the differential system (1.1) we will understand a function \( u : [0, T] \to \mathbb{R}^N \) of class \( C^1 \) with \( h_p(u') \) absolutely continuous, which satisfies the equality in (1.1) a.e. on \( (0, T) \).

Existence results for various differential systems involving the ordinary vector \( p \)-Laplacian \( [h_p(u')]' \) associated with classical boundary conditions, such as Dirichlet, Neumann and periodic, have been obtained in recent time. See, for example, [4,13–16] and references therein. We note that the nonlinear multivalued boundary condition (1.2) includes as particular cases the above mentioned classical boundary conditions; these are obtained by appropriate choices of \( j \) (see, e.g., Chapter 2 in [11]). Recently, Gasinski and Papageorgiou [9] extended a result of Mawhin [15] concerning a Hartman type condition for the periodic problem to the case of a nonpotential system of differential inclusions with ordinary vector \( p \)-Laplacian, \( p \geq 2 \), subjected to a boundary condition of type (1.2) with a general maximal monotone mapping instead of \( \partial j \). We also recall that earlier works deal with differential equations with boundary conditions of type (1.2). In this respect, let us remark that Section 5.2 in [3] is devoted to the study of second-order multivalued equations in Hilbert spaces, of the form

\[
-\left[ h(u') \right]' + Au \ni f(t)
\]
with two-point boundary conditions of type (1.2), where \( h \) is the subdifferential of a convex function \( \phi \) and \( A \) is a maximal monotone operator. Also, in [17] higher order scalar differential equations are considered with boundary conditions in terms of a nonlinear maximal monotone mapping that is not necessarily a subdifferential. The monotonicity property of the data plays a key role in the approach of [3, 17].

Our Eq. (1.1) involves a potential nonlinear (not necessarily monotone) perturbation \( \nabla F(t, u) \). This together with the fact that the right-hand side of (1.2) is a subdifferential allows us to employ a variational method. This is based on Szulkin’s critical point theory [20] and enables us to obtain the existence of solutions in a coercive case (Theorem 4.2) as well as the existence of nontrivial solutions when the corresponding Euler–Lagrange functional has a “mountain pass” geometry (Theorem 4.5). Our approach is different from that in [17] (which is based on results in multivalued analysis and the theory of nonlinear monotone operators) and the results we obtain are of a different type and even more general in some respects (\( p \) is not restricted to be \( \geq 2 \), and we do not require their condition \( H(\xi) \) on the nonlinear mapping \( \xi \) in the boundary condition).

The rest of the paper is organized as follows. Section 2 contains some elements of critical point theory that are necessary in the sequel. In Section 3 a variational formulation for problem (1.1), (1.2) is given. The main existence results (Theorems 4.2, 4.5 and 4.6) are stated and proved in Section 4. These extend to the case of problem (1.1), (1.2) known results for the classical \( p \)-Laplacian operator associated with homogeneous Dirichlet boundary conditions [2, 5, 6]. Some applications are also presented.

2. Preliminaries

Our approach for problem (1.1), (1.2) is a variational one and it relies on Szulkin’s critical point theory [20]. For the convenience of the reader, in this section we briefly recall some notions and results in the framework of this theory which are needed in the sequel.

Let \((X, \| \cdot \|)\) be a real Banach space and \( I : X \to (-\infty, +\infty] \) be a functional of the type
\[
I = \Phi + \psi,
\]
where \( \Phi \in C^1(X, \mathbb{R}) \) and \( \psi \) is proper, convex and l.s.c. A point \( u \in X \) is said to be a critical point of \( I \) if it satisfies the inequality
\[
\langle \Phi'(u), v - u \rangle + \psi(v) - \psi(u) \geq 0, \quad \forall v \in X.
\]
A number \( c \in \mathbb{R} \) such that \( I^{-1}(c) \) contains a critical point is called a critical value of \( I \).

**Proposition 2.1** (Proposition 1.1 in [20]). If \( I \) satisfies (2.1), each local minimum point of \( I \) is necessarily a critical point of \( I \).

The functional \( I \) is said to satisfy the Palais–Smale (in short, (PS)) condition if every sequence \( \{u_n\} \subset X \) for which \( I(u_n) \to c \in \mathbb{R} \) and
\[
\langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in X,
\]

is bounded.
where $\varepsilon_n \to 0$, possesses a convergent subsequence. The next theorem extends the well-known mountain pass theorem of Ambrosetti and Rabinowitz [1].

**Theorem 2.2** (Theorem 3.2 in [20]). Suppose that $I$ satisfies (2.1), the (PS) condition and

(i) $I(0) = 0$ and there exist $\alpha, \rho > 0$ such that $I(u) \geq \alpha$ if $\|u\| = \rho$,

(ii) $I(e) \leq 0$ for some $e \in X$, with $\|e\| > \rho$.

Then $I$ has a critical value $c \geq \alpha$ which can be characterized by

$$c = \inf_{f \in \Gamma} \sup_{t \in [0, 1]} I(f(t)),$$

where $\Gamma = \{f \in C([0, 1], X) : f(0) = 0, f(1) = e\}$.

We conclude this section by recalling that a Banach space $X$ is said to have the Kadec–Klee property if for any sequence $\{u_n\}$ such that $u_n \to u$, weakly in $X$, and $\|u_n\| \to \|u\|$, we have $u_n \to u$, strongly in $X$. It is well known that a locally uniformly convex Banach space has the Kadec–Klee property (see, e.g., [10, p. 233]).

3. A variational approach for problem (1.1), (1.2)

The Sobolev space $W^{1,p} := W^{1,p}(0, T; \mathbb{R}^N)$ will be considered to be endowed with the norm

$$\|u\|_\eta = (\eta \|u\|_{L^p}^p + \|u'\|_{L^p}^p)^{1/p},$$

where $\eta > 0$ and $\| \cdot \|_{L^p}$ stands for the norm on $L^p := L^p(0, T; \mathbb{R}^N)$, i.e.,

$$\|u\|_{L^p} = \left( \int_0^T |u|^p \right)^{1/p}.$$

We set $C = C([0, T]; \mathbb{R}^N)$ and the usual norm in $C$ will be denoted by $\| \cdot \|_C$, i.e.,

$$\|u\|_C = \max\{|u(t)| : t \in [0, T]\}.$$

For $\varepsilon \geq 0$, let $\varphi_\varepsilon : W^{1,p} \to \mathbb{R}$ be given by

$$\varphi_\varepsilon(u) := \frac{1}{p} \left( \|u'\|_{L^p}^p + \varepsilon \|u\|_{L^p}^p \right), \quad \forall u \in W^{1,p}.$$ (3.1)

It is easy to see that $\varphi_\varepsilon$ is convex and l.s.c. on $W^{1,p}$. Moreover, arguments similar to those from the proof of Theorem 5.3 in [12] (see also the proof of Theorem 9 in [6]) can be used to show that $\varphi_\varepsilon \in C^1(W^{1,p}, \mathbb{R})$ and

$$\langle \varphi_\varepsilon'(u), v \rangle = \int_0^T (h_p(u')v') + \varepsilon \int_0^T (h_p(u)v), \quad \forall u, v \in W^{1,p},$$ (3.2)
where \((\cdot, \cdot)\) denotes the usual inner product in \(\mathbb{R}^N\). We also consider the functional
\[
J(u) = j(u(0), u(T)), \quad \forall u \in W^{1,p}.
\]
(3.3)

Note that, as \(j\) is proper, convex and l.s.c., the same hold true for \(J\). Then, setting
\[
\psi_{\varepsilon} = \varphi_{\varepsilon} + J,
\]
with \(\varphi_{\varepsilon}\) in (3.1) and \(J\) in (3.3), it is clear that \(\psi_{\varepsilon}\) is proper, convex and l.s.c. on \(W^{1,p}\).

Further, let us assume that the Carathéodory mapping \(F : (0, T) \times \mathbb{R}^N \to \mathbb{R}\) satisfies (H1)–(H3). Note that for each \(\rho > 0\) one has
\[
\|F(t, x)\| \leq \rho \alpha_{\rho}(t) + \|F(t, 0)\|, \quad \text{for a.e. } t \in (0, T), \forall x \in \mathbb{R}^N \text{ with } |x| \leq \rho,
\]
with \(\alpha_{\rho} \in L^1(0, T)\) from (H3). Indeed, (3.5) is immediate from (1.3) and the estimate
\[
\|F(t, x)\| = \left| \int_0^1 \frac{d}{ds} F(t, sx) \, ds + F(t, 0) \right| = \left| \int_0^1 (\nabla F(t, sx)|x) \, ds + F(t, 0) \right|
\leq |x| \int_0^1 \|\nabla F(t, sx)\| \, ds + |F(t, 0)|.
\]

On account of (3.5) and the embedding \(W^{1,p} \subset C\), we can introduce the functional
\[
\Phi_F : W^{1,p} \to \mathbb{R}, \text{ defined by}
\]
\[
\Phi_F(u) = -\int_0^T F(t, u) \, dt + F(t, 0), \quad \forall u \in W^{1,p}.
\]
(3.6)

Standard reasonings from the theory of Nemytskii’s operator (see, e.g., Chapter 2 in [8]) show that \(\Phi_F \in C^1(W^{1,p}, \mathbb{R})\) and \(\Phi'_F(u) = -\nabla F(\cdot, u)\), i.e.,
\[
\{\Phi'_F(u), v\} = -\int_0^T (\nabla F(t, u)|v) \, dt, \quad \forall u, v \in W^{1,p}.
\]
(3.7)

Now, the functional framework of Section 2 fits the following choices: \(X = W^{1,p}\), \(\Phi = \Phi_F\) in (3.6), \(\psi = \psi_{\varepsilon}\) in (3.4) and \(I = I_{F,\varepsilon}\),
\[
I_{F,\varepsilon} = \Phi_F + \psi_{\varepsilon}.
\]
(3.8)

**Proposition 3.1.** Let the Carathéodory function \(F : (0, T) \times \mathbb{R}^N \to \mathbb{R}\) satisfies (H1)–(H3) and let \(u \in W^{1,p}\). If \(u\) is a critical point of the functional \(I_{F,\varepsilon}\) defined by (3.8), in the sense of (2.2), i.e.,
\[
\{\Phi'_F(u), v - u\} + \psi_{\varepsilon}(v) - \psi_{\varepsilon}(u) \geq 0, \quad \forall v \in W^{1,p},
\]
then \(u\) is a solution of problem (1.1), (1.2). The converse implication is also true.
Proof. Assume that \( u \) is a critical point of \( I_{F,\varepsilon} \). In (3.9) we take \( v = u + sw, s > 0 \); then, dividing by \( s \) and letting \( s \to 0^+ \), we get
\[
\langle \Phi'_F(u), w \rangle + \langle \varphi'_\varepsilon(u), w \rangle + J'(u; w) \geq 0, \quad \forall w \in W^{1,p},
\]
where \( J'(u; w) \) is the directional derivative of the convex function \( J \) at \( u \) in the direction \( w \); this is known to exist. By virtue of (3.3), inequality (3.10) becomes
\[
\langle \Phi'_F(u), w \rangle + \langle \varphi'_\varepsilon(u), w \rangle + j'((u(0), u(T)); (w(0), w(T))) \geq 0, \\
\forall w \in W^{1,p}.
\]
(3.11)

Since \( C^\infty_0 := C^\infty_0(0, T; \mathbb{R}^N) \subset W^{1,p} \), from (3.11) we infer
\[
\langle \Phi'_F(u), w \rangle + \langle \varphi'_\varepsilon(u), w \rangle = 0, \quad \forall w \in C^\infty_0,
\]
which, taking into account (3.2) and (3.7), yields
\[
\int_0^T (h_p(u')|w'|) = \int_0^T (-\varepsilon h_p(u) + \nabla F(t, u)|w), \quad \forall w \in C^\infty_0.
\]
(3.12)

Next, as \( u \in W^{1,p} \), we have
\[
h_p(u), h_p(u') \in L^{p'},
\]
(3.13)
with \( 1/p + 1/p' = 1 \). Also, \((H_3)\) implies
\[
\nabla F(\cdot, u) \in L^1.
\]
(3.14)

From (3.12)–(3.14) it follows that
\[
h_p(u') \in W^{1,1}
\]
(3.15)
and
\[
-[h_p(u')]' = -\varepsilon h_p(u) + \nabla F(t, u), \quad \text{a.e. } t \in (0, T).
\]
(3.16)

Since \( h_p \) is a homeomorphism, (3.15) ensures that \( u \) is of class \( C^1 \). This together with (3.16) shows that \( u \) is a solution of the differential system (1.1).

To prove that \( u \) satisfies the boundary condition (1.2), we note that (3.11) and (3.16) yield
\[
j'((u(0), u(T)); (w(0), w(T))) \geq (h_p(u')(0)|w(0)) - (h_p(u')(T)|w(T)), \\
\forall w \in W^{1,p}.
\]
Thus
\[
j'((u(0), u(T)); (x, y)) \geq (h_p(u')(0)|x) + (-h_p(u')(T)|y), \quad \forall x, y \in \mathbb{R}^N,
\]
which, by a standard result from convex analysis (see, e.g., Theorem 23.2 in [19]), means that (1.2) holds true. The proof of the converse implication is not difficult and it is left to the reader. \( \Box \)
4. Existence results for problem (1.1), (1.2)

We begin by introducing the constant

$$
\lambda_1 = \lambda_1(p, j, \varepsilon) := \varepsilon + \inf \left\{ \frac{\|u^\prime\|_{L^p}^p}{\|u\|_{L^p}^p} : u \in W^{1,p} \setminus \{0\}, \ (u(0), u(T)) \in D(j) \right\},
$$

(4.1)

for $\varepsilon \geq 0$. It should be noticed that $\lambda_1(p, j, 0)$ can be either equal to 0 (e.g., if $D(j) = \mathbb{R}^N \times \mathbb{R}^N$) or $> 0$ (e.g., if $D(j) = \{(0, 0)\})$.

The existence results will be obtained under the main hypothesis:

$$(H_{\lambda_1}) \ \lambda_1(p, j, \varepsilon) > 0.$$ 

**Proposition 4.1.** If $(H_{\lambda_1})$ holds true, then

$$
2^{−1/p} \|u\|_{\lambda_1} \leq (\varepsilon \|u\|_{L^p}^p + \|u^\prime\|_{L^p}^p)^{1/p} \leq \|u\|_{\lambda_1}, \ \forall u \in D(J),
$$

(4.2)

where $J$ is defined by (3.3).

**Proof.** Since $D(J) = \{u \in W^{1,p} : (u(0), u(T)) \in D(j)\}$, by (4.1) we have

$$
\lambda_1 = \inf \left\{ \frac{\varepsilon \|u\|_{L^p}^p + \|u^\prime\|_{L^p}^p}{\|u\|_{L^p}^p} : u \in W^{1,p} \setminus \{0\}, \ u \in D(J) \right\},
$$

(4.3)

and a straightforward computation shows that (4.2) holds true. \square

If the nonlinearity $F$ lies asymptotically on the left of $\lambda_1$ then problem (1.1), (1.2) is solvable. In this view, the theorem below extends to the boundary value problem (1.1), (1.2) known results in the case of the $p$-Laplacian operator associated with homogeneous Dirichlet boundary conditions [2,6], or for the ordinary vector $p$-Laplacian with periodic boundary conditions [4].

**Theorem 4.2.** Assume $(H_1)$–$(H_3)$ and $(H_{\lambda_1})$. If

$$
\limsup_{|x| \to \infty} \frac{pF(t, x)}{|x|^p} < \lambda_1, \ \text{uniformly for a.e.} \ t \in (0, T),
$$

(4.4)

then problem (1.1), (1.2) has at least a solution.

**Proof.** We shall prove that the functional $I_{F, \varepsilon}$ in (3.8) is sequentially weakly l.s.c. and coercive on the space $(W^{1,p}, \| \cdot \|_{\lambda_1})$. Then, by a well-known result from calculus of variations, $I_{F, \varepsilon}$ is bounded from below and attains its infimum at some $u \in W^{1,p}$, which by Propositions 2.1 and 3.1 is a solution of problem (1.1), (1.2).

Let us begin by noting that $\Phi_F$ in (3.6) is sequentially weakly continuous. This can be shown as follows. Let $u, v \in W^{1,p}$ be such that $\|u\|_C, \|v\|_C \leq M$, with some $M > 0$. By $(H_3)$ there is an $a_{2M} \in L^1(0, T)$ such that

$$
|\nabla F(t, x)| \leq a_{2M}(t), \ \text{for a.e.} \ t \in (0, T), \ \forall x \in \mathbb{R}^N \text{ with } |x| \leq 2M.
$$

(4.5)

We estimate
\[ |\Phi_F(u) - \Phi_F(v)| = \left| \int_0^T (F(t, v) - F(t, u)) \right| \]
\[ = \left| \int_0^T \int_0^1 \frac{d}{ds} F(t, u + s(v - u)) \, ds \right| \]
\[ = \left| \int_0^T \int_0^1 |\nabla F(t, u + s(v - u))| |v - u| \, ds \right| \]
\[ \leq \left( \int_0^T \int_0^1 |\nabla F(t, u + s(v - u))| \, ds \right) \|v - u\|_C \]

and by (4.5) it follows that
\[ |\Phi_F(u) - \Phi_F(v)| \leq \left( \int_0^T \alpha_2 M(t) \, dt \right) \|v - u\|_C. \]  

(4.6)

By the compactness of the embedding \( W^{1,p} \subset C \) and (4.6) we get that \( \Phi_F \) is sequentially weakly continuous on \( W^{1,p} \). Then, by the weak lower semicontinuity of \( \psi_\epsilon \) in (3.4), \( I_{F,\epsilon} \) is sequentially weakly lower semicontinuous.

Further, from (4.4) there are constants \( \sigma \in (0, \lambda_1) \) and \( \rho > 0 \) such that
\[ F(t, x) \leq \frac{\lambda_1 - \sigma}{p} |x|^p, \quad \text{for a.e. } t \in (0, T), \; \forall x \in \mathbb{R}^N \text{ with } |x| > \rho. \]  

(4.7)

Then, (3.5) and (4.7) yield
\[ F(t, x) \leq \rho \alpha_\rho(t) + |F(t, 0)| + \frac{\lambda_1 - \sigma}{p} |x|^p, \quad \text{for a.e. } t \in (0, T), \; \forall x \in \mathbb{R}^N, \]

which, by (3.6), gives
\[ \Phi_F(u) \geq -K(\rho) - \frac{\lambda_1 - \sigma}{p} \|u\|_{L^p}^p, \quad \forall u \in W^{1,p}, \]  

(4.8)

where \( K(\rho) = \rho \int_0^T \alpha_\rho(t) + 2 \int_0^T |F(t, 0)| \). Using (3.8), (4.8), (4.3) and Proposition 4.1, we estimate \( I_{F,\epsilon} \) on \( D(J) \) as follows:
\[ I_{F,\epsilon}(u) = \Phi_F(u) + \psi_\epsilon(u) \]
\[ \geq -K(\rho) - \frac{\lambda_1 - \sigma}{p} \|u\|_{L^p}^p + \frac{1}{p} \left( \|u'\|_{L^p}^p + \epsilon \|u\|_{L^p}^p \right) + J(u) \]
\[ \geq -K(\rho) - \frac{\lambda_1 - \sigma}{p} \|u'\|_{L^p}^p + \frac{1}{p} \left( \|u'\|_{L^p}^p + \epsilon \|u\|_{L^p}^p \right) + J(u) \]
\[ = -K(\rho) + \frac{\sigma}{p \lambda_1} \left( \|u'\|_{L^p}^p + \epsilon \|u\|_{L^p}^p \right) + J(u) \]
\[ \geq -K(\rho) + \frac{\sigma}{2p \lambda_1} \|u\|_{L^p}^p + J(u), \quad \forall u \in D(J). \]
Since \( j \) is convex and l.s.c. it is bounded from below by an affine functional. Therefore, by (3.3) there are constants \( k_1, k_2, k_3 \geq 0 \) such that
\[
I_{F,\varepsilon}(u) \geq -K(\rho) + \frac{\sigma}{2p\lambda_1}\|u\|_{\lambda_1}^p - k_1|u(0)| - k_2|u(T)| - k_3, \quad \forall u \in D(J). \tag{4.9}
\]
From (4.9) and the continuity of the embedding \( W^{1,p} \subset C \), one obtains
\[
I_{F,\varepsilon}(u) \geq -K(\rho) + \frac{\sigma}{2p\lambda_1}\|u\|_{\lambda_1}^p - \tilde{k}_1\|u\|_{\lambda_1} - \tilde{k}_2\|u\|_{\lambda_1} - k_3, \quad \forall u \in D(J),
\]
with some constants \( \tilde{k}_1, \tilde{k}_2 \geq 0 \). Consequently,
\[
I_{F,\varepsilon}(u) \to +\infty, \quad \text{as } \|u\|_{\lambda_1} \to \infty,
\]
meaning that \( I_{F,\varepsilon} \) is coercive on \((W^{1,p}, \| \cdot \|_{\lambda_1})\) and the proof is complete. \( \square \)

In the sequel we are concerned with existence of nontrivial solutions for problem (1.1), (1.2). The main tool in obtaining such existence results will be Theorem 2.2.

**Lemma 4.3.** Let \( \psi_\varepsilon \) be defined by (3.4) and assume that \( D(j) \) is closed and \((H_{\lambda_1})\) holds true. Then, for each sequence \( \{u_n\} \subset D(J) \) such that \( u_n \to u \), weakly in \( W^{1,p} \), and
\[
\liminf_{n \to \infty} \psi_\varepsilon'(u_n; u - u_n) \geq 0, \tag{4.10}
\]
one has that \( u_n \to u \), strongly in \( W^{1,p} \).

**Proof.** Let us begin by noting that \( u \in D(J) \) because the closed convex set \( D(J) \) is weakly closed in \((W^{1,p}, \| \cdot \|_{\lambda_1})\), hence \( J'(u_n; u - u_n) < \infty \), for all \( n \in \mathbb{N} \). We have
\[
\limsup_{n \to \infty} \langle \varphi_\varepsilon'(u_n), u_n - u \rangle = \limsup_{n \to \infty} \big[ -\psi_\varepsilon'(u_n; u - u_n) + J'(u_n; u - u_n) \big] \\
\leq -\liminf_{n \to \infty} \psi_\varepsilon'(u_n; u - u_n) + \limsup_{n \to \infty} J'(u_n; u - u_n) \\
\leq -\liminf_{n \to \infty} \psi_\varepsilon'(u_n; u - u_n) + \limsup_{n \to \infty} [J(u) - J(u_n)] \\
\leq -\liminf_{n \to \infty} \psi_\varepsilon'(u_n; u - u_n) + J(u) - \liminf_{n \to \infty} J(u_n),
\]
and, as \( J \) is l.s.c., from (4.10) we get
\[
\limsup_{n \to \infty} \langle \varphi_\varepsilon'(u_n), u_n - u \rangle \leq 0. \tag{4.11}
\]
Let \( \varphi_{\lambda_1} \) be defined by (3.1) with \( \lambda_1 \) instead of \( \varepsilon \), i.e.,
\[
\varphi_{\lambda_1}(v) = \frac{1}{p}\|v\|_{\lambda_1}^p, \quad \forall v \in W^{1,p}.
\]
Clearly, one has
\[
\varphi_{\lambda_1}(v) = \varphi_\varepsilon(v) + \frac{\lambda_1 - \varepsilon}{p}\|v\|_{L^p}^p, \quad \forall v \in W^{1,p},
\]
hence,
\[ \langle \phi'_{\lambda_1}(v), w \rangle = \langle \phi'_{\varepsilon}(v), w \rangle + (\lambda_1 - \varepsilon) \int_0^T (h_p(v)|w|) \, \forall v, w \in W^{1,p}. \] (4.12)

Taking into account the compact embedding \( W^{1,p} \subset C \), we see that
\[ \int_0^T (h_p(u_n)|u_n - u|) \to 0, \quad \text{as } n \to \infty. \] (4.13)

From (4.12), (4.11) and (4.13) we obtain
\[ \limsup_{n \to \infty} \langle \phi'_{\lambda_1}(u_n), u_n - u \rangle \leq 0. \] (4.14)

Using the Hölder inequality, standard computations show that
\[ 0 \leq (\|u_n\|_{\lambda_1}^{p-1} - \|u\|_{\lambda_1}^{p-1})(\|u_n\|_{\lambda_1} - \|u\|_{\lambda_1}) \leq \langle \phi'_{\lambda_1}(u_n) - \phi'_{\lambda_1}(u), u_n - u \rangle. \]

This, together with (4.14) yields
\[ \|u_n\|_{\lambda_1} \to \|u\|_{\lambda_1}, \quad \text{as } n \to \infty. \] (4.15)

Finally, since \( (W^{1,p}, \| \cdot \|_{\lambda_1}) \) is uniformly convex [4], it has the Kadec–Klee property. Consequently, as \( u_n \to u \), weakly in \( W^{1,p} \), from (4.15) it follows that \( u_n \to u \), strongly in \( W^{1,p} \).

**Lemma 4.4.** Assume \((H_1)-(H_5)\) and \((H_{\lambda_1})\). If \( D(j) \) is closed and there are constants \( \theta > p \) and \( K, M > 0 \) such that
\[ j'(z; z) \leq \theta j(z) + K, \quad \forall z \in D(j), \] (4.16)

and
\[ \theta F(t, x) \leq (\nabla F(t, x)|x|), \quad \text{for a.e. } t \in (0, T), \forall x \in \mathbb{R}^N \text{ with } |x| > M, \] (4.17)

then the functional \( I_{F,\varepsilon} \) in (3.8) satisfies the (PS) condition on \( (W^{1,p}, \| \cdot \|_{\lambda_1}) \), i.e., every sequence \( \{u_n\} \subset W^{1,p} \) for which \( I_{F,\varepsilon}(u_n) \to c \in \mathbb{R} \) and
\[ \{\Phi'_{F}(u_n), v - u_n\} + \psi_{\varepsilon}(v) - \psi_{\varepsilon}(u_n) \geq -\varepsilon_n \|v - u_n\|_{\lambda_1}, \quad \forall v \in W^{1,p}, \] (4.18)

where \( \varepsilon_n \to 0 \), possesses a convergent subsequence.

**Proof.** By (4.16) and (3.3) it follows
\[ J(v) - \frac{1}{\theta} J'(v; v) \geq -K_1, \quad \forall v \in D(J), \] (4.19)

with \( K_1 = K/\theta \). By (3.5) there is some \( \alpha_M \in L^1(0, T) \) such that
\[ F(t, x) \leq M\alpha_M(t) + |F(t, 0)|, \quad \text{for a.e. } t \in (0, T), \forall x \in \mathbb{R}^N \text{ with } |x| \leq M. \] (4.20)

Using (3.6), (4.20), (4.17), (1.3) and (3.7), we obtain
\[
\int_0^T F(t, 0) - \Phi_F(v) = \int_{[|v| \leq M]} F(t, v) + \int_{[|v| > M]} F(t, v) \\
\leq \int_{[|v| \leq M]} (M \alpha_M(t) + |F(t, 0)|) + \frac{1}{\theta} \int_{[|v| > M]} (\nabla F(t, v)|v|) \\
= \int_{[|v| \leq M]} (M \alpha_M(t) + |F(t, 0)|) + \frac{1}{\theta} \left( \int_0^T (\nabla F(t, v)|v|) - \int_{[|v| \leq M]} (\nabla F(t, v)|v|) \right) \\
\leq \int_0^T \left( (M \alpha_M(t) + |F(t, 0)|) + \frac{1}{\theta} M \alpha_M(t) \right) - \frac{1}{\theta} \langle \Phi'_F(v), v \rangle,
\]

yielding
\[
\Phi_F(v) - \frac{1}{\theta} \langle \Phi'_F(v), v \rangle \geq -K_2, \quad \forall v \in W^{1,p},
\]
with \(K_2 = K_2(M, \theta)\) a positive constant.

Next, the proof resembles the proof of Lemma 3.5 in [7]. Let \(\{u_n\} \subset W^{1,p}\) be a sequence for which \(I_{F, \varepsilon}(u_n) \rightarrow c \in \mathbb{R}\) and (4.18) holds true with \(\varepsilon_n \rightarrow 0\). Clearly, \(\{u_n\} \subset D(I_{F, \varepsilon}) = D(J)\) and there is a constant \(K_3 > 0\), such that
\[
|I_{F, \varepsilon}(u_n)| \leq K_3, \quad \forall n \in \mathbb{N}.
\]
In (4.18) we set \(v = u_n + su_n, s > 0\), then dividing by \(s\) and letting \(s \rightarrow 0^+\), we obtain
\[
\langle \Phi'_F(u_n), u_n \rangle + \psi'_\varepsilon(u_n; u_n) \geq -\varepsilon_n \|u_n\|_{\lambda_1}, \quad \forall n \in \mathbb{N}.
\]
Using (4.22), (3.4) and (4.23) one obtains
\[
K_3 + \frac{\varepsilon_n}{\theta} \|u_n\|_{\lambda_1} \geq \Phi_F(u_n) + \psi_\varepsilon(u_n) + \frac{\varepsilon_n}{\theta} \|u_n\|_{\lambda_1} \\
= \Phi_F(u_n) + \psi_\varepsilon(u_n) + J(u_n) + \frac{\varepsilon_n}{\theta} \|u_n\|_{\lambda_1} \\
\geq \Phi_F(u_n) - \frac{1}{\theta} \langle \Phi'_F(u_n), u_n \rangle + \psi_\varepsilon(u_n) - \frac{1}{\theta} \langle \psi'_\varepsilon(u_n), u_n \rangle \\
+ J(u_n) - \frac{1}{\theta} J'(u_n; u_n)
\]
and by virtue of (4.21), (4.19), (3.1), (3.2) and Proposition 4.1 we deduce
\[
K_1 + K_2 + K_3 + \frac{\varepsilon_n}{\theta} \|u_n\|_{\lambda_1} \geq \left( \frac{1}{p} - \frac{1}{\theta} \right) (\varepsilon \|u_n\|^p_{L^p} + \|u_n\|^p_{L^p}) \\
\geq \frac{1}{2} \left( \frac{1}{p} - \frac{1}{\theta} \right) \|u_n\|^p_{\lambda_1},
\]
Since \(\theta > p\), from (4.24) it follows that \(\|u_n\|_{\lambda_1}\) is bounded. By the compactness of the embedding \(W^{1,p} \subset C\), the sequence \(\{u_n\}\) has a subsequence, again denoted by \(\{u_n\}\), such that \(u_n \rightharpoonup u\), weakly in \(W^{1,p}\) and strongly in \(C\). Similarly to (4.23) we derive
\[
\langle \Phi'_F(u_n), u - u_n \rangle + \psi'_\varepsilon(u - u_n; u_n) \geq -\varepsilon_n \|u - u_n\|_{\lambda_1}, \quad \forall n \in \mathbb{N}.
\]
As $u_n \to u$, strongly in $C$, by (3.7) and (1.3) it follows that
\[ \{ \Phi_F'(u_n), u - u_n \} \to 0, \quad \text{as } n \to \infty. \] (4.26)

Then, since $\|u - u_n\|_{\lambda_1}$ is bounded and $\varepsilon_n \to 0$ from (4.25) and (4.26) we obtain
\[ \lim \inf_{n \to \infty} \psi'_\varepsilon(u_n; u - u_n) \geq 0 \]
and Lemma 4.3 applies showing that $u_n \to u$, strongly in $W^{1,p}$. □

The theorem below extends to the case of the boundary value problem (1.1), (1.2) the well-known result of Ambrosetti and Rabinowitz for the Laplace operator associated with homogeneous Dirichlet boundary conditions stated in Theorem 3.10 in [1] (see also Theorem 2.15 in [18]), as well as Theorem 3.6 in [5] (Theorem 18 in [6]) for the $p$-Laplacian operator.

**Theorem 4.5.** Let the Carathéodory function $F : (0, T) \times \mathbb{R}^N \to \mathbb{R}$ satisfy (H\(_1\))–(H\(_3\)). We assume (H\(_{\lambda_1}\)) and, in addition,

(a) the set $D(j)$ is a closed cone and $(0, 0) \in \partial j(0, 0)$;
(b) $\limsup_{|x| \to 0} \frac{\rho(F(t, x) - F(t, 0))}{|x|^p} < \lambda_1$, uniformly for a.e. $t \in (0, T)$;
(c) there are constants $\theta > p$ and $K, M > 0$ such that (4.16) holds true and
\[ 0 < \theta F(t, x) \leq (\nabla F(t, x)|x), \]
for a.e. $t \in (0, T)$, $\forall x \in \mathbb{R}^N$ with $|x| > M$. (4.27)

Then problem (1.1), (1.2) has a nontrivial solution

**Proof.** We shall prove that the functional $I_{F,\varepsilon}$ defined in (3.8) has the geometry required by Theorem 2.2. First, without loss of generality, we may assume
\[ j(0, 0) = 0. \] (4.28)

Since $(0, 0) \in \partial j(0, 0)$, from (3.3) and (4.28) it follows
\[ J(u) \geq J(0) = 0, \quad \forall u \in D(J). \] (4.29)

Then, it is clear that
\[ I_{F,\varepsilon}(0) = 0. \] (4.30)

By Lemma 4.4, condition (c) ensures that $I_{F,\varepsilon}$ satisfies (PS) condition on $(W^{1,p}, \|\cdot\|_{\lambda_1})$. We fix a constant $K_0$ such that
\[ \|u\|_C \leq K_0 \|u\|_{\lambda_1}, \quad \forall u \in W^{1,p}; \] (4.31)
this is known to exist by the continuity of the embedding $W^{1,p} \subset C$. From (b) there are constants $\sigma \in (0, \lambda_1)$ and $\rho > 0$, such that
\[ F(t, x) - F(t, 0) \leq \frac{\lambda_1 - \sigma}{p} |x|^p, \]
for a.e. $t \in (0, T)$, $\forall x \in \mathbb{R}^N$ with $|x| \leq \rho K_0$, (4.32)
with $K_0$ in (4.31).
Let \( u \in W^{1,p} \), with \( \|u\|_{\lambda_1} \leq \rho \), be arbitrarily chosen. From (4.31) and (4.32) we have

\[
F(t, u) - F(t, 0) \leq \frac{\lambda_1 - \sigma}{p} |u|^p, \quad \text{for a.e. } t \in (0, T).
\]

This implies

\[
-\Phi_F (u) \leq \frac{\lambda_1 - \sigma}{p} \|u\|^p_{L^p},
\]

which, using (3.1), (4.3) and Proposition 4.1, gives

\[
\Phi_F (u) + \varphi_{\varepsilon}(u) \geq -\frac{\lambda_1 - \sigma}{p} \varepsilon \|u\|^p_{L^p} + \frac{1}{\lambda_1} \left( \varepsilon \|u\|^p_{L^p} + \|u'\|^p_{L^p} \right) = \frac{\sigma}{p\lambda_1} \varepsilon \|u\|^p_{L^p} + \frac{1}{2p\lambda_1} \|u'\|^p_{L^p}, \quad \forall u \in D(j).
\]

By virtue of (4.29),

\[
I_{F,\varepsilon}(u) = \Phi_F (u) + \varphi_{\varepsilon}(u) + J(u) \geq \alpha, \quad \text{if } \|u\|_{\lambda_1} = \rho,
\]

with \( \alpha = \frac{\sigma \rho^p}{2p\lambda_1} > 0 \). Then, (4.30) and (4.33) show that condition (i) in Theorem 2.2 is fulfilled with \( I = I_{F,\varepsilon} \).

Our next task is to prove that \( I_{F,\varepsilon} \) satisfies condition (ii) in Theorem 2.2. To this end, let us first observe that by virtue of (4.27), for a.e. \( t \in (0, T) \) and each \( x \in \mathbb{R}^N \), with \( |x| > M \), the mapping

\[
s \mapsto \frac{F(t, sx)}{s^\theta}
\]

is increasing on \([1, \infty)\). It follows that

\[
F(t, sx) \geq s^\theta F(t, x), \quad \text{for a.e. } t \in (0, T), \ \forall x \in \mathbb{R}^N, \ |x| > M \text{ and } \forall s \geq 1.
\]

Let \( \tilde{e} \in C_0^\infty \) be such that \( |	ilde{e}| > M \) on a set of positive measure. From (3.3) and (4.28), we have

\[
J(s\tilde{e}) = 0, \quad \forall s \in \mathbb{R}.
\]

Using (3.5), (4.27) and (4.34), for \( s \geq 1 \), we obtain

\[
\int_0^T F(t, s\tilde{e}) = \int_{|s\tilde{e}| \leq M} F(t, s\tilde{e}) + \int_{|s\tilde{e}| > M} F(t, s\tilde{e}) \geq -\int_0^T \left( M \alpha_M(t) + |F(t, 0)| \right) + \int_{|\tilde{e}| > M} F(t, s\tilde{e}) \geq -\int_0^T \left( M \alpha_M(t) + |F(t, 0)| \right) + s^\theta \int_{|\tilde{e}| > M} F(t, \tilde{e}),
\]

respectively.
i.e.,
\[ \int_{0}^{T} F(t, s \bar{e}) \geq -K_1 + s^\theta K_2, \quad (4.36) \]
with constants \( K_1 = K_1(M) \geq 0 \) and \( K_2 = K_2(M, \bar{e}) > 0 \). By (3.8), (4.35), (4.36) and Proposition 4.1 we get
\[ I_{F, \varepsilon}(s \bar{e}) \leq -s^\theta K_2 + K_1 + \int_{0}^{T} F(t, 0) + \frac{s^p}{p} ||\bar{e}||^p_{\lambda_1} \rightarrow -\infty, \quad as \ s \rightarrow +\infty. \quad (4.37) \]
Now, by (4.37), we can choose \( s_1 \) sufficiently large to satisfy
\[ I_{F, \varepsilon}(s_1 \bar{e}) \leq 0 \]
and
\[ ||s_1 \bar{e}||_{\lambda_1} > \rho, \]
with \( \rho \) intervening in (4.33). This means that condition (ii) in Theorem 2.2 is fulfilled for \( I = I_{F, \varepsilon} \) with \( e = s_1 \bar{e} \).

We conclude that the functional \( I_{F, \varepsilon} \) has a nontrivial critical point \( u \in W^{1,p} \), which by Proposition 3.1 is a nontrivial solution of problem (1.1), (1.2).  \( \Box \)

**Remark.** In the above proof the assumption that \( D(j) \) is a cone was not explicitly used, so it could seem that hypothesis (a) can be weakened by asking only “\( D(j) \) is closed” instead of “\( D(j) \) is a closed cone.” But, in fact, if \( D(j) \) is closed, \((0, 0) \in \partial j(0, 0) \) and (4.16) holds then necessarily \( D(j) \) is a cone.

We conclude this section by some applications of Theorem 4.5.

Let \( b : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \) be a convex, Gâteaux differentiable function, \( b'(0, 0) = (0, 0) \), and let \( S \subset \mathbb{R}^N \times \mathbb{R}^N \) be a nonempty closed convex cone. We denote by \( N_S(z) \) the normal cone to \( S \) at \( z \in S \), i.e.,
\[ N_S(z) = \{ \xi \in \mathbb{R}^N \times \mathbb{R}^N : (\xi | \xi - z) \leq 0, \forall \xi \in S \}, \quad \forall z \in S. \]

The differential system (1.1) is considered to be associated with the boundary conditions
\[ (u(0), u(T)) \in S, \]
\[ (h_p(u')(0), -h_p(u')(T)) - b'(u(0), u(T)) \in N_S(u(0), u(T)). \quad (4.38) \]

We set
\[ \bar{\lambda}_1 = \bar{\lambda}_1(p, S, \varepsilon) := \varepsilon + \inf \left\{ \frac{||u'||_{L_p}^p}{||u||_{L_p}^p} : u \in W^{1,p} \setminus [0], \quad (u(0), u(T)) \in S \right\}. \]

**Theorem 4.6.** Assume the Carathéodory function \( F : (0, T) \times \mathbb{R}^N \rightarrow \mathbb{R} \) satisfies \((H_1)-(H_3)\). If, in addition, \( \bar{\lambda}_1 > 0 \) and
(i) \( \limsup_{|x| \to 0} \frac{|F(t,x) - F(t,0)|}{|x|^p} < \bar{\lambda}, \) uniformly for a.e. \( t \in (0, T); \)

(ii) there are constants \( \theta > p \) and \( K, M > 0, \) such that (4.27) holds and

\[
\langle b'(z), z \rangle \leq \theta b(z) + K, \quad \forall z \in S,
\]

then the differential system (1.1) has a nontrivial solution which satisfies (4.38).

**Proof.** Let \( I_S \) be the indicator function of the set \( S. \) Since \( NS(z) = \partial IS(z), \forall z \in S, \) Theorem 4.5 applies with \( j(z) = b(z) + IS(z), \forall z \in \mathbb{R}^{2N}. \)

It should be noticed that (4.38) allows various possible choices of \( b \) and \( S, \) which, among others, recover classical boundary conditions. For instance, if \( b = 0 \) then the Dirichlet, Neumann and periodic boundary conditions are obtained by choosing \( S = \{ (0, 0) \}, \)

\( S = \mathbb{R}^N \times \mathbb{R}^N \) and \( S = \{ (x, x) : x \in \mathbb{R}^N \}, \) respectively. In these three cases condition (4.39) is automatically satisfied with any \( \theta \in \mathbb{R} \) and \( K = 0, \) therefore, sufficient conditions ensuring the existence of nontrivial solutions of system (1.1) can easily be stated by means of Theorem 4.6. Also, if \( b(z) = \frac{1}{2} (Az|z), \) \( \forall z \in \mathbb{R}^{2N}, \) where \( A \) is a symmetric, positive \( (2N \times 2N) \)-matrix, and \( S = \mathbb{R}^N \times \mathbb{R}^N, \) then (4.38) reads

\[
\begin{pmatrix}
  h_p(u'(0)) \\
  -h_p(u'(T))
\end{pmatrix}
= A
\begin{pmatrix}
  u(0) \\
  u(T)
\end{pmatrix}.
\]

(4.40)

In this case condition (4.39) is fulfilled with any \( \theta \geq 2 \) and \( K = 0. \) From Theorem 4.6 we have the following

**Corollary 4.7.** Assume the Carathéodory function \( F : (0, T) \times \mathbb{R}^N \to \mathbb{R} \) satisfies (H1)–(H3) and let \( \varepsilon > 0. \) If, in addition,

(i) \( \limsup_{|x| \to 0} \frac{|F(t,x) - F(t,0)|}{|x|^p} < \varepsilon, \) uniformly for a.e. \( t \in (0, T); \)

(ii) there are constants \( \theta \) and \( M, \theta \geq 2, \theta > p, M > 0, \) such that (4.27) holds,

then the differential system (1.1) has a nontrivial solution which satisfies (4.40). If \( A = 0 \) then it suffices that \( \theta \) in (ii) be > \( p. \)

As a simple example, if \( q \geq 2, q > p, f \in L^\infty((0, T); \mathbb{R}), \) \( f > 0 \) a.e. on \( (0, T), \) and \( \varepsilon > 0, \) then the differential equation

\[
-(|u'|^{p-2}u')' + \varepsilon |u|^{p-2}u = f(t)|u|^{q-2}u, \quad \text{in} \ (0, T),
\]

has a nontrivial solution satisfying the boundary condition (4.40). To see this, Corollary 4.7 applies with

\[
F(t, x) = \frac{1}{q} f(t)|x|^q, \quad \forall t \in (0, T), \ x \in \mathbb{R}^N,
\]

and \( \theta = q, M = 1. \) If \( A = 0 \) then it suffices that \( q > p. \)
References


