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J. Math. Anal. Appl. 339 (2008) 384–398

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Large time behavior of solutions to parabolic equations with Neumann boundary conditions

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Received 5 February 2007

Available online 14 July 2007

Submitted by C. Gutierrez

Abstract

In this paper we are interested in the large time behavior as $t \to +\infty$ of the viscosity solutions of parabolic equations with nonlinear Neumann type boundary conditions in connection with ergodic boundary problems which have been recently studied by Barles and the author in [G. Barles, F. Da Lio, On the boundary ergodic problem for fully nonlinear equations in bounded domains with general nonlinear Neumann boundary conditions, Ann. Inst. H. Poincaré Anal. Non Linèaire 22 (5) (2005) 521–541].

Keywords: Asymptotic behavior; Ergodic problems; Viscosity solutions; Boundary value problems for parabolic PDEs

1. Introduction

In this note we study the large time behavior as $t \to +\infty$ of the viscosity solutions to two different types of Neumann boundary value problems

$$\chi_t + F\big(x, D\chi, D^2\chi\big) = \lambda \quad \text{in } \mathcal{O} \times (0, \infty), \tag{1}$$

$$L(x, D\chi) = \mu \quad \text{on } \partial\mathcal{O} \times (0, \infty), \tag{2}$$

$$\chi(x, 0) = \chi_0(x) \quad \text{in } \mathcal{O}, \tag{3}$$

and

$$w_t + F\big(x, Dw, D^2w\big) = 0 \quad \text{in } \mathcal{O} \times (0, +\infty), \tag{4}$$

$$w_t + L(x, Dw) = 0 \quad \text{on } \partial\mathcal{O} \times (0, +\infty), \tag{5}$$

$$w(x, 0) = w_0(x) \quad \text{in } \mathcal{O}, \tag{6}$$

where, say, $\mathcal{O} \subset \mathbb{R}^n$ is a smooth domain, F and L are, at least, continuous functions defined, respectively, on $\overline{\mathcal{O}} \times \mathbb{R}^n \times \mathcal{S}^n$ and $\overline{\mathcal{O}} \times \mathbb{R}^n$ with values in $\mathbb{R}$, $\mathcal{S}^n$ denotes the space of real, $n \times n$, symmetric matrices, $\chi_0, w_0 \in C(\overline{\mathcal{O}})$ and

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.06.052

λ, μ are real constant. The solution u of these nonlinear problems is scalar and u_t, Du, D^2u denote, respectively, the partial derivative with respect to t, the gradient and the Hessian matrix of u.

We recall that the boundary condition $L = 0$ is said to be a nonlinear Neumann boundary condition if the function L satisfies the following conditions:

(L1) There exists $\nu > 0$ such that, for every $(x, p) \in \partial O \times \mathbb{R}^n$, and $s > 0$, we have

$$L\big(x, p + sn(x)\big) - L(x, p) \geqslant \nu s, \tag{7}$$

where $n(x)$ denotes the unit outward normal vector to ∂O at $x \in \partial O$.

(L2) There is a constant $K > 0$ such that, for all $x, y \in \partial O,\ p, q \in \mathbb{R}^n$, we have

$$\big|L(x, p) - L(y, q)\big| \leqslant K\big[\big(1 + |p| + |q|\big)|x - y| + |p - q|\big]. \tag{8}$$

The main examples of boundary conditions we have in mind are the following: first, linear type boundary conditions like oblique derivative boundary conditions, in which L is given by

$$L(x, p) = \big\langle p, \gamma(x)\big\rangle + g(x), \tag{9}$$

where $\gamma : \partial O \to \mathbb{R}^n$ is a bounded, Lipschitz continuous vector field such that

$$\big\langle \gamma(x), n(x)\big\rangle \geqslant \beta > 0 \quad \text{for all } x \in \partial O,$$

and g is a Lipschitz function. Here and below, "$\langle p, q\rangle$" denotes the usual scalar product of the vectors p and q of $\mathbb{R}^n$.

Next nonlinear boundary conditions: the first example is capillarity type boundary conditions for which L is given by

$$L(x, p) = \big\langle p, n(x)\big\rangle - \theta(x)\sqrt{1 + |p|^2}, \tag{10}$$

where $\theta : \partial O \to \mathbb{R}^n$ is a Lipschitz scalar function, such that $|\theta(x)| < 1$ for every $x \in \partial O$. A second example is the boundary condition arising in the optimal control of processes with reflection when there is control on the reflection, namely

$$L(x, p) = \sup_{\alpha \in A}\big\{\big\langle \gamma_\alpha(x), p\big\rangle - g_\alpha(x)\big\}, \tag{11}$$

where A is a compact metric space, $\gamma_\alpha : \partial O \to \mathbb{R}^n$ are Lipschitz continuous vector fields such that $\langle \gamma_\alpha(x), n(x)\rangle \geqslant \beta > 0$ for all $x \in \partial O$, and $g_\alpha : \partial O \to \mathbb{R}$ is a Lipschitz continuous, scalar function.

The interest in these two evolution problems is motivated by some results recently obtained by Barles and the author in [4] on what can be called "the boundary ergodic problems" which consist in solving the following type of fully nonlinear elliptic equations associated with nonlinear Neumann boundary conditions

$$F\big(x, Du, D^2u\big) = \lambda \quad \text{in } \mathcal{O}, \tag{12}$$

$$L(x, Du) = \mu \quad \text{on } \partial\mathcal{O}. \tag{13}$$

The key point in these ergodic problems is that the constant μ, which is called the "boundary ergodic cost," is part of the unknowns while λ is considered as a given constant.

If we consider only Eq. (12) without boundary condition, i.e. the case when $\mathcal{O} = \mathbb{R}^n$ the typical result one expects, under suitable assumptions on F, is the existence of a unique constant λ such that (12) has a bounded solution. Such results were first proved for first-order equations by Lions, Papanicolaou and Varadhan [25] in the case of periodic equations and solutions. General results for second-order equations in the periodic setting are proved by Evans [16,17] and results in the evolution case, when the equation is periodic both in space and time, were obtained by Barles and Souganidis [9]. Recently, Ishii [22] generalizes these results in the almost periodic case.

We refer the reader to the introduction of the paper [4] for a complete description of the connection of such types of results with the applications (ergodic control problems, homogenization of elliptic and parabolic PDEs, asymptotic behavior of solutions to parabolic equations).

Since in this paper we are interested in the large time behavior of the solutions of (1)–(2) and (4)–(5) we recall which is a typical result in $\mathbb{R}^n$. One consider a solution $u(x, t)$ of the corresponding evolutive equation

$$u_t + F(x, Du, D^2u) = 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty). \tag{14}$$

If there exists a unique λ such that (12) has a bounded solution v_∞, then one should have

$$\frac{u(x,t)}{t} \to -\lambda \quad \text{locally uniformly as } t \to \infty. \tag{15}$$

Therefore the ergodic constant governs the asymptotic behavior of the associated evolution equation and in good cases, one can even show that

$$u(x,t) + \lambda t \to v_\infty(x) \quad \text{locally uniformly as } t \to \infty. \tag{16}$$

It is worth pointing out that if a property like (15) can be obtained rather easily as a consequence of standard comparison results for Eq. (12), the more precise asymptotic behavior (16) is a far more difficult result.

Recently a lot of works have been devoted to study the large time behavior of the solutions of first-order Hamilton–Jacobi equations. Fathi [18–20] and Namah and Roquejoffre [26] were the first who established quite general convergence results for $u_t + H(x, Du) = 0$ in $\mathbb{R}^n \times (0, +\infty)$, in the case when H is convex in Du, it is periodic in x and the solutions are bounded. We recall that Fathi's approach is based on dynamical systems arguments and in particular on the so-called Mather's set which is roughly speaking an attractor for the geodesics associated to the representation formula of the solution. By using more PDEs techniques Barles and Souganidis [8] extended the asymptotic results to a nonconvex framework. Barles and Roquejoffre [7] and Fujita, Ishii and Loreti [23] have recently investigated the case when the Hamiltonian is not periodic in x and the solutions of the evolution and stationary equations are unbounded, (see also Fathi and Maderna [21] for results without periodicity assumption of H). To the best of our knowledge, there are not a lot of general results in the case of second-order equations: the uniformly elliptic case seems the only one which can be done through the use of the Strong Maximum Principle and the methods of [9] which are used in the paper to prove the convergence to space–time periodic solutions but which can be used to show the convergence to solutions of the stationary equations.

As far as the connection with the large time behavior of the problems (1)–(2) and (4)–(5) is concerned, in [4] the following results are proved. In the case of (1)–(2), it is shown that the ergodic constant $\mu(\lambda)$ is characterized as the only constant μ for which the solution χ remains bounded. In the case of (4)–(5), the expected behavior is to have $t^{-1}w(x,t)$ converging to a constant $-\tilde{\lambda}$ which has to be such that (12)–(13) has a solution for $\tilde{\lambda} = \lambda = \mu(\lambda)$. It is proved that, under suitable conditions, such a constant $\tilde{\lambda}$, i.e. a fixed point of the map $\lambda \mapsto \mu(\lambda)$, does exist and that we have the expected behavior at infinity for w.

The aim of this paper is to complete the results in [4] by showing a more precise asymptotic behavior as $t \to +\infty$ of the solutions of the two evolutions problems (1)–(2) and (4)–(5). Under the same assumptions in [4] we prove that

$$\chi(x,t) \to u_\infty(x) \quad \text{as } t \to +\infty \text{ uniformly in } \overline{\mathcal{O}}$$

and

$$w(x,t) + \tilde{\lambda} t \to u_\infty(x) \quad \text{as } t \to +\infty \text{ uniformly in } \overline{\mathcal{O}},$$

where u_∞ is a solution of the stationary problem (12)–(13), χ, w and $\tilde{\lambda}$ are as above.

The main ingredients to get such convergence results are the following. The first one is the $C^{0,\alpha}_{\mathrm{loc}}(\mathcal{O})$ estimates for solutions of (1)–(2) and (4)–(5), which provide the compactness in $C(\mathcal{O})$ of the functions $\chi(\cdot,t)$ and $w(\cdot,t) - \tilde{\lambda} t$. The second one is the half-relaxed limits method introduced by Barles and Perthame [6]. The third one is the Strong Maximum Principle which is extended here to viscosity solutions to evolution equations with Neumann boundary conditions (see Lemma 2.1).

In view of the results obtained here and in [4] it would be interesting, in the case when F is a Hamilton–Jacobi operator, to investigate the connections between ergodic properties of diffusion processes with reflection and their invariant measures. This will be the aim of a future work.

The paper is organized as follows. In Section 2 we list the main assumptions which are used in the paper, we provide some preliminary results and state the main results. In Section 3 we show the interior Hölder estimates in the x variable (uniformly with respect to $t > 0$). In Section 4 we prove the main results of the paper namely the large time behavior as $t \to +\infty$ of the solutions to (1)–(2) and (4)–(5).

2. Preliminary results

In this section we list the main assumptions and prove some preliminary results.

The main assumptions we will use are the following:

(O1) $\mathcal{O}$ is a bounded domain with a $W^{3,\infty}$ boundary.

We point out that such an assumption on the regularity of the boundary is needed to use the comparison and existence results of [2].

We denote by d the sign-distance function to $\partial\mathcal{O}$ which is positive in $\mathcal{O}$ and negative in $\mathbb{R}^n \setminus \overline{\mathcal{O}}$. If $x \in \partial\mathcal{O}$, we recall that $Dd(x) = -n(x)$ where $n(x)$ is the outward unit normal vector to $\partial\mathcal{O}$ at x. The main consequence of (O1) is that d is $W^{3,\infty}$ in a neighborhood of $\partial\mathcal{O}$.

The operator F satisfies the following assumptions.

(F1) (*Regularity*) The function F is locally Lipschitz continuous on $\overline{\mathcal{O}} \times \mathbb{R}^n \times \mathcal{S}^n$ and there exists a constant $K > 0$ such that, for any $x, y \in \overline{\mathcal{O}}$, $p, q \in \mathbb{R}^n$, $M, N \in \mathcal{S}^n$,

$$\left|F(x,p,M) - F(y,q,N)\right| \leqslant K\left\{|x-y|\left(1+|p|+|q|+\|M\|+\|N\|\right)+|p-q|+\|M-N\|\right\}.$$

(F2) (*Uniform ellipticity*) There exists $\kappa > 0$ such that, for any $x \in \overline{\mathcal{O}}$, $p \in \mathbb{R}^n$, $M, N \in \mathcal{S}^n$ with $N \geqslant 0$,

$$F(x,p,M+N) - F(x,p,M) \leqslant -\kappa \operatorname{Tr}(N).$$

(F3) There exists a continuous function F_∞ such that

$$t^{-1}F(x,tp,tM) \to F_\infty(x,p,M) \quad \text{locally uniformly, as } t \to +\infty.$$

The operator L satisfies (L1), (L2) and

(L3) There exists a continuous function L_∞ such that

$$t^{-1}L(x,tp) \to L_\infty(x,p) \quad \text{locally uniformly, as } t \to +\infty.$$

We want to emphasize the fact that the above assumptions are very well adapted for applications to stochastic control and differential games: indeed (F1)–(L1) are clearly satisfied as soon as the dynamic has bounded and Lipschitz continuous drift, diffusion matrix and direction of reflection and when the running and boundary cost satisfies analogous properties (maybe these assumptions are not optimal but they are rather natural) while (F3)–(L3) are almost obviously satisfied because of the structure of the Bellman or Isaac Equations (“sup” or “inf sup” of affine functions in p and M).

We recall here some results obtained in [4] by Barles and the author concerning the connection between the evolution problems (1)–(2) and (4)–(5) and the boundary ergodic problem (12)–(13).

Theorem 2.1. *(See [4].) Assume* (O1), (F1)–(F3) *and* (L1)–(L3)*, then, for any $\lambda \in \mathbb{R}$, there exists a unique $\mu \in \mathbb{R}$ such that* (12)–(13) *has a continuous viscosity solution. Moreover the map $\lambda \mapsto \mu(\lambda)$ is continuous, decreasing and therefore there exists a unique $\lambda := \tilde{\lambda}$ such that $\mu(\tilde{\lambda}) = \tilde{\lambda}$.*

Theorem 2.2. *(See [4].) Under the assumptions of Theorem* 2.1*, there exists a unique viscosity solution χ of* (1)–(3) *which is defined for all time. Moreover, χ remains uniformly bounded in time if and only if $\mu = \mu(\lambda)$.*

Theorem 2.3. *(See [4].) Under the assumptions of Theorem* 2.1*, there exists a unique viscosity solution of* (4)–(6) *which is defined for all time. Moreover, as $t \to +\infty$, we have*

$$\frac{w(x,t)}{t} \to -\tilde{\lambda} \quad \textit{uniformly on } \overline{\mathcal{O}},$$

where $\tilde{\lambda}$ is as in Theorem 2.1.

Remark 2.1. If L is a linear operator of the form (9) in the above theorems we can weaken in a suitable sense the uniform ellipticity of F (see hypothesis (F5) in Section 3) and we refer the reader to [4] for the details.

The main results of this note are the following two theorems.

Theorem 2.4. *Assume* (O1), (F1)–(F3) *and* (L1)–(L3). *Let χ be the bounded solution of* (1)–(3) *corresponding to λ and $\mu = \mu(\lambda)$. Then there is a solution u_∞ of* (12)–(13) *such that*

$$\chi(x,t) \to u_\infty(x) \quad \text{as } t \to +\infty \text{ uniformly in } \overline{\mathcal{O}}. \tag{17}$$

Theorem 2.5. *Assume* (O1), (F1)–(F3) *and* (L1)–(L3). *Let w be the solution of* (4)–(6) *with $\lambda = \mu = \tilde{\lambda}$. Then there is a solution u_∞ of* (12)–(13) *such that*

$$w(x,t) + \tilde{\lambda} t \to u_\infty(x), \quad \text{as } t \to +\infty \text{ uniformly in } \overline{\mathcal{O}}. \tag{18}$$

The proofs of Theorems 2.4 and 2.5 are postponed to Section 4 and we continue by showing some preliminary results.

In the following lemma we show that under the current hypotheses the difference of a lower semi-continuous supersolution and an upper semi-continuous subsolution of either (1)–(2) or (4)–(5) is a supersolution of a problem involving a positively homogeneous uniformly elliptic operator and boundary condition.

We recall the definition of extremal Pucci operators [11,27], with parameters $0 < \kappa_1 \leqslant \kappa_2$, defined by

$$\mathcal{M}^+_{\kappa_1,\kappa_2}(M) = \kappa_2 \sum_{e_i>0} e_i + \kappa_1 \sum_{e_i<0} e_i, \qquad \mathcal{M}^-_{\kappa_1,\kappa_2}(M) = \kappa_1 \sum_{e_i>0} e_i + \kappa_2 \sum_{e_i<0} e_i,$$

for any symmetric $N \times N$ matrix M. Here $e_i = e_i(M)$, $i = 1, \dots, N$, denote the eigenvalues of M. Pucci's operators are extremal in the sense that $\mathcal{M}^+_{\kappa_1,\kappa_2}(M) = \sup_{A \in \mathcal{A}_{\kappa_1,\kappa_2}} \operatorname{tr}(AM)$, $\mathcal{M}^-_{\kappa_1,\kappa_2}(M) = \inf_{A \in \mathcal{A}_{\kappa,\kappa_2}} \operatorname{tr}(AM)$, where $\mathcal{A}_{\kappa_1,\kappa_2}$ denotes the set of all symmetric matrices whose eigenvalues lie in the interval $[\kappa_1, \kappa_2]$.

In the sequel we will denote by $BUSC(\overline{\mathcal{O}} \times [0,+\infty))$ and $BLSC(\overline{\mathcal{O}} \times [0,+\infty))$, respectively, the set of bounded upper and lower semi-continuous functions in $\overline{\mathcal{O}} \times [0,+\infty)$.

Lemma 2.1. *Assume* (O1), (F1)–(F2) *and* (L1)–(L2). *Let $u \in BUSC(\overline{\mathcal{O}} \times [0,+\infty))$ and $v \in BLSC(\overline{\mathcal{O}} \times [0,+\infty))$ be respectively sub- and supersolution of either* (1)–(2) *or* (4)–(5). *Then the function $\omega = u - v$ is a viscosity subsolution of*

$$\omega_t - \mathcal{M}^+\big(D^2\omega\big) - K|D\omega| = 0 \quad \text{in } \mathcal{O} \times [0,+\infty), \tag{19}$$

$$\omega_t + \nu \frac{\partial \omega}{\partial n} - C|D_T\omega| = 0 \quad \text{on } \partial\mathcal{O} \times (0,+\infty), \tag{20}$$

where $C > \max(K, \bar{K})$, $K, \bar{K}, \nu$ being the constants appearing in (F1) *and* (L1)–(L2).

Proof. The strategy of proof in both cases is very similar to the one of Lemma 4.1 in [4] (see also [15]), thus we provide here the main arguments only in the case when u, v are respectively sub- and supersolution of problem (4)–(5).

Let $\phi \in C^2(\overline{\mathcal{O}} \times [0,+\infty))$ be such that $\omega - \phi$ has a local maximum at $(\bar{x}, \bar{t}) \in \overline{\mathcal{O}} \times (0,+\infty)$. We suppose that $\bar{x} \in \partial\mathcal{O}$, the case $\bar{x} \in \mathcal{O}$ being similar and even simpler.

For all $\varepsilon > 0$, α and $\eta > 0$, we introduce the auxiliary function

$$\Phi_{\varepsilon,\eta,\alpha}(x,y,t,s) = u(x,t) - v(y,s) - \psi_{\varepsilon,\eta,\alpha}(x,y,t,s) - \phi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) - |x-\bar{x}|^4 - |t-\bar{t}|^2, \tag{21}$$

where $\psi_{\varepsilon,\eta,\alpha}(x,y,t,s)$ is the test function built in Barles [2] relative to the boundary condition (13). Let $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)$ be the maximum point of $\Phi_{\varepsilon,\eta,\alpha}(x,y,t,s)$ in $\overline{\mathcal{O}} \times \overline{\mathcal{O}} \times [0,+\infty) \times [0,+\infty)$. Since $(\bar{x}, \bar{t})$ is a strict local maximum point of $(x,t) \mapsto w(x,t) - \phi(x,t) - |x-\bar{x}|^4 - |t-\bar{t}|^2$, standard arguments show that

$$(x_\varepsilon, y_\varepsilon) \to (\bar{x}, \bar{x}) \quad \text{and} \quad (t_\varepsilon, s_\varepsilon) \to \bar{t} \quad \text{as } \varepsilon \to 0.$$

On the other hand, by construction we have

$$D_t\psi(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) + L\big(x_\varepsilon, D_x\psi_{\varepsilon,\eta,\alpha}(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)\big) > 0 \quad \text{if } x_\varepsilon \in \partial\mathcal{O},$$
$$-D_s\psi(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) + L\big(y_\varepsilon, -D_y\psi_{\varepsilon,\eta,\alpha}(x_\varepsilon, y_\varepsilon)\big) < 0 \quad \text{if } y_\varepsilon \in \partial\mathcal{O}.$$

Moreover, if $\zeta_{\varepsilon,\eta,\alpha}(x, y, t, s) := \psi_{\varepsilon,\eta,\alpha}(x, y, t, s) + \phi(\frac{x+y}{2}, \frac{t+s}{2}) + |x - \bar{x}|^4 + |t - \bar{t}|^2$, by standard arguments (cf. [12]), we know that, for every $\rho > 0$, there exist $X, Y \in \mathcal{S}^n$ such that

$$\big(D_t\zeta_{\varepsilon,\alpha,\eta}(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon), D_x\zeta_{\varepsilon,\alpha,\eta}(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon), X\big) \in \overline{P}^{2,+}_{\overline{\mathcal{O}}} u(x_\varepsilon, t_\varepsilon),$$
$$\big(-D_s\zeta_{\varepsilon,\alpha,\eta}(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon), -D_y\zeta_{\varepsilon,\alpha,\eta}(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon), Y\big) \in \overline{P}^{2,-}_{\overline{\mathcal{O}}} v(y_\varepsilon, s_\varepsilon),$$

and

$$-\left(\frac{1}{\rho} + \big\|D^2\zeta_{\varepsilon,\alpha,\eta}(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)\big\|\right)\mathrm{Id} \leqslant \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \big(\mathrm{Id} + \rho D^2\zeta_{\varepsilon,\eta,\alpha}(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)\big)D^2\zeta_{\varepsilon,\alpha,\eta}(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon).$$

Now suppose that

$$D_t\phi(\bar{x}, \bar{t}) + \nu\frac{\partial\phi}{\partial n}(\bar{x}, \bar{t}) - C\big|D_T\phi(\bar{x}, \bar{t})\big| > 0.$$

If $x_\varepsilon \in \partial\mathcal{O}$, then, for ε small enough, we have

$$\begin{aligned} D_t\zeta_{\varepsilon,\alpha,\eta} + L(x_\varepsilon, D_x\zeta_{\varepsilon,\alpha,\eta}) &\geqslant D_t\psi_{\varepsilon,\eta,\alpha} + L(x_\varepsilon, D_x\psi_{\varepsilon,\eta,\alpha}) \\ &\quad + \frac{1}{2}\left(D_t\phi_t + \nu\frac{\partial\phi(\bar{x}, \bar{t})}{\partial n} - C\big|D_T\phi(\bar{x}, \bar{t})\big|\right) + o_\varepsilon(1) > 0, \end{aligned}$$

while if $y_\varepsilon \in \partial\mathcal{O}$,

$$\begin{aligned} -D_s\zeta_{\varepsilon,\alpha,\eta}(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) + L(y_\varepsilon, -D_y\zeta_{\varepsilon,\alpha,\eta}) &\leqslant -D_s\psi_{\varepsilon,\eta,\alpha} + L(y_\varepsilon, -D_y\psi_{\varepsilon,\alpha,\eta}) \\ &\quad - \frac{1}{2}\left(D_t\phi_t(\bar{x}, \bar{t}) + \nu\frac{\partial\phi(\bar{x}, \bar{t})}{\partial n} - C\big|D_T\phi(\bar{x}, \bar{t})\big|\right) + o_\varepsilon(1) < 0. \end{aligned}$$

Therefore, if ε is small enough, wherever $x_\varepsilon, y_\varepsilon$ lie we have

$$D_t\zeta_{\varepsilon,\alpha,\eta} + F(x_\varepsilon, D_x\zeta_{\varepsilon,\eta,\alpha}, X) \leqslant 0, \qquad -D_s\zeta_{\varepsilon,\alpha,\eta} + F(y_\varepsilon, -D_y\zeta_{\varepsilon,\eta,\alpha}, Y) \geqslant 0.$$

By subtracting the above inequalities, using the above estimates on X, Y together with the comparison arguments of Section A.1 in [4] (see also [15]), the assumption (F1) and (F2) and the definition of the Pucci's extremal operator $\mathcal{M}^+$, by letting first $\varepsilon \to 0$ and then $\alpha, \eta \to 0$, we are lead to

$$\phi_t(\bar{x}, \bar{t}) - \mathcal{M}^+\big(D^2\phi(\bar{x}, \bar{t})\big) - K\big|D\phi(\bar{x}, \bar{t})\big| \leqslant 0,$$

and the conclusion follows. □

Next we prove the Strong Maximum Principle for the subsolutions to the problem (19)–(20).

Proposition 2.1. *Assume the hypotheses of Lemma* 2.1. *Let w be a bounded subsolution of* (19)–(20) *that attains its maximum at $(\bar{x}, \bar{t})$ in $\overline{\mathcal{O}} \times (0, +\infty)$. Then $w(x, t)$ is a constant in $\overline{\mathcal{O}} \times [0, \bar{t}]$.*

Proof. Let $(\bar{x}, \bar{t}) \in \overline{\mathcal{O}} \times (0, +\infty)$ be such that $w(\bar{x}, \bar{t}) = \sup_{\overline{\mathcal{O}} \times [0,+\infty)} w(x, t) =: M$. If $\bar{x} \in \mathcal{O}$, then the result follows from the Strong Maximum Principle for viscosity solutions to parabolic equations proved by the author [14]. Thus $\bar{x} \in \partial\mathcal{O}$ and there is $\bar{r} > 0$ such that $w(y, s) < M$ in $\overline{B}((\bar{x}, \bar{t}), \bar{r}) \cap (\overline{\mathcal{O}} \times (0, +\infty))$.

Now we are going to use the following lemma whose proof is postponed to the end of this section.

Lemma 2.2. *There exists $0 < \tilde\theta < \bar r$ and a smooth function φ on $\overline{B}((\bar x, \bar t), \tilde\theta) \cap (\overline{\mathcal{O}} \times (0, +\infty))$ such that $\varphi(\bar x, \bar t) = 0$, $\varphi(y, s) > 0$ on $\overline{B}((\bar x, \bar t), \tilde\theta) \cap (\partial\mathcal{O} \times (0, +\infty))$,*

$$\varphi_t - \mathcal{M}^+\big(D^2 w\big) - K|D\varphi| > 0 \quad \text{on } \overline{B}\big((\bar x, \bar t), \tilde\theta\big) \cap \big(\mathcal{O} \times (0, +\infty)\big), \tag{22}$$

and

$$\varphi_t - \nu\frac{\partial\varphi}{\partial n} - C|D_T\varphi| > 0 \quad \text{on } \overline{B}\big((\bar x, \bar t), \tilde\theta\big) \cap \big(\partial\mathcal{O} \times (0, +\infty)\big). \tag{23}$$

We continue with the proof of Proposition 2.1.

By choosing $\tau > 0$ small enough, we have $w(y, s) - \tau\varphi(y, s) < M = w(\bar x, \bar t) - \tau\varphi(\bar x, \bar t)$ for $(y, s) \in \partial B((\bar x, \bar t), \tilde\theta) \cap (\overline{\mathcal{O}} \times (0, +\infty))$. Indeed, for (y, s) close to $\partial\mathcal{O} \times (0, +\infty)$, $\varphi(y, s) > 0$ while in $\mathcal{O} \times (0, +\infty)$ we have $w(y, s) < M$.

We deduce from this property that, if we consider $\max_{\overline{B}((\bar x, \bar t), \theta) \cap (\overline{\mathcal{O}} \times (0, +\infty))}(w - \tau\varphi)$, this maximum is achieved in $(x', t') \in B((\bar x, \bar t), \tilde\theta) \cap (\overline{\mathcal{O}} \times (0, +\infty))$ and therefore it is a local maximum point of $\bar w - \tau\varphi$. Since the operators in (19)–(20) are positively homogeneous of degree 1, the function $\tau\varphi$ is still a strict supersolution of (22)–(23). Thus by applying to the function w the definition of viscosity solution at (x', t') with $\tau\varphi$ as test function we get a contradiction with the inequalities (22)–(23). □

Remark 2.2. In the same way one can prove under the assumption of Lemma 2.1 that if w is a bounded supersolution of (19)–(20) that attains its minimum at $(\bar x, \bar t)$ in $\overline{\mathcal{O}} \times (0, +\infty)$, then $w(x, t)$ is constant in $\overline{\mathcal{O}} \times [0, \bar t]$.

Proof of Lemma 2.2. We use here arguments which are borrowed from [14]. Since $\mathcal{O}$ is a C^2 domain, for $\delta > 0$ small enough, $d((\bar x, \bar t) - \delta n(\bar x)) = \delta$ where d is the distance to the boundary $\partial\mathcal{O}$. We set $x_0 = \bar x - \delta n(\bar x)$ for such a δ and we build a function φ of the following form

$$\varphi(y, s) = \exp\big(-\rho\delta^2\big) - \exp\big(-\rho\big(|y - x_0|^2 + |s - \bar t|^2\big)\big),$$

where ρ has to be chosen later. Let us set $\theta = \delta/2$. Since $\delta = |\bar x - x_0|$, we have $\varphi(\bar x, \bar t) = 0$ and if $(y, s) \in (\partial\mathcal{O} \times (0, +\infty)) \cap \overline{B}((\bar x, \bar t), \theta) - \{(\bar x, \bar t)\}$, $|y - x_0| \geqslant \delta/2$ and therefore $\varphi(y, s) > 0$. Moreover

$$D_x\varphi(y, s) = 2\rho(y - x_0)\exp\big(-\rho\big(|y - x_0|^2 + |s - \bar t|^2\big)\big),$$
$$D_t\varphi(y, s) = 2\rho(s - t)\exp\big(-\rho\big(|y - x_0|^2 + |s - \bar t|^2\big)\big).$$

We note that by the definition of x_0, $D\varphi(\bar x, \bar t) = kn(\bar x)$ with $k = 2\delta\rho\exp(-\rho\delta^2) > 0$.

Using the notations $\ell(y, s) = 2\rho\exp(-\rho(|y - x_0|^2 + |s - \bar t|^2))$ and $p(y) = y - x_0$, we have

$$D_t\varphi - \mathcal{M}^+\big(D^2\varphi\big) - K|D\varphi| = \ell(y, s)\big[(s - \bar t) - \mathcal{M}^+\big(\mathrm{Id} - 2\rho p(y) \otimes p(y)\big) - K\big|p(y)\big|\big],$$
$$D_t\varphi + \nu\frac{\partial\varphi}{\partial n} - C|D_T\varphi| = (s - \bar t)\ell(y, s) + \nu\ell(y, s)\big\langle p(y), n(y)\big\rangle - C\big|D_T\big(\ell(y, s)p(y)\big)\big|.$$

Now we observe that for ρ large enough we have $1 - 2\rho|p| < 0$ and thus

$$\mathcal{M}^+\big(\mathrm{Id} - 2\rho p(y) \otimes p(y)\big) = \kappa\big(1 - 2\rho\big|p(y)\big|^2\big) + \tilde\kappa(n - 1)$$

for suitable $\kappa, \tilde\kappa$ depending on the ellipticity constants of F. Moreover for ρ large enough and for some $\eta > 0$ we have

$$\begin{aligned} D_t\varphi(\bar x, \bar t) - \mathcal{M}^+\big(D^2\varphi(\bar x, \bar t)\big) - K\big|D\varphi(\bar x, \bar t)\big| &= -\ell(\bar x, \bar t)\mathcal{M}^+\big(\mathrm{Id} - 2\rho p(\bar x) \otimes p(\bar x)\big) - K\ell(y)\big|p(\bar x)\big| \\ &= \ell(\bar x, \bar t)\big[-\Lambda(n - 1) - \lambda\big(1 - \rho\big|p(\bar x)\big|^2\big) - K\big|p(\bar x)\big|\big] > \eta, \end{aligned}$$

and

$$D_t\varphi(\bar x, \bar t) + \nu\frac{\partial\varphi}{\partial n}(\bar x, \bar t) - C\big|D_T\varphi(\bar x, \bar t)\big| = \nu\ell(\bar x, \bar t) > \eta.$$

Thus there is $\tilde\theta < \theta$ such that

$$D_t\varphi(y,s) - \mathcal{M}^+\big(D^2\varphi(y,s)\big) - K\big|D\varphi(y,s)\big| > \eta/2, \quad \text{for all } (y,s) \in \overline{B}\big((\bar{x},\bar{t}),\tilde{\theta}\big) \cap \big(\mathcal{O} \times (0,+\infty)\big),$$

and

$$D_t\varphi(y,s) + \nu\frac{\partial\varphi}{\partial n}(y,s) - C\big|D_T\varphi(y,s)\big| > \eta/2, \quad \text{for all } (y,s) \in \overline{B}\big((\bar{x},\bar{t}),\tilde{\theta}\big) \cap \big(\partial\mathcal{O} \times (0,+\infty)\big).$$

The proof is complete and we conclude. □

3. Interior Hölder estimates

The regularity of viscosity solutions to fully nonlinear uniformly parabolic equations has been studied by several authors (see for instance [10,13,28] and the references therein). What is known is that a bounded solution of (1)–(2) and (4)–(5) is in $C^{1+\alpha,\frac{1+\alpha}{2}}_{\mathrm{loc}}(\mathcal{O} \times (0,+\infty))$ and for every interior subset $\mathcal{O}' \subset \mathcal{O}$ and for every $c > 0$ we have

$$\|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\mathcal{O}' \times [c,+\infty))} \leqslant C\big(1 + \|u\|_\infty\big),$$

for some $\alpha \in (0,1)$, where C is a constant depending on the operator F (through the constants appearing in F_1 and F_2), the distance from $\mathcal{O}'$ to $\partial\mathcal{O}$ and the diameter of $\mathcal{O}$.

In order to prove the asymptotic of the solutions to the problems (1)–(2) and (4)–(5) we will just need the local Hölder continuity with respect the x variable uniformly in $t \in [c,+\infty)$. We would like to mention that the proofs in the literature of the interior Hölder continuity of the solutions are in general consequences of Harnack type inequalities (see e.g. [13,28]) or of comparison and continuous dependence type results under suitable regularity assumption of the initial data (see e.g. [10]).

In this section we would like to show, for the reader's convenience, another proof of the local Hölder continuity with respect the x variable uniformly in $t \in [c,+\infty)$ which is based on an idea introduced by Ishii and Lions [24]. This idea have been already used for instance in [1,9] to show gradient estimates of viscosity solutions to quasilinear elliptic and parabolic PDEs with Lipschitz initial conditions, and by Barles and the author [5] to prove local Hölder estimates up to the boundary of bounded solutions to fully nonlinear elliptic PDEs with Neumann boundary conditions. We point out that this method works also for quasilinear and possibly degenerate parabolic equations. To this purpose we consider only in this section operators F satisfying the following two weaker assumptions:

(F4) (*Growth Condition on F*) There exist positive constants C_1, C_2, C_3 and functions $\omega_1, \omega_2, \varpi^R : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\omega_1(0+) = 0$, $\omega_2(r) = O(r)$ as $r \to +\infty$, $\varpi(t) \to 0$ as $t \to +\infty$, and for any $x, y \in \overline{O}$, $p, q \in \mathbb{R}^n$, $M \in \mathcal{S}^n$ and $K > 0$,

$$\begin{aligned} F(x,p,M) - F(y,q,M+K\mathrm{Id}) \leqslant{}& \omega_1\big(|x-y|\big(1+|p|+|q|\big) + \varpi\big(|p| \wedge |q|\big)|p-q|\big)\|M\| \\ &+ \omega_2(K) + C_1 + C_2\big(|p|^2 + |q|^2\big) + C_3|x-y|\big(|p|^3 + |q|^3\big), \end{aligned}$$

where $|p| \wedge |q| = \min(|p|,|q|)$.

(F5) There exist $\kappa > 0$ such that, for all $x \in \overline{O}$, $M, N \in \mathcal{S}^n$ with $N \geqslant 0$, we have

$$F(x,p,M+N) - F(x,p,M) \leqslant -\kappa\langle N\hat{p},\hat{p}\rangle + o(1)\|N\|, \tag{24}$$

where $o(1)$ denotes a function of the real variable $|p|$ which converges to 0 as $|p|$ tends to infinity.

One of the main examples we have in mind is the case of standard quasilinear equations

$$u_t - \mathrm{Tr}\big[b(x,Du)D^2u\big] + H(x,Du) = 0 \quad \text{in } O \times (0,+\infty), \tag{25}$$

where b is an $n \times n$ matrix and H a continuous function. In this case, the assumptions (F4) and (F5) are easily checkable.

(F5) is equivalent to: there exists $\kappa > 0$ such that, for any $x \in \overline{O}$, $p \in \mathbb{R}^n$,

$$b(x,p) \geqslant \kappa\hat{p} \otimes \hat{p} - o(1)\mathrm{Id},$$

where, as in (F5), $o(1)$ is a function of $|p|$ which converges to 0 as $|p| \to +\infty$.

We observe that the assumption (F5) without the $o(1)$ term would be essentially reduced to

$$b(x,p) \geqslant \kappa \hat{p} \otimes \hat{p},$$

for any $x \in \overline{\mathcal{O}}$ and $p \in \mathbb{R}^n - \{0\}$, while, with this term, (F5) is satisfied if

$$b(x,p) \geqslant \kappa \widehat{q(x,p)} \otimes \widehat{q(x,p)},$$

where q is a continuous function such that $|p|^{-1}(q(x,p)-p) \to 0$ as $p \to \infty$, uniformly with respect to $x \in \overline{\mathcal{O}}$.

As far as the hypothesis (F4) is concerned it is satisfied if

(i) b is a bounded, continuous function of x and p and there exists a modulus of continuity $\omega_1 : \mathbb{R}^+ \to \mathbb{R}^+$ and a function $\varpi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varpi(t) \to 0$ as $t \to +\infty$ and

$$\left|b(x,p) - b(y,q)\right| \leqslant \omega_1\bigl(|x-y|\bigl(1+|p|+|q|\bigr) + \varpi\bigl(|p| \wedge |q|\bigr)|p-q|\bigr).$$

Moreover the uniform bound on b provides ω_2 with a linear growth.

(ii) The function H satisfies: there exist positive constants C_1, C_2, C_3 such that, for any $x, y \in \overline{\mathcal{O}}$, and $p, q \in \mathbb{R}^n$,

$$H(x,p) - H(y,q) \leqslant C_1 + C_2\bigl(|p|^2 + |q|^2\bigr) + C_3|x-y|\bigl(|p|^3 + |q|^3\bigr).$$

We mention that the assumption (F4) is classical when one wants to get interior regularity (see for instance Ishii and Lions [24], Barles [1]).

In the next theorem we will show that a bounded viscosity solution of either (1)–(3) or (4)–(6) is in $C^{0,\alpha}_{\mathrm{loc}}(\mathcal{O})$ uniformly in $t \in [c, +\infty)$, $c > 0$. We adapt to the parabolic case the strategy of proof of Theorem 2.1 in [5]. Here the situation is simpler since we want to obtain interior estimates. We give the proof in detail for the reader's convenience.

Theorem 3.1. *Assume* (O1), (F4)–(F5) *and* (L1)–(L2). *Let* $u \in C(\overline{\mathcal{O}} \times [0,+\infty))$ *be a bounded viscosity solution of either* (1)–(3) *or* (4)–(6). *Then for all* $c > 0$, *for all* $t \geqslant c$, *and for every* $0 < \alpha < 1$, $u(\cdot,t) \in C^{0,\alpha}_{\mathrm{loc}}(O)$. *Moreover for every interior subset* $\mathcal{O}' \subset \mathcal{O}$ *and for every* $c > 0$ *the* $C^{0,\alpha}$*-norm of* u *in* $\mathcal{O}' \times [c,+\infty)$ *depends only on* $\|u\|_\infty$, c, *the distance from* $\mathcal{O}'$ *to* $\partial\mathcal{O}$, F (*through the constants appearing in* (F4)–(F5)).

Proof. We suppose that u is a solution of (1)–(3) (the proof of the other case is the same). We fix $\delta > 0$, $c > 0$, $x_0 \in \mathcal{O}$, $d(x_0, \partial\mathcal{O}) > \delta$. We want to show that there exists $C > 0$ (depending on δ and the data of the problem) such that for all $x \in B(x_0, \frac{\delta}{4})$, $t_0 \geqslant c$ we have

$$u(x,t_0) - u(x_0,t_0) \leqslant C|x-x_0|^\alpha. \tag{26}$$

We note that the condition (26) implies the Hölder continuity of $u(\cdot,t_0)$ in $B(x_0,\frac{\delta}{4})$. Indeed if $x, y \in B(x_0, \frac{\delta}{4})$, then $B(x,\frac{\delta}{4}) \subseteq \mathcal{O}$, $B(y,\frac{\delta}{4}) \subseteq \mathcal{O}$ as well. If $|x-y| < \frac{\delta}{4}$, then the result follows from (26), otherwise

$$u(x,t_0) - u(y,t_0) \leqslant 2\|u\|_\infty \frac{|x-y|^\alpha}{(\delta/4)^\alpha}. \tag{27}$$

In order to prove (26) we consider the auxiliary function

$$\Phi(x,y) = u(x) - u(y) - \Theta(x,y),$$

where the function Θ has the following form:

$$\Theta(x,y) = C|x-y|^\alpha + L\bigl(|x-x_0|^4 + |t-t_0|^2\bigr),$$

where $\alpha \in (0,1)$ is a fixed constant, C, L are some large constants to be chosen later on.

We show that for a suitable choice of L, chosen large enough in order to localize in the space and time variables, then for $C > 0$ large enough we have

$$M_{L,C} := \max_{\overline{B}(x_0,\delta/4) \times \overline{B}(x_0,\delta/4) \times [0,+\infty)} \Phi(x,y,t) \leqslant 0. \tag{28}$$

Indeed if (28) holds, then by choosing $x = x_0$ and $t = t_0$ we get (26).

We first choose L large enough so that $\Phi(x,y) \leqslant 0$ is $|t-t_0| > \frac{c}{2}$ and $x \in \partial B(x_0, \frac{\delta}{4})$. This is possible since u is bounded in $\overline{O} \times [0,+\infty)$. We fix such an L and we argue by contradiction assuming that for all $C>0$, $M_{L,C}>0$. Since Φ is a continuous function, the maximum is achieved at some $(\bar{x},\bar{y},\bar{t}) \in \overline{B}(x_0,\delta/4) \times \overline{B}(x_0,\delta/4) \times [c,+\infty)$ and we observe that, by the choice of L, C, we may even assume that $\bar{x} \in B(x_0,\delta/8)$ and $\bar{y} \in B(x_0,\delta/8)$ and $\bar{t}>0$. Here we have dropped the dependence of $\bar{x},\bar{y},\bar{t}$ on C for simplicity of notations.

Two quantities are going to play a key role in the proof

$$Q_1 := C|\bar{x}-\bar{y}|^\alpha,$$
$$Q_2 := L|\bar{x}-x_0|^4$$

(again we have dropped the dependence of Q_1, Q_2 in C for the sake of simplicity of notations). The reason for that is the following: by using only the local boundedness of u, we are only able to show that Q_1, Q_2 are uniformly bounded when C becomes very large while if we use the local modulus of continuity of u, we can show that $Q_1, Q_2 \to 0$ as $C \to +\infty$. The idea of the proof can therefore be described in the following way: we first show that u is locally in $C^{0,\alpha}$ for α small enough with suitable estimates depending only on the L^∞ norm of u and on the data, and this is done by using only the uniform boundedness of Q_1, Q_2. Then this first step provides us with a local modulus of continuity for u and we obtain the full result using this time that $Q_1, Q_2 \to 0$ as $C \to +\infty$.

From the fact that $\Phi(\bar{x},\bar{y},\bar{t})>0$ by using classical arguments we get

$$C|x-y|^\alpha \leqslant 2\|u\|_\infty,$$
$$L\big(|x-x_0|^4 + |t-t_0|^2\big) \leqslant 2\|u\|_\infty.$$

In particular it follows that $|\bar{x}-\bar{y}| \to 0,$ as $C \to +\infty$. We may also suppose without loss of generality that $|\bar{x}-\bar{y}|>0$ for C large enough.

By the arguments of User's Guide [12], for all $\varepsilon>0$, there exist $(a,p,B_1) \in \overline{P}^{2,+}u(x,t)$, $(b,q,B_2) \in \overline{P}^{2,-}u(y,t)$ such that

$$p = D_x\Theta(x,y,t), \qquad q = -D_y\Theta(x,y,t), \qquad a-b = D_t\Theta,$$
$$-\big(\varepsilon^{-1} + \big\|D^2\Theta(x,y,t)\big\|\big)\mathrm{Id} \leqslant \begin{pmatrix} B_1 & 0 \\ 0 & -B_2 \end{pmatrix} \leqslant D^2\Theta(x,y,t) + \varepsilon\big(D^2\Theta(x,y)\big)^2, \tag{29}$$

and

$$a + F\big(x,u(x,t),p,B_1\big) \leqslant 0, \qquad b + F\big(y,u(y,t),q,B_2\big) \geqslant 0. \tag{30}$$

We choose below $\varepsilon = \rho\|D^2\Theta_\delta(x,y,t)\|^{-1}$ for ρ small enough but fixed. Its size is determined in the proofs below. The following estimates hold:

$$D_x\Theta = C\alpha|x-y|^{\alpha-2}(x-y) + 4L|x-x_0|^2(x-x_0),$$
$$D_y\Theta = -C\alpha|x-y|^{\alpha-2}(x-y),$$
$$D^2_{xx}\Theta_0 = C\alpha\mathrm{Id}|x-y|^{\alpha-2} + C\alpha(\alpha-2)|x-y|^{\alpha-4}(x-y)\otimes(x-y)$$
$$\qquad + 8L(x-x_0)\otimes(x-x_0) + 4L|x-x_0|^2\mathrm{Id},$$
$$D^2_{xy}\Theta = -C\alpha\mathrm{Id}|x-y|^{\alpha-2} - C\alpha(\alpha-2)|x-y|^{\alpha-4}(x-y)\otimes(x-y),$$
$$D^2_{yy}\Theta = C\alpha\mathrm{Id}|x-y|^{\alpha-2} + C\alpha(\alpha-2)|x-y|^{\alpha-4}(x-y)\otimes(x-y),$$
$$D_t\Theta = 2L(t-t_0).$$

We denote $Y = x-y$ and $\Psi(Y) = |Y|^\alpha$.

The right-hand side of inequality (29) can be rewritten as: for all $\xi,\zeta \in \mathbb{R}^n$,

$$\langle B_1\xi,\xi\rangle - \langle B_2\zeta,\zeta\rangle \leqslant \big(1+O(\rho)\big)\big[C\big\langle D_{YY}\Psi(\xi-\zeta),(\xi-\zeta)\big\rangle + 2L|x-x_0|^2|\xi|^2\big]. \tag{31}$$

Choosing $\xi = \zeta$ in the above inequality, we first deduce that

$$B_1 - B_2 \leqslant \big(1+O(\rho)\big)8L|x-x_0|^2\mathrm{Id} =: \widetilde{K}\mathrm{Id}.$$

We next choose in (31), $\xi = -\zeta = \widehat{Y}$ and we get

$$\begin{aligned}\big\langle (B_1 - B_2)\widehat{Y}, \widehat{Y}\big\rangle &\leqslant \big(1 + O(\rho)\big)\big[8L|x - x_0|^2 + \langle D_{YY}\Psi \widehat{Y}, \widehat{Y}\rangle\big] \\ &= \big(1 + O(\rho)\big)\big[8L|x - x_0|^2 + C\alpha(\alpha - 1)|Y|^{\alpha-2}\big].\end{aligned}$$

In the sequel, K always denotes a positive constant which may vary from line to line and depends only on the data of the problem.

Now by using the estimates on the first and second derivatives of Θ we get, for some $K > 0$,

$$\begin{aligned}&|p|, |q| \geqslant C\alpha K^{-1}|Y|^{\alpha-1} + K, \\ &|p|, |q| \leqslant C\alpha K|Y|^{\alpha-1} + O\big(|x - x_0|^3\big), \\ &|p - q| = O\big(|x - x_0|^3\big), \\ &\|B_1\|, \|B_2\| \leqslant K\left(1 + \frac{1}{O(\rho)}\right)\big(1 + C\alpha|Y|^{\alpha-2} + O\big(|x - x_0|^2\big)\big), \\ &\|B_1 - B_2\| \leqslant K\left(1 + \frac{1}{O(\rho)}\right)\big[C\alpha|Y|^{\alpha-2} + 8L|x - x_0|^2\big],\end{aligned}$$

as $C \to +\infty$.

We subtract the two inequalities (30) and write the difference in the following way:

$$F(x, p, B_1) - F(x, p, B_2 + \widetilde{K}\mathrm{Id}) \leqslant F(y, q, B_2) - F(x, p, B_2 + \widetilde{K}\mathrm{Id}) + b - a,$$

and, using the fact that $B_1 - B_2 \leqslant \widetilde{K}\mathrm{Id}$, we apply (F4) to the left-hand side and (F5) to the right-hand side of (32). Recalling also that $|p|, |q| \to +\infty$ as $C \to +\infty$, this yields

$$\begin{aligned}&\kappa\,\mathrm{Tr}\big[(B_2 - B_1 + \widetilde{K}\mathrm{Id})(\hat{p} \otimes \hat{p})\big] + o(1)\|B_2 - B_1 + \widetilde{K}\mathrm{Id}\| \\ &\quad\leqslant \omega_1\big(|x - y|\big(1 + |p| + |q|\big) + \varpi\big(|p| \wedge |q|\big)|p - q|\big)\|B_2\| + \omega_2(\widetilde{K}) + C_1 \\ &\qquad + C_2\big(|p|^2 + |q|^2\big) + C_3|x - y|\big(|p|^3 + |q|^3\big).\end{aligned}$$

Now one can easily see that

$$\hat{p} = \widehat{Y} + o_Y(1) \quad \text{as } |Y| \to 0.$$

We point out that, $|Y| \to 0$ is in fact equivalent to C going to infinity.

We have

$$\mathrm{Tr}\big[(B_2 - B_1 + \widetilde{K}\mathrm{Id})(\hat{p} \otimes \hat{p})\big] \geqslant \big\langle (B_2 - B_1)\widehat{Y}, \widehat{Y}\big\rangle + \widetilde{K} - \|B_2 - B_1 + \widetilde{K}\|\big(o_Y(1)\big).$$

Therefore, by using the estimates on $\|B_2\|$, $\|B_1 - B_2\|$, $|p|, |q|$ and $|p - q|$, we are lead to

$$\mathrm{Tr}\big[(B_2 - B_1 + \widetilde{K}\mathrm{Id})(\hat{p} \otimes \hat{p})\big] \geqslant CK^{-1}\alpha(1 - \alpha)|Y|^{\alpha-2} - K - \big(CK\alpha|Y|^{\alpha-2} + K\big)o_Y(1).$$

On the other hand, for the right-hand side of (32), we first look at the ω_1 term. By some computations, we get

$$|x - y|\big(1 + |p| + |q|\big) + \varpi\big(|p| \wedge |q|\big)|p - q| = K\alpha Q_1 + K\varpi\big(|p| \wedge |q|\big)Q_2^{3/4} + o_Y(1),$$

since $O(|x - x_0|^3)$ is like $Q_2^{3/4}$. This estimate is emphasizing the role of Q_1, Q_2 and the necessity of having the ϖ term.

The complete estimate of the right-hand side of (32) is

$$\begin{aligned}&K\omega_1\big(K\alpha Q_1 + K\varpi\big(|p| \wedge |q|\big)Q_2^{3/4} + o_Y(1)\big)C\alpha|Y|^{\alpha-2} \\ &\quad + KC^2\alpha^2|Y|^{2\alpha-2} + C^3\alpha^3|Y|^{3\alpha-2} + K + o_Y(1) + 2L(t - t_0),\end{aligned}$$

where we (partially) use the fact that $Q_1 = C|Y|^{\alpha}$ and $Q_2 = L|x - x_0|^4$ are bounded for C large enough.

By dividing all the above inequalities by the (very large) term $C\alpha|Y|^{\alpha-2}$, we obtain the following (almost) final estimate

$$\kappa(1-\alpha)K^{-1} \leqslant K\omega_1\big(K\alpha Q_1 + K\varpi\big(|p|\wedge|q|\big)Q_2^{3/4} + o_Y(1)\big) + K\alpha Q_1 + K\alpha^2 Q_1^2 + o_Y(1).$$

And by using the fact that $|p|, |q| \to +\infty$ as C tends to $+\infty$, this yields

$$\kappa(1-\alpha)K^{-1} \leqslant K\omega_1\big(K\alpha Q_1 + o_Y(1)Q_2^{3/4} + o_Y(1)\big) + K\alpha Q_1 + K\alpha^2 Q_1^2 + o_Y(1). \tag{32}$$

On one hand, by using the uniform control on Q_1, Q_2, we can choose α small enough (depending only on the L^∞ norm of u and the data) in order to have

$$\kappa(1-\alpha)K^{-1} \geqslant \frac{3}{2}\big(K\omega_1(K\alpha Q_1) + K\alpha Q_1 + K\alpha^2 Q_1^2\big) > K\omega_1(K\alpha Q_1) + K\alpha Q_1 + K\alpha^2 Q_1^2.$$

With this choice, it is clear that the inequality (32) cannot holds for C large enough (depending again only on the local L^∞ norm of u and the data) and the local $C^{0,\alpha}$ estimate is proved for small enough α.

This $C^{0,\alpha}$ property provides us with a modulus of continuity in $B(x_0, \delta/4)$ (which depends only on the L^∞ norm of u and the data). By using this modulus of continuity we can show that for any $0 < \alpha < 1$, $Q_1, Q_2 \to 0$ as $C \to +\infty$. Hence arguing as above, we obtain the $C^{0,\alpha}$ estimate for any $\alpha < 1$. The proof of Theorem 3.1 is complete. □

Remark 3.1. If we suppose that F satisfies (F4)–(F5) and for every $R > 0$ there is $L_R > 0$ such that

$$\big|F(x,p,M)\big| \leqslant L_R\big(1 + \|M\|\big), \quad \text{for all } (x,p,M) \in \overline{\mathcal{O}} \times \overline{B}(0,R) \times \mathcal{S}^n,$$

then by using for instance the same strategy of proof of Lemma 9.1 in [3] one can show that u is in $C^{0,\alpha/2}_{\mathrm{loc}}(0,+\infty)$ locally uniformly in x.

4. Convergence as $t \to +\infty$ to the stationary solution

In this section we will prove Theorems 2.4 and 2.5.

Proof of Theorem 2.4. We split the proof in several steps

1. Let $\tilde{u}$ be a solution of (12)–(13). Since χ and $\tilde{u}$ are both solutions of (1)–(2), by applying the comparison result in $\mathcal{O} \times (t,+\infty)$ instead of $\mathcal{O} \times (0,+\infty)$, we get for all $s \geqslant t$,

$$\max_{\overline{\mathcal{O}}}\big(\chi(x,s) - \tilde{u}(x)\big) \leqslant \max_{\overline{\mathcal{O}}}\big(\chi(x,t) - \tilde{u}(x)\big),$$
$$\min_{\overline{\mathcal{O}}}\big(\chi(x,s) - \tilde{u}(x)\big) \geqslant \min_{\overline{\mathcal{O}}}\big(\chi(x,t) - \tilde{u}(x)\big).$$

Thus the functions $t \mapsto M(t) = \max_{\overline{\mathcal{O}}}(\chi(x,t) - \tilde{u}(x))$, $t \mapsto m(t) = \min_{\overline{\mathcal{O}}}(\chi(x,t) - \tilde{u}(x))$ are respectively decreasing and increasing in t. Since $M(t)$ and $m(t)$ are also bounded, we have $m(t) \to \bar{m}$ and $M(t) \to \overline{M}$ as $t \to +\infty$.

2. Let $x_0 \in \mathcal{O}$ and $r > 0$ such that $\overline{B}(x_0, r) \subset \mathcal{O}$. From Theorem 3.1 it follows that there exists a sequence $t_n \to +\infty$ as $n \to +\infty$ such that

$$\chi(x, t_n) \to v(x), \quad \text{uniformly on } \overline{B}(x_0, r).$$

We define

$$\phi_n(x,t) := \chi(x, t + t_n).$$

We notice that ϕ_n is a solution of the problem (1)–(2) in $\mathcal{O} \times (-t_n, +\infty)$. We are going to use the half-relaxed limits of ϕ_n introduced by Barles and Perthame [6] and defined by

$$\bar{\phi}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ n\to+\infty}} \phi_n(y,s), \qquad \underline{\phi}(x,t) = \liminf_{\substack{(y,s)\to(x,t)\\ n\to+\infty}} \phi_n(y,s).$$

The two functions $\bar{\phi}$ and $\underline{\phi}$ are respectively sub- and supersolutions of the problem on $\mathcal{O} \times (-\infty, +\infty)$.

3. We claim that $\max_{\overline{\mathcal{O}}}(\bar{\phi}(x,t) - \tilde{u}(x))$ and $\min_{\overline{\mathcal{O}}}(\underline{\phi}(x,t) - \tilde{u}(x))$ are constant in time.

In order to prove the claim it is enough to show that

$$\lim_{n\to\infty}\max_{\overline{\mathcal{O}}}\big(\phi_n(x,t)-\tilde{u}(x)\big)=\max_{\overline{\mathcal{O}}}\big(\bar{\phi}(x,t)-\tilde{u}(x)\big) \tag{33}$$

and

$$\lim_{n\to\infty}\min_{\overline{\mathcal{O}}}\big(\phi_n(x,t)-\tilde{u}(x)\big)=\min_{\overline{\mathcal{O}}}\big(\underline{\phi}(x,t)-\tilde{u}(x)\big). \tag{34}$$

Indeed if the (33) and (34) hold, then Lemma 2.1 and Proposition 2.1 imply

$$\underline{\phi}(x,t)=\tilde{u}(x)+\bar{m},\quad \bar{\phi}(x,t)=\tilde{u}(x)+\overline{M}\quad \text{for all } (x,t)\in\overline{\mathcal{O}}\times[0,+\infty).$$

Now we show (33) (the proof of (34) is similar).

For all $n\in\mathbb{N}$, let $x_n\in\overline{\mathcal{O}}$ and $\bar{x}\in\overline{\mathcal{O}}$ be such that

$$\chi(x_n,t+t_n)-\tilde{u}(x_n)=\max_{\overline{\mathcal{O}}}\big(\chi(x,t+t_n)-\tilde{u}(x)\big)=\max_{\overline{\mathcal{O}}}\big(\phi_n(x,t)-\tilde{u}(x)\big)$$

and

$$\bar{\phi}(\bar{x},t)-\tilde{u}(\bar{x})=\max_{\overline{\mathcal{O}}}\big(\bar{\phi}(x,t)-\tilde{u}(x)\big).$$

We have $x_n\to\tilde{x}$ up to subsequence. Thus the following estimate holds:

$$\begin{aligned}\lim_{n\to+\infty}M(t+t_n)&\leqslant\limsup_{n\to+\infty}\big(\phi_n(x_n,t)-\tilde{u}(x_n)\big)\\&\leqslant\bar{\phi}(\tilde{x},t)-\tilde{u}(\tilde{x})\leqslant\max_{\overline{\mathcal{O}}}\big(\bar{\phi}(x,t)-\tilde{u}(x,t)\big).\end{aligned}$$

On the other hand, let $(x_n,s_n)\to(\bar{x},\bar{t})$ be such that

$$\phi_n(x_n,s_n)\to\bar{\phi}(\bar{x},t)\quad\text{as } n\to+\infty.$$

We have

$$\begin{aligned}\max_{\overline{\mathcal{O}}}\big(\bar{\phi}(x,t)-\tilde{u}(x,t)\big)&=\bar{\phi}(\bar{x},t)-\tilde{u}(\bar{x})=\lim_{n\to+\infty}\phi_n(x_n,s_n)-\tilde{u}(x_n)\\&=\limsup_{s\to t,n\to\infty}\phi_n(x_n,s)-\tilde{u}(x_n)=\limsup_{n\to\infty}\phi_n(x_n,t)-\tilde{u}(x_n)\\&\leqslant\lim_n M(t+t_n).\end{aligned}$$

Thus we have proved the (33) and the claim.

4. Now we observe that $\bar{\phi}(x,t)=\underline{\phi}(x,t)$ in $B(x_0,r)\times\{0\}$. Indeed since the problem is invariant by translation, the operators F and L being independent of the time, we have by the maximum principle

$$\big\|\chi(y,t_n+s)-\chi(y,t_n)\big\|_{L^\infty(\overline{\mathcal{O}})}\leqslant\big\|\chi(y,s)-\chi(y,0)\big\|_{L^\infty(\overline{\mathcal{O}})}.$$

Thus

$$\begin{aligned}\bar{\phi}(x,0)&=\limsup_{\substack{(y,s)\to(x,0)\\ n\to+\infty}}\phi_n(y,s)=\limsup_{\substack{(y,s)\to(x,0)\\ n\to+\infty}}\chi(y,t_n+s)\\&=\limsup_{\substack{(y,s)\to(x,0)\\ n\to+\infty}}\big[\chi(y,t_n+s)-\chi(y,t_n)+\chi(y,t_n)\big]\\&=\limsup_{\substack{(y,s)\to(x,0)\\ n\to+\infty}}\big[\chi(y,t_n+s)-\chi(y,t_n)\big]+\limsup_{\substack{(y,s)\to(x,0)\\ n\to+\infty}}\chi(y,t_n)=v(x).\end{aligned}$$

Therefore $\bar{m}=\overline{M}$ and in particular $v=\tilde{u}+\bar{m}$. The convergence (17) holds with $\tilde{u}$ replaced by $u_\infty=\tilde{u}+\bar{m}$. Indeed for $t>t_n$ we have

$$\left\|\chi(x,t)-\tilde{u}-\bar{m}\right\|_{L^\infty(\overline{\mathcal{O}})}\leqslant\left\|\chi(x,t_n)-\tilde{u}+\bar{m}\right\|_{L^\infty(\overline{\mathcal{O}})}=o_n(1)$$

as $n\to+\infty$.

The proof is complete and we conclude. □

Proof of Theorem 2.5. We just give a sketch of proof, the arguments being similar to ones of proof of Theorem 2.4. The existence and the uniqueness of a solution w of (4)–(5) is a consequence of the results in [2]. We observe that $w+\tilde{\lambda}t$ and $\tilde{u}$ are both solutions of

$$w_t+F\big(x,Dw,D^2w\big)=\tilde{\lambda}\quad\text{in }\mathcal{O}\times(0,+\infty),\tag{35}$$

$$w_t+G(x,Dw)=\tilde{\lambda}\quad\text{on }\partial\mathcal{O}\times(0,+\infty).\tag{36}$$

The comparison principle for this evolution problem yields

$$\left\|w(x,t)-u(x)+\tilde{\lambda}t\right\|_\infty\leqslant\|\Phi-u\|_\infty.$$

Therefore $w(x,t)+\tilde{\lambda}t$ remains bounded. Theorem 3.1 yields that for all $t\geqslant c>0$ we have $w(\cdot,t)+\tilde{\lambda}t\in C^{0,\alpha}(\overline{O})$. Thus the conclusion follows arguing exactly as in the proof of Theorem 2.4. □

Acknowledgments

This work was influenced by the discussions the author had with Prof. Guy Barles during the preparation of the joint paper [4] and the author is very grateful to him. The author would like to thank the referee for some useful remarks. This work was partially supported by MIUR, project "Viscosity, metric, and control theoretic methods for nonlinear partial differential equations."

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