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Closed description of arbitrariness in resolving quantum master equation

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A B S T R A C T

In the most general case of the Delta exact operator valued generators constructed of an arbitrary Fermion operator, we present a closed solution for the transformed master action in terms of the original master action in the closed form of the corresponding path integral. We show in detail how that path integral reduces to the known result in the case of being the Delta exact generators constructed of an arbitrary Fermion function.

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1. Introduction

It is recognized commonly that the field–antifield formalism in its present form provides for the most powerful BRST-inspired methods for covariant (Lagrangian) quantization as applied to complex relativistic gauge-invariant dynamical systems.

It is well known that the gauge invariant status of the general field–antifield formalism is completely under control of the quantum master equation. The existence of the Fermion nilpotent Delta-operator makes it possible to expect that the transformations with Delta exact generators do act transitively on the set of allowed solutions to the quantum master equation. These generators have the form of $[\Delta, F]$, where F is a Fermion operator, in general. Usually, one considers a simple case of being the F an arbitrary Fermion function $F(Z)$ rather than an operator [1–3]. In the latter case the corresponding arbitrariness is a set of finite anticanonical master transformations [1–3]. In the simplest case of being $F(Z)$ only quadratic in Z, these linear transformations preserve the antisymplectic metric, so that we call them an antisymplectic rescaling. We conjecture that the field renormalizations can be included naturally into the group of antisymplectic rescaling. In the present article, our main purpose is to give a closed description to the arbitrariness in resolving the quantum master equation in the most general case of being the F an arbitrary Fermion ZP ordered operator $F(Z, P)$, where P is a canonically conjugate for Z.

Of course, an explicit solution is impossible in that case. However, by making use of the symbol calculus, together with the functional methods [4–6], we express the transformed master action in terms of the original master action in the closed form of the corresponding path integral. In principle, the latter path integral can be calculated, in general, in the form of quasi-classical loop expansion. On the other hand, it is an interesting question, how the path integral suggested reproduces the explicit solution for the transformed master action in the previous simple case of being the F an arbitrary Fermion function $F(Z)$. It appears that in the latter case, there happens exactly the phenomenon of quantum localization of classical mechanics [7], so that the P integration yields the delta functional concentrated exactly on the explicit anticanonically F-transformed Z, which results in precise reconstruction to the previous explicit solution.

2. Antisymplectic rescaling to the quantum master equation

Let us proceed with the standard quantum master equation

$$\Delta \exp\left\{\frac{i}{\hbar}W\right\} = 0, \quad \varepsilon(\Delta) = 1, \quad \Delta^2 = 0 \qquad (2.1)$$

to be resolved for the quantum action W, $\varepsilon(W) = 0$. Its natural automorphisms are given by the well-known formula [1–3]

$$\exp\left\{\frac{i}{\hbar}W\right\} \rightarrow \exp\left\{\frac{i}{\hbar}W'\right\} =: \exp\{[\Delta, F]\} \exp\left\{\frac{i}{\hbar}W\right\}. \qquad (2.2)$$

The form of a supercommutator of two Fermion operators, with being at least one of them nilpotent, is rather characteristic for the unitarizing Hamiltonian in the generalized Hamiltonian for-

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malism [8–12], especially, in the formulation invariant under time reparametrizations [13,14]. That form is also known to yield the Heisenberg equations of motion whose right-hand side is proportional to the sum of the two dual quantum antibrackets [15–17] generated, respectively, by each of the two operators involved.

It seems natural to conjecture that the renormalization can be included into the group of antisymplectic rescalings extracted from (2.2) by choosing a quadratic ansatz for F,

$$F =: \frac{1}{2} Z^A F_{AB} Z^B, \tag{2.3}$$

$$F_{AB} = \text{const}(Z), \quad \varepsilon(F_{AB}) = \varepsilon_A + \varepsilon_B + 1, \tag{2.4}$$

$$F_{AB} = F_{BA}(-1)^{\varepsilon_A \varepsilon_B}. \tag{2.5}$$

Given a constant invertible antisymplectic metric,

$$E^{AB} = \text{const}(Z), \quad \varepsilon(E^{AB}) = \varepsilon_A + \varepsilon_B + 1, \tag{2.6}$$

$$E^{AB} = -E^{BA}(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}, \tag{2.7}$$

the Delta-operator is defined as to the case of trivial measure density, $\rho = 1$,

$$\Delta =: \frac{1}{2}(-1)^{\varepsilon_A} \partial_A E^{AB} \partial_B. \tag{2.8}$$

Then a remarkable formula holds

$$[\Delta, F] = (\Delta F) - \text{ad}(F) \tag{2.9}$$

with F being an arbitrary Fermion function $F(Z)$ (Section 3), as well as an arbitrary ZP ordered Fermion operator $F(Z, P)$, where P is canonically conjugate to Z (Section 4).

In terms of E^{AB} and F_{AB}, let us define the antisymplectic generator [18],

$$G^A_{\ B} = -G_B^{\ A}(-1)^{(\varepsilon_A+1)\varepsilon_B}, \tag{2.10}$$

where

$$G^A_{\ B} =: E^{AC} F_{CB}, \quad G_B^{\ A} =: F_{BC} E^{CA}. \tag{2.11}$$

In terms of the $G_A^{\ B}$, the right-hand side in (2.2) rewrites as

$$\exp\left\{\frac{i}{\hbar} W'\right\} = \exp\left\{-\frac{1}{2} G_A^{\ A}(-1)^{\varepsilon_A} - Z^A G_A^{\ B} \overrightarrow{\partial_B}\right\} \exp\left\{\frac{i}{\hbar} W\right\} =$$
$$= \exp\left\{-\frac{1}{2} G_A^{\ A}(-1)^{\varepsilon_A} + \frac{i}{\hbar} W_R\right\}, \tag{2.12}$$

where

$$W_R =: W(Z_R), \quad Z^A_R =: Z^B(\exp\{-G\})_B^{\ A} = (\exp\{G\})^A_{\ B} Z^B. \tag{2.13}$$

Here in (2.13), the Z^A_R is just the antisymplectic rescaling as applied to Z^A. Of course, the matrix

$$S^A_{\ B} =: (\exp\{G\})^A_{\ B}, \tag{2.14}$$

preserves the antisymplectic metric,

$$S^A_{\ C} E^{CD} S^B_{\ D}(-1)^{\varepsilon_D(\varepsilon_B+1)} = E^{AB}. \tag{2.15}$$

3. The general case of an arbitrary Fermion function $F(Z)$

Now, let us describe in short the general case of arbitrary Fermion function $F(Z)$ in formula (2.2). Then the formula (2.12) generalizes as

$$\exp\left\{\frac{i}{\hbar} W'\right\} = J^{1/2} \exp\left\{\frac{i}{\hbar} W_R\right\}, \tag{3.1}$$

where

$$W_R = W(Z_R), \quad Z^A_R = \exp\{-\text{ad}(F)\} Z^A, \tag{3.2}$$

$$J =: \text{sDet}[(Z^A_R \overleftarrow{\partial}_B)], \quad J^{1/2} = \exp\{(E(-\text{ad}(F))\Delta F)\}, \tag{3.3}$$

$$E(X) =: \int_0^1 dt \exp\{tX\} = \frac{\exp\{X\}-1}{X}. \tag{3.4}$$

4. The most general case of an arbitrary Fermion operator $F(Z, P)$

Finally, let us mention in short the case of being the F an operator,

$$F = F(Z, P), \quad [Z^A, P_B] = i\hbar \delta^A_B 1, \tag{4.1}$$

with P_A being momenta canonically conjugate to Z^A,

$$P_A =: -i\hbar \overrightarrow{\partial_A}(-1)^{\varepsilon_A}. \tag{4.2}$$

In terms of the symbol chosen, say ZP symbol, the formula (2.2) rewrites as

$$\exp\left\{\frac{i}{\hbar} W'(Z)\right\}$$
$$= \left(\exp_*\{[\text{symbol}\Delta, \text{symbol}F]_*\} * \exp\left\{\frac{i}{\hbar} W(Z)\right\}\right)\Big|_{\text{symbol } P=0}, \tag{4.3}$$

where $*$ means the symbol multiplication,

$$\text{operator} A \ \leftrightarrow \ \text{symbol} A,$$

$$\text{operator} A \ \text{operator} B \ \leftrightarrow \ \text{symbol} A * \text{symbol} B, \tag{4.4}$$

$[\ ,\ ]_*$ means the respective symbol supercommutator, and $\exp_*$ means the symbol exponential. Given the operator F in ZP normal form, let us denote its ZP symbol in short as $F(Z, P)$, while the respective symbol multiplication is given by

$$* =: \exp\left\{-i\hbar \frac{\overleftarrow{\partial}}{\partial P_A}(-1)^{\varepsilon_A} \frac{\overrightarrow{\partial}}{\partial Z^A}\right\}. \tag{4.5}$$

Then, by proceeding with the symbol representation (4.3), and using the standard functional methods [4–6], one can derive the following path integral solution

$$\exp\left\{\frac{i}{\hbar} W'(Z)\right\}$$
$$= \left\langle \exp\left\{-\frac{i}{\hbar} \int_0^1 dt H(Z(t), P(t-0)) + \frac{i}{\hbar} W(Z(0))\right\}\right\rangle, \tag{4.6}$$

$$H(Z, P) =: (F(Z, P), Z^A) P_A (-1)^{\varepsilon_A} - \frac{\hbar}{i}(\Delta F(Z, P)), \tag{4.7}$$

where the functional average is defined as

$$\langle(...)\rangle =: \frac{\int \mathcal{D}Z \mathcal{D}P (...) \exp\left\{\frac{i}{\hbar} \int_0^1 dt P_A \dot{Z}^A\right\}}{\int \mathcal{D}Z \mathcal{D}P \exp\left\{\frac{i}{\hbar} \int_0^1 dt P_A \dot{Z}^A\right\}}, \tag{4.8}$$

where the integration trajectory $Z^A(t)$ is restricted to satisfy the condition

$$Z^A(t+0=1) = Z^A. \tag{4.9}$$

One can take the latter condition into account explicitly by introducing the well defined representation

$$Z^A(t) =: Z^A - \int_{t+0}^{1} dt' V^A(t').$$ (4.10)

Then one can change for the integration over unrestricted velocities $V^A(t)$, $\mathcal{D}Z \to \mathcal{D}V$.

Now, let us return temporary to the case of P-independent F, $F = F(Z)$. Then the P integration in (4.6) yields the delta functional

$$\delta[\dot{Z}^A - (F, Z^A)],$$ (4.11)

so that

$$Z^A(t) = \exp\{-(1-t)\mathrm{ad}(F)\}Z^A, \quad Z^A(0) = \exp\{-\mathrm{ad}(F)\}Z^A.$$ (4.12)

Let us represent the Jacobian of the delta functional (4.11) via the unrestricted velocity $V^A(t)$,

$$\mathrm{sDet}\,[\delta_B^A \delta(t-t') + (F, Z^A)\overleftarrow{\partial}_B(Z(t))\,\theta(t'-t-0)].$$ (4.13)

By expanding the logarithm of the Jacobian (4.13) in powers of the second term, one can easily see that all orders are zero due to the specific products of the theta functions. For the first order we have

$$-\int_0^1 dt (\Delta F)(Z(t))\theta(-0) = 0.$$ (4.14)

For the second order we get

$$\int_0^1 dt \int_0^1 dt'((F, Z^A)\overleftarrow{\partial}_B(Z(t)))((F, Z^B)\overleftarrow{\partial}_A(Z(t')))(-1)^{\varepsilon_A}$$
$$\times \theta(t'-t-0)\,\theta(t-t'-0) = 0,$$ (4.15)

and so on (for closed derivation see Appendix A). Thus, the Jacobian (4.13) equals to one. Then, by substituting the solution (4.12), we arrive at the formula (3.1). That is a particular case of the phenomenon of quantum localization of classical mechanics [7].

In a purely formal sense, the path integral (4.6) resolves the Schrödinger equation

$$i\hbar\partial_t \Psi(t, Z) = H(Z, P)\Psi(t, Z),$$ (4.16)

with $H(Z, P)$ being the operator valued Hamiltonian (see (4.2) for momenta P_A),

$$H(Z, P) = (i\hbar)^{-1}[F(Z, P), \frac{1}{2}P_A E^{AB} P_B(-1)^{\varepsilon_B}],$$ (4.17)

where $F(Z, P)$ is assumed ZP ordered,

$$F(Z, P) =: F(Z, Y)\exp\left\{\overleftarrow{\frac{\partial}{\partial Y_A}}P_A\right\}\bigg|_{Y=0}.$$ (4.18)

Then we have

$$\Psi(0, Z) = \exp\left\{\frac{i}{\hbar}W(Z)\right\}, \quad \Psi(1, Z) = \exp\left\{\frac{i}{\hbar}W'(Z)\right\}.$$ (4.19)

Thus, we see that in the case of being the $F(Z, P)$ just an operator valued quantity, the arbitrariness in resolving the quantum master equation can only be described comprehensively by applying the quantum-mechanical treatment in its precise form.

Notice that the path integral solution (4.6) rewrites naturally into its variation-derivative form,

$$\exp\left\{\frac{i}{\hbar}W'(Z)\right\}$$
$$= \exp\left\{-\frac{i}{\hbar}\int_0^1 dt H\left(i\hbar\frac{\delta}{\delta J(t)}, i\hbar\frac{\delta}{\delta K(t-0)}\right)\right\}$$
$$\times \exp\left\{-\frac{i}{\hbar}\int_0^1 dt J_A(t)\left(Z^A - \int_t^1 dt' K^A(t')(-1)^{\varepsilon_A}\right)\right.$$
$$\left. + \frac{i}{\hbar}W\left(Z - \int_0^1 dt' K(t')(-1)^\varepsilon\right)\right\}\bigg|_{J=0, K=0}.$$ (4.20)

One can always return back to (4.6) by inserting the factor

$$1 = \mathrm{const}\int \mathcal{D}V \int \mathcal{D}P \exp\left\{\frac{i}{\hbar}\int_0^1 dt P_A(V^A - K^A(-1)^{\varepsilon_A})\right\},$$ (4.21)

to the right of the first exponential in the right-hand side in (4.20). Here in (4.21), const is a normalization constant.

In the most general case of a non-constant antisymplectic metric $E^{AB}(Z)$ and a measure density $\rho(Z)$, where the Delta operator (2.8) becomes

$$\Delta =: \frac{1}{2}(-1)^{\varepsilon_A}\rho^{-1}\partial_A\rho E^{AB}\partial_B,$$ (4.22)

the ZP symbol (4.7) generalizes as

$$i\hbar H(Z, P) =: \Pi(Z, \tilde{P})F(Z, P) + F(Z, P)\Pi(\tilde{Z}, P),$$ (4.23)

$$\Pi(Z, P) = \frac{1}{2}(E^{AB}(Z)P_B P_A - i\hbar(\mathrm{div}\,E)^B(Z)P_B)(-1)^{\varepsilon_B},$$ (4.24)

where we have denoted

$$\tilde{P}_A =: P_A - i\hbar\frac{\overrightarrow{\partial}}{\partial Z^A}(-1)^{\varepsilon_A}, \quad \tilde{Z}^A =: Z^A - i\hbar\frac{\overleftarrow{\partial}}{\partial P_A}(-1)^{\varepsilon_A},$$ (4.25)

$$(\mathrm{div}\,E)^B(Z) =: \rho^{-1}(Z)\left(\frac{\partial}{\partial Z^A}\rho(Z)E^{AB}(Z)\right)(-1)^{\varepsilon_A}.$$ (4.26)

In its turn, the operator valued Hamiltonian (4.17) generalizes as

$$H(Z, P) = (i\hbar)^{-1}[F(Z, P), \Pi(Z, P)].$$ (4.27)

In its general features, the above consideration was addressed to the case of the Delta operator (4.22) as assumed to be a nilpotent one. However, there exists a bit modified version as to the Delta operator [19] (see also the references therein). One cancels the nilpotency assumption for the original Delta operator (4.22), and then defines a new nilpotent operator by adding a Fermion function to the (4.22), so that the new Fermion function is determined just via the nilpotency condition for the new Delta operator. In this way, the measure density becomes independent of the antisymplectic metric. By proceeding with the new Delta operator, one can apply the above consideration in a quite similar way. As a result, there will be no modifications, being the $F(Z)$ a function. In the case of being the $F(Z, P)$ an actual operator, a simple new term should be added in the right-hand side in (4.23), that is $(i\hbar)^2 \nu(Z)$, with $\nu(Z)$ being just the new Fermion function added to the (4.22).

In the case of a non-constant $\rho(Z)$, as the scalar product is defined with respect to the invariant integration measure $d\mu(Z) =: \rho(Z)dZ$, to make the operator (4.2) Hermitian, the latter should be transformed:

$$P_A \rightarrow \rho^{-1/2}P_A\rho^{1/2} = P_A - \frac{i\hbar}{2}(\ln\rho)\overset{\leftarrow}{\partial}_A, \qquad (4.28)$$

which results in the same shift as to the P-argument in every ZP symbol. The latter common shift can be canceled by the opposite shift for P in the kinetic exponential in the *nominator* in (4.8). As a result, one acquires the factor

$$\frac{\rho^{1/2}(Z(0))}{\rho^{1/2}(Z)} \qquad (4.29)$$

in front of the exponential inside the overage in the right-hand side in (4.6). Thereby, the equation (4.6) takes the form of a transformation law as formulated for the semi-density $\exp\{\frac{i}{\hbar}W\}\rho^{1/2}$. Notice that for a symbol $F(Z, P)$, the above shift (4.28) does *not* coincide with the precise form of the symbol transformation

$$F(Z, P) \rightarrow \rho^{-1/2}(Z) * F(Z, P) * \rho^{1/2}(Z)$$
$$= F(Z, P - \frac{i\hbar}{2}(\ln\rho)\overset{\leftarrow}{\partial}_A(\tilde{Z})), \qquad (4.30)$$

with the $\tilde{Z}^A$ being given by the second in (4.25). In contrast to the latter formula (4.30), the second term in (4.28) is taken at the original argument Z^A, *not* at the $\tilde{Z}^A$. However, in the integrand of the *regularized* functional integral in (4.6), just the above shift (4.28) in the symbol $H(Z, P)$ results, when having been canceled via the opposite shift in P_A, in appearance of the correct factor (4.29) as having it come from the kinetic exponential in the functional integrand in the nominator in (4.8). Notice also that the components of the second argument of F in (4.30) do commute among themselves.

Let us notice by the way that the above consideration extends naturally as to the case of the $Sp(2)$ symmetric quantum master equation [20–24],

$$\Delta_+^a \exp\left\{\frac{i}{\hbar}W\right\} = 0, \quad \Delta_+^a\Delta_+^b + (a \leftrightarrow b) = 0, \qquad (4.31)$$

where $\Delta_\pm^a$ is a pair of the $Sp(2)$-vector valued Delta operator together with its transposed,

$$\Delta_\pm^a =: \Delta^a \pm \frac{i}{\hbar}\mathcal{V}^a, \quad \varepsilon(\Delta^a) = \varepsilon(\mathcal{V}^a) = 1, \qquad (4.32)$$

$$\Delta^a =: \frac{1}{2}(-1)^{\varepsilon_A}\rho^{-1}\partial_A\rho E^{aAB}\partial_B$$
$$= \frac{1}{2}\left((-1)^{\varepsilon_A}E^{aAB}\partial_B\partial_A + (\text{div }E^a)^B\partial_B\right), \qquad (4.33)$$

$$(\text{div }E^a)^B =: (-1)^{\varepsilon_A}\rho^{-1}(\partial_A\rho E^{aAB}), \qquad (4.34)$$

$$\mathcal{V}^a =: V^a + \frac{1}{2}\text{div }V^a, \quad V^a =: V^{aA}\partial_A, \quad \text{div }V^a =: \rho^{-1}\partial_A(\rho V^{aA}), \qquad (4.35)$$

with V^a being the special vector field. A counterpart to the formula (2.2) has the form [25]

$$\exp\left\{\frac{i}{\hbar}W\right\} \rightarrow \exp\left\{\frac{i}{\hbar}W'\right\}$$
$$=: \exp\left\{i\hbar\frac{1}{2}\varepsilon_{ab}[\Delta_+^b, [\Delta_+^a, B]]\right\}\exp\left\{\frac{i}{\hbar}W\right\}, \quad \varepsilon(B) = 0. \qquad (4.36)$$

By making use of the methods quite similar to the above, one can derive in a simple way a natural counterpart to the formulae (4.6), (4.23).

Of course, the general formula (4.6) remains valid, while the formula (4.23) generalizes as to take the form

$$i\hbar H(Z, P) = \Pi^b(Z, \tilde{P})F_b(Z, P) + F_b(Z, P)\Pi^b(\tilde{Z}, P), \qquad (4.37)$$

where

$$i\hbar F_b(Z, P) = \Pi^a(Z, \tilde{P})B(Z, P)\frac{1}{2}\varepsilon_{ab} - B(Z, P)\frac{1}{2}\varepsilon_{ab}\Pi^a(\tilde{Z}, P), \qquad (4.38)$$

and

$$\Pi^a(Z, P) = \frac{1}{2}\Big[\big(E^{aAB}(Z)P_BP_A + (2V^{aB}(Z) -$$
$$- i\hbar(\text{div }E^a)^B(Z))P_B\big)(-1)^{\varepsilon_B} - i\hbar\text{ div }V^a(Z)\Big]. \qquad (4.39)$$

In the main body of the present paper, we have used the normal ZP-symbol, which is the simplest one technically. In principle, one could use another type of symbols, say, the Weyl symmetric symbol. At least, in the case of being the generator an arbitrary Fermion function $F(Z)$, it can be shown that with the use of a new symbol one reproduces the same formula (3.3) in new coordinates.

Acknowledgements

Appendix A. Closed derivation to the Jacobian (4.13)

Here we present in short a closed derivation to the Jacobian (4.13). Let us denote

$$X^A_B(t) =: (F, Z^A)\overset{\leftarrow}{\partial}_B(Z(t)). \qquad (A.1)$$

Then, in short matrix notations, logarithm of the Jacobian (4.13) reads

$$\ln J =: \int_0^1 d\lambda \int_0^1 dt \int_0^1 dt'\, \text{str}\,(G(t, t'; \lambda)X(t'))\theta(t - t' - 0), \qquad (A.2)$$

where the Green's function $G(t, t'; \lambda)$ is defined by the integral equation

$$\int_0^1 dt'[\delta(t - t')1 + \lambda X(t)\theta(t' - t - 0)]G(t', t''; \lambda) = \delta(t - t'')1. \qquad (A.3)$$

Let us denote

$$\Gamma(t, t''; \lambda) =: \int_{t+0}^1 dt'G(t', t''; \lambda), \qquad (A.4)$$

then the equation (A.3) rewrites in its differential form

$$[-\partial_t + \lambda X(t)]\Gamma(t, t''; \lambda) = \delta(t - t''), \quad \Gamma(t + 0 = 1, t''; \lambda) = 0, \qquad (A.5)$$

which resolves in the form

$$\Gamma(t, t''; \lambda) = \theta(t'' - t - 0)U(t, t''; \lambda), \tag{A.6}$$

where the holonomy matrix U is defined by the equation

$$[-\partial_t + \lambda X(t)]U(t, t''; \lambda) = 0, \quad U(t = t'', t''; \lambda) = 1. \tag{A.7}$$

At $Z^A(t) = Z^A(t; \lambda)$ in (A.1), the latter Cauchy problem (A.7) resolves in the form

$$U(t, t''; \lambda) =: U(t; \lambda)U^{-1}(t''; \lambda), \quad U(t; \lambda) =: Z(t; \lambda) \otimes \frac{\overleftarrow{\partial}}{\partial Z}, \tag{A.8}$$

where $Z^A(t; \lambda)$ is given by the first in (4.12) with the F being λ-rescaled as $F \to \lambda F$.

It follows from (A.4), (A.6) that

$$G(t, t'; \lambda) = -\partial_t \Gamma(t, t'; \lambda)$$
$$= \delta(t - t')1 - \theta(t' - t - 0)\lambda X(t)U(t, t'; \lambda). \tag{A.9}$$

By inserting (A.9) into (A.2), we arrive at

$$\ln J = \int_0^1 d\lambda \int_0^1 dt \int_0^1 dt' \, \text{str} \left((\delta(t - t')1 - \right.$$
$$\left. - \theta(t' - t - 0)\lambda X(t)U(t, t'; \lambda))X(t') \right)\theta(t - t' - 0) = 0. \tag{A.10}$$

Thus, we have confirmed via closed derivation that $J = 1$. Notice also that (A.2) rewrites directly in terms of (A.4) as

$$\ln J = \int_0^1 d\lambda \int_0^1 dt' \, \text{str} \left(\Gamma(t', t'; \lambda)X(t') \right). \tag{A.11}$$

On the other hand, it follows from (A.6) together with the second in (A.7) that

$$\Gamma(t', t'; \lambda) = \theta(-0)1 = 0, \tag{A.12}$$

which, when being substituted into (A.11), confirms (A.10), even in a simpler way.

It is worth to mention that the result (A.10) seems a bit paradoxical, as the Delta functional (4.11) is concentrated on the solution (4.12) being an anti-canonical transformation as applied to Z^A. On the other hand, at the level of the Z^A-space, an anti-canonical transformation is known to yield a nontrivial Jacobian, in general (see the formula (3.3)). It appears, however, that at the level of the functional space of trajectories $Z^A(t)$, the corresponding functional Jacobian is trivial, just due to the presence of the theta-functions regularized in accordance with the ZP normal ordering chosen.

Finally, let us consider in more details the relation between the functional Jacobian (A.11) and the finite-dimensional Jacobian in the second in (3.3). Due to (A.12), we have for (A.11), now redenoted as $(J')^{-1}$ for further convenience,

$$\ln J' = -\theta(-0) \int_0^1 d\lambda \int_0^1 dt' \, \text{str} \left(X(t') \right), \tag{A.13}$$

where $X(t)$ is defined in (A.1) with the $Z^A(t)$ given by the first in (4.12). Thus, the λ-integral is trivial, and we rewrite (A.13)

in the form

$$(J')^{1/2} = \exp\{\theta(-0)(E(-\text{ad}(F))\Delta F)\}. \tag{A.14}$$

By comparing the latter to the second in (3.3), we conclude that

$$J' = (J)^{\theta(-0)} = 1. \tag{A.15}$$

It is just the relation that shows us a miraculous phenomenon of the functional Jacobian J', inverse to (A.11), as having gulped the finite-dimensional Jacobian J, the second in (3.3). The inverse functional Jacobian J' just comes to stand in front of the delta functional (A.11), and thus the J' is a natural candidate to be compared to J in (3.3).

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