Available online at www.sciencedirect.com

SCIENCE DIRECT

ELSEVIER

Linear Algebra and its Applications 415 (2006) 31–51

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Continuous methods for extreme and interior eigenvalue problems ☆

Gene H. Golub [a], Li-Zhi Liao [b,*]

[a] *Scientific Computing and Computational Mathematics Program, Stanford University, Stanford, CA 94305-9025, United States*

[b] *Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Kowloon, Hong Kong, China*

Received 17 May 2003; accepted 14 January 2005

Available on line 2 March 2005

Submitted by H.A. Van der Vorst

Abstract

In this paper, continuous methods are introduced to compute both the extreme and interior eigenvalues and their corresponding eigenvectors for real symmetric matrices. The main idea is to convert the extreme and interior eigenvalue problems into some optimization problems. Then a continuous method which includes both a merit function and an ordinary differential equation (ODE) is introduced for each resulting optimization problem. The convergence of each ODE solution is proved for any starting point. The limit of each ODE solution for any starting point is fully studied. Both the extreme and the interior eigenvalues and their corresponding eigenvectors can be easily obtained under a very mild condition. Promising numerical results are also presented.

AMS classification: 65F15; 65K10; 65L15; 65M12

Keywords: Continuous method; Eigenvalue; Eigenvector

☆ The first author was supported in part by DOE-FC02-01ER4177. The second author was supported in part by grants from Hong Kong Baptist University and the Research Grant Council of Hong Kong.

* Corresponding author.

E-mail addresses: golub@sccm.stanford.edu (G.H. Golub), liliao@hkbu.edu.hk (L.-Z. Liao).

0024-3795/$ - see front matter © 2005 Elsevier Inc. All rights reserved.

doi:10.1016/j.laa.2005.01.009

1. Introduction

Let $A \in R^{n\times n}$ be a symmetric matrix. From the Real Schur Theorem, we know that all eigenvalues of A are real and there exists an orthonormal matrix $U = (u_1, u_2, \ldots, u_n)$ and a diagonal matrix Λ such that

$$A = U\Lambda U^{\mathrm{T}}, \tag{1.1}$$

where

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \quad \lambda_1 = \cdots = \lambda_s < \lambda_{s+1} \leqslant \cdots \leqslant \lambda_n, \tag{1.2}$$

and $1 \leqslant s \leqslant n$. As a result, we denote

$$S = \left\{x \in R^n \mid Ax = \lambda_1 x, x^{\mathrm{T}}x = 1\right\}. \tag{1.3}$$

Obviously, the columns of U form an orthonormal basis in R^n and S is the subset containing all the eigenvectors with l_2-norm one corresponding to the smallest eigenvalue of A. The extreme eigenvalue problem that we are interested in is to find λ_1 and an $x \in S$. While the interior eigenvalue problem is to find an eigenvalue in a predefined interval $[a, b]$, i.e.

$$Ax = \lambda x, \quad x^{\mathrm{T}}x = 1, \ \ \lambda \in [a, b], \tag{1.4}$$

and its corresponding eigenvector. It should be mentioned that we do not assume that there must exist an eigenvalue in $[a, b]$. Our new method is capable of detecting if there is any eigenvalue in $[a, b]$ or not.

The eigenvalue problem is a classical but very important problem (see [12]). Besides the conventional methods in numerical analysis for the eigenvalue problem (see [12] and the references therein), some continuous methods have been discussed in [1–3] for the extreme eigenvalue problem. In [1,2], various ODE systems are introduced for many numerical analysis problems. Sparked by Hopfield's neural network approach [6–8], Cichocki and Unbehauen [3] introduced a neural network model for computing the minimum eigenvalue and the corresponding eigenvector. The idea in [3] is to convert the minimum eigenvalue problem into a constrained optimization problem. Then a neural network model was introduced to solve this constrained problem by using either the penalty method or the Lagrange multiplier method. However, the optimization problems formulated in [3] are not easy to solve. Therefore, the application of their methods is quite limited. The interior eigenvalue problem is relatively difficult comparing with the extreme eigenvalue problem. So far, there are not any continuous methods for the interior eigenvalue problem in the literature.

In this paper, we also convert both the extreme and interior eigenvalue problems into some optimization problems (Section 2). However, our optimization problems are to minimize a strictly concave function over a unit ball. Therefore, our optimization problems are very easy to solve. For each resulting optimization problem, a continuous method which consists of a merit function and an ordinary differential equation (ODE) is introduced. The convergence of each ODE solution is proved for any starting point

(Section 3). Some promising numerical results are reported (Section 4). Finally, some conclusions are drawn (Section 5).

2. Equivalent optimization problems

To distinguish the two eigenvalue problems, we will discuss them in separate subsections.

2.1. Extreme eigenvalue problem

To formulate the extreme eigenvalue problem into an optimization problem, we consider

$$\begin{aligned} &\min_{x\in R^n} \quad x^{\mathrm{T}}Ax \\ &\text{s.t.} \quad x^{\mathrm{T}}x = 1. \end{aligned} \tag{2.1}$$

For any $x \in R^n$, there exist α_i, $i = 1, \dots, n$ such that

$$x = \sum_{i=1}^{n} \alpha_i u_i, \tag{2.2}$$

where u_i's are the column vectors of U.

Problem (2.1) is to minimize a quadratic function on the surface of a ball. The difficulty for problem (2.1) is its constraint where the feasible region is not a convex set. Now we further convert problem (2.1) into another optimization problem which is much easier to solve. First, let us select a constant c such that

$$c \geqslant \lambda_n + 1. \tag{2.3}$$

Since A is symmetric, from Corollary 2.3.2 in [4], we have

$$\max_{1\leqslant i\leqslant n} |\lambda_i| = \|A\|_2 \leqslant \|A\|_1, \tag{2.4}$$

we can always choose $c = \|A\|_1 + 1$ (we will adopt this formula for c as the default value in our numerical computation). Then we can establish the following problem:

$$\begin{aligned} &\min_{x\in R^n} \quad x^{\mathrm{T}}Ax - cx^{\mathrm{T}}x \\ &\text{s.t.} \quad x^{\mathrm{T}}x \leqslant 1. \end{aligned} \tag{2.5}$$

Problem (2.5) differs from problem (2.1) in that the objective function is quadratic and strictly concave but the constraint is a simple ball constraint. The feasible region for (2.5) is a closed convex set. Therefore, it is much easier to solve (2.5) than (2.1).

Lemma 2.1

(i) *Every local minimizer of* (2.5) *is also a global minimizer of* (2.5).

(ii) x *is a global minimizer of* (2.5) $\iff x \in S$.

Proof. Using (2.2), problem (2.5) becomes

$$\begin{aligned} \min \quad & \sum_{i}^{n} \alpha_i^2(\lambda_i - c) \\ \text{s.t.} \quad & \sum_{i=1}^{n} \alpha_i^2 \leqslant 1. \end{aligned} \tag{2.6}$$

Since

$$0 > \lambda_i - c \geqslant \lambda_1 - c, \quad i = 1, \ldots, n, \tag{2.7}$$

the results in (i) and (ii) can be easily established. □

From Lemma 2.1, we can easily see that the minimum value of (2.5) is $\lambda_1 - c$ and any optimal solution is an eigenvector corresponding to λ_1.

2.2. Interior eigenvalue problem

First, let us consider the following optimization problem:

$$\begin{aligned} \min_{x \in R^n} \quad & x^{\mathrm{T}}(A - aI_n)(A - bI_n)x \\ \text{s.t.} \quad & x^{\mathrm{T}}x = 1. \end{aligned} \tag{2.8}$$

In problem (2.8), there is not any restriction on the values of a and b.

From (2.2), problem (2.8) becomes

$$\begin{aligned} \min_{\alpha_1,\ldots,\alpha_n} \quad & \sum_{i=1}^{n} \alpha_i^2(\lambda_i - a)(\lambda_i - b) \\ \text{s.t.} \quad & \sum_{i=1}^{n} \alpha_i^2 = 1. \end{aligned} \tag{2.9}$$

From (2.9), it is easy to see that

$$\kappa \geqslant x^{\mathrm{T}}(A - aI_n)(A - bI_n)x \geqslant \eta, \tag{2.10}$$

where

$$\kappa = \max_{1 \leqslant i \leqslant n} (\lambda_i - a)(\lambda_i - b), \tag{2.11}$$

$$\eta = (\lambda_k - a)(\lambda_k - b) = \min_{1 \leqslant i \leqslant n} (\lambda_i - a)(\lambda_i - b), \tag{2.12}$$

and k is an index achieving the minimum.

Eq. (2.10) indicates that the objection function in (2.9) is always bounded. In addition, the constraint in (2.8) or (2.9) can be easily satisfied. Therefore, problem (2.8) or problem (2.9) is well defined. Now we explore an important property for problem (2.8).

Lemma 2.2. *Every local minimizer of* (2.8) *is also a global minimizer of* (2.8).

Proof. Obviously, the global minimum value for (2.8) and (2.9) is η which is achievable. Let x be a local but not global minimizer for (2.8). From (2.9)–(2.12), we know

$$\sum_{i=1}^{n} \alpha_i^2 (\lambda_i - a)(\lambda_i - b) > \eta. \tag{2.13}$$

Therefore, there must exist an index j such that $\alpha_j \neq 0$ and $(\lambda_j - a)(\lambda_j - b) > \eta$. Then, by reducing $|\alpha_j|$ and increasing α_k while maintaining $\sum_{i=1}^n \alpha_i^2 = 1$, we can reduce $\sum_{i=1}^n \alpha_i^2 (\lambda_i - a)(\lambda_i - b)$. This contradicts with our assumption that x is a local minimizer. This completes the proof. □

Based on the result of Lemma 2.2, we can establish the following result.

Theorem 2.3. *Let x^* be a global minimizer of* (2.8), *then we have the following results.*

(i) $\eta = (x^*)^{\mathrm{T}}(A - aI_n)(A - bI_n)x^*$.
(ii) *If $\eta > 0$, then there exists no eigenvalue of A in the interval $[a, b]$.*
(iii) *If $\eta \leqslant 0$, then there exists at least one eigenvalue of A in the interval $[a, b]$. In addition, if there exists exactly one eigenvalue of A in the interval $[a, b]$, then $(x^*)^{\mathrm{T}}Ax^* = \lambda_k$ is the eigenvalue and x^* is the corresponding eigenvector.*
(iv) *If $\eta = 0$, then one of the eigenvalues of A must be either a or b.*

Proof. (i) From the proof of Lemma 2.2, we know that the global minimum value for (2.8) or (2.9) is η which is achievable at $\alpha_k = 1$ and $\alpha_i = 0$, $i = 1, \ldots, k-1, k+1, \ldots, n$.

(ii) If $\eta > 0$, from its definition in (2.12), then $(\lambda_i - a)$ and $(\lambda_i - b)$ have the same sign for all i's. Therefore, there exists no eigenvalue of A in the interval $[a, b]$.

(iii) If $\eta \leqslant 0$, from (2.12), we know there exists an index k such that

$$a \leqslant \lambda_k \leqslant b. \tag{2.14}$$

This guarantees that there exists at least one eigenvalue of A in the interval $[a, b]$. The rest of (iii) are straightforward.

(iv) If $\eta = 0$, then $(\lambda_k - a)(\lambda_k - b) = 0$. Therefore, either a or b must be an eigenvalue of A. □

From Theorem 2.3, we know that if there are more than one eigenvalue of A in the interval $[a, b]$. $(x^*)^{\mathrm{T}}Ax^*$ may not be an eigenvalue of A. This can be easily checked from $\|Ax^* - (x^*)^{\mathrm{T}}Ax^*x^*\|$. In the case that there is more than one eigenvalue in the interval $[a, b]$, we can solve

$$(\lambda_i - a)(\lambda_i - b) = \eta$$

to obtain two λ_i's. Then by checking $\|Ax^* - \lambda_i x^*\|$, the desired eigenvalue can be easily located.

Now the key point is to solve problem (2.8). Problem (2.8) is to minimize a quadratic function on the surface of a ball. The difficulty for solving problem (2.8) is that its constraint set is not a convex set. Now we further convert problem (2.8) into the following optimization problem which is much easier to solve.

$$\begin{aligned} \min_{x \in R^n} \quad & x^{\mathrm{T}}(A - aI_n)(A - bI_n)x - cx^{\mathrm{T}}x \\ \text{s.t.} \quad & x^{\mathrm{T}}x \leqslant 1, \end{aligned} \tag{2.15}$$

where $c > \kappa + 1$. From the requirement on c, we can see that $[cI_n - (A - aI_n)(A - bI_n)]$ is a positive definite matrix.

Problem (2.15) differs from problem (2.8) in that the objective function is still a quadratic function, but it is a strictly concave function. In addition, the constraints are a simple ball constraint. Therefore, it is much easier to solve (2.15) than (2.8).

Lemma 2.4

(i) *Every local minimizer of* (2.15) *is also a global minimizer of* (2.15).

(ii) x *is a global minimizer of* (2.8) $\Longleftrightarrow$ x *is a global minimizer of* (2.15).

Proof. It is easy to see that problem (2.15) is equivalent to

$$\begin{aligned} \min_{\alpha_1,\ldots,\alpha_n} \quad & \sum_{i=1}^{n} \alpha_i^2[(\lambda_i - a)(\lambda_i - b) - c] \\ \text{s.t.} \quad & \sum_{i=1}^{n} \alpha_i^2 \leqslant 1. \end{aligned} \tag{2.16}$$

Obviously, the global minimum value for (2.16) is $\eta - c$ which is achievable at $\alpha_k = 1$, $\alpha_i = 0, \forall i \neq k$.

(i) Assume that x is a local but not a global minimizer of (2.15). Since (2.15) and (2.16) are equivalent, we have

$$0 \geqslant \sum_{i=1}^{n} \alpha_i^2[(\lambda_i - a)(\lambda_i - b) - c] > \eta - c, \tag{2.17}$$

where α_i's are defined in (2.2).

From (2.17), we know that there must exist an index j such that $\alpha_j \neq 0$ and $(\lambda_j - a)(\lambda_j - b) > \eta$. (Otherwise the objective function in (2.16) would be $\sum_{i=1}^n \times \alpha_i^2(\eta - c)$. This would contradict with our assumption on x.) Thus, by reducing α_j and increasing α_k, we can maintain $\sum_{i=1}^n \alpha_i^2 \leqslant 1$ and reduce the objective function value in (2.16). This contracts with our assumption on x. Therefore every local minimizer of (2.15) must also be a global minimizer of (2.15).

(ii) Since (2.15) and (2.16) are equivalent, then we have

$$0 \geqslant \sum_{i=1}^n \alpha_i^2[(\lambda_i - a)(\lambda_i - b) - c] \geqslant (\eta - c)\sum_{i=1}^n \alpha_i^2. \tag{2.18}$$

But the global minimum value for (2.16) is $\eta - c$. Therefore from (2.18), we know that any global minimizer of (2.16) must satisfy $\sum_{i=1}^n \alpha_i^2 = 1$. Again from the equivalence of (2.8) and (2.9), we can easily see that (ii) is true. □

3. Continuous methods

Similar to our discussion in the previous section, we will consider the continuous methods for problem (2.5) (extreme eigenvalue problem) and problem (2.15) (interior eigenvalue problem) in the following two subsections.

3.1. A continuous method for extreme eigenvalue problem

Now we focus on problem (2.5). Generally speaking, a continuous method for an optimization problem consists of two components: a merit function (bounded below) and a dynamical system. In addition, the merit function must be monotonically nonincreasing along the solution of the dynamical system. Following the model developed in [9], we have our continuous method for problem (2.5):

Merit function:

$$f(x) = x^{\mathrm{T}}Ax - cx^{\mathrm{T}}x. \tag{3.1}$$

Dynamical system:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -[x - P_\Omega(x - \nabla f(x))], \tag{3.2}$$

where $\Omega = \{x \in R^n \mid x^{\mathrm{T}}x \leqslant 1\}$ and $P_\Omega(\cdot)$ is the projection onto Ω defined as

$$P_\Omega(y) = \operatorname*{argmin}_{x\in\Omega} \|x - y\|_2, \quad \forall y \in R^n.$$

To simplify the following discussion, we define

$$e(x) = x - P_\Omega(x - \nabla f(x)). \tag{3.3}$$

First, let us reveal an important property for $e(x)$.

Lemma 3.1. *$e(x) = 0$ with $x \neq 0 \iff x$ is an eigenvector of A with $\|x\| = 1$.*

Proof. "$\Leftarrow$" This is straightforward. "$\Rightarrow$" $e(x) = 0$ implies

$$x = \begin{cases} [x - \nabla f(x)]/\|x - \nabla f(x)\|, & \text{if } \|x - \nabla f(x)\| > 1, \\ x - \nabla f(x), & \text{if } \|x - \nabla f(x)\| \leqslant 1. \end{cases} \tag{3.4}$$

If $\|x - \nabla f(x)\| \leqslant 1$, from (3.4), we have

$$\nabla f(x) = 2(A - cI_n)x = 0. \tag{3.5}$$

But $(cI_n - A)$ is a positive definite matrix, (3.5) implies $x = 0$ which contradicts with $x \neq 0$. Therefore, it must be true that $\|x - \nabla f(x)\| > 1$. Let $\gamma = \|x - \nabla f(x)\| - 1 > 0$, then from (3.4), we have

$$(A - cI_n)x = -\frac{\gamma}{2}x. \tag{3.6}$$

Eq. (3.6) indicates that x is an eigenvector of A. From (3.4), it is easy to see that $\|x\| = 1$ if $\|x - \nabla f(x)\| > 1$. This completes our proof. □

Now we are ready to analyze the convergence properties for the solution of (3.2). These results will be summarized in the following theorems.

Theorem 3.2. *For any $x_0 \in R^n$, there exists a unique solution $x(t)$ of the dynamical system (3.2) with $x(t = t_0) = x_0$ in $[t_0, +\infty)$.*

Proof. Since the right-hand-side of (3.2) is continuous in R^n, the Cauchy–Peano theorem ensures that there exists a solution $x(t)$ of the dynamical system (3.2) with $x(t = t_0) = x_0$. For this solution $x(t)$, we define

$$E(x(t)) = \|x(t) - P_\Omega(x(t))\|^2. \tag{3.7}$$

Obviously, $E(x(t))$ is the square of the distance of $x(t)$ to set Ω. Then we have from (3.2) and (3.3) that

$$\frac{\mathrm{d}E(x(t))}{\mathrm{d}t} = \begin{cases} -2\left(1 - \frac{1}{\|x\|}\right) x^{\mathrm{T}} e(x), & \text{if } \|x\| > 1, \\ 0, & \text{if } \|x\| \leqslant 1. \end{cases} \tag{3.8}$$

From (3.3), we have

$$x^{\mathrm{T}} e(x) = \begin{cases} x^{\mathrm{T}} x - \frac{(2c+1)x^{\mathrm{T}}x - 2x^{\mathrm{T}}Ax}{\|(2c+1)x - 2Ax\|}, & \text{if } \|(2c+1)x - 2Ax\| > 1, \\ 2x^{\mathrm{T}}Ax - 2cx^{\mathrm{T}}x, & \text{if } \|(2c+1)x - 2Ax\| \leqslant 1. \end{cases} \tag{3.9}$$

From the requirement on c in (2.3), we have

$$\|(2c+1)x - 2Ax\| \geqslant \|2cx - 2Ax\| - \|x\| \geqslant \|2x\| - \|x\| = \|x\|. \tag{3.10}$$

Eqs. (3.9) and (3.10) indicate

$$x^{\mathrm{T}}e(x) = x^{\mathrm{T}}x - \frac{(2c+1)x^{\mathrm{T}}x - 2x^{\mathrm{T}}Ax}{\|(2c+1)x - 2Ax\|} > 0 \quad \text{if } \|x\| > 1. \tag{3.11}$$

Eqs. (3.8) and (3.11) indicate that $E(x(t))$ is monotonically nonincreasing in t. Therefore, we have

$$\|e(x)\| \leqslant \|x - P_\Omega(x)\| + \|P_\Omega(x) - P_\Omega(x - \nabla f(x))\| \leqslant \|x(t_0)\| + 3. \tag{3.12}$$

Eq. (3.12) indicates that the right-hand-side of (3.2) is bounded for any given x_0. Again the Cauchy–Peano theorem ensures that the solution $x(t)$ exists in $[t_0, +\infty)$.

Since Ω is a closed convex set, from the nonexpansive property of the projection operator, we have

$$\|P_\Omega(u) - P_\Omega(v)\| \leqslant \|u - v\|, \quad \forall u, v \in R^n.$$

Therefore,

$$\begin{aligned}\|e(x) - e(y)\| &= \|x - P_\Omega(x - \nabla f(x)) - y + P_\Omega(y - \nabla f(y))\| \\ &\leqslant \|x - y\| + \|P_\Omega(x - \nabla f(x)) - P_\Omega(y - \nabla f(y))\| \\ &\leqslant \|x - y\| + \|x - y\| + \|\nabla f(x) - \nabla f(y)\| \\ &\leqslant (2 + 2\|A\| + 2c)\|x - y\|, \quad \forall x, y \in R^n. \end{aligned} \tag{3.13}$$

Eq. (3.13) implies that $e(x)$ in (3.3) is Lipschitz continuous in R^n. From the Picard–Lindelöf theorem, the proof is completed. □

The result of Theorem 3.2 indicates that our dynamical system (3.2) is well defined. In the proof of Theorem 3.2, we can see that if $x_0 \notin \Omega$, then the solution $x(t)$ of (3.2) will move towards the feasible region, and if $x_0 \in \Omega$, then the solution $x(t)$ of (3.2) will stay in Ω from then on. Before we prove the convergence of the solution of (3.2), we need to observe the following properties. First, from (2.2), we can define

$$x(t) = \sum_{i=1}^{n} \alpha_i(t)u_i = U\alpha(t), \tag{3.14}$$

where $x(t)$ is the solution of (3.2), u_i's are the column vectors of U defined in (2.1), and $\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))^{\mathrm{T}}$. Then from (3.1), we have

$$g(\alpha) \equiv f(x) = \sum_{i=1}^{n} \alpha_i^2(t)(\lambda_i - c). \tag{3.15}$$

Therefore, it is straightforward to see that (3.2) is equivalent to

$$\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} = -[\alpha - P_\Omega(\alpha - \nabla g(\alpha))]. \tag{3.16}$$

In order to prove the convergence of the ODE solution $x(t)$, we need the following lemma.

Lemma 3.3. *Suppose scalar function $h(t)$ is differentiable on $[t_0, T]$ with $h(t_0) = 0$. If there exists an $M > 0$ such that $\left|\frac{\mathrm{d}h}{\mathrm{d}t}\right| \leqslant M|h(t)|$, $t \in [t_0, T]$, then $h(t) = 0$, $t \in [t_0, T]$.*

Proof. See Exercise 26, p. 119, [10]. □

It is easy to see that T can be extended to $+\infty$. Now we prove the following important convergence results for the solution of (3.2).

Theorem 3.4. *For any $x_0 \in \Omega$, let $x(t)$ be the solution of* (3.2) *with $x(t = t_0) = x_0$. Then* (i) *if $e(x_0) = 0$, $x(t) \equiv x_0$, $\forall t \geqslant t_0$;* (ii) *if $e(x_0) \neq 0$, then $\lim_{t\to+\infty} e(x(t)) = 0$.*

Proof. (i) From (3.16), we have for $i = 1, \ldots, n$,

$$\frac{\mathrm{d}\alpha_i(t)}{\mathrm{d}t} = \begin{cases} -\alpha_i(t) + \frac{(2c+1-2\lambda_i)\alpha_i(t)}{\|\alpha(t)-\nabla g(\alpha)\|}, & \text{if } \|\alpha(t) - \nabla g(\alpha)\| > 1, \\ 2(c - \lambda_i)\alpha_i(t), & \text{if } \|\alpha(t) - \nabla g(\alpha)\| \leqslant 1. \end{cases} \tag{3.17}$$

In both cases, we have for $i = 1, \ldots, n$,

$$\left|\frac{\mathrm{d}\alpha_i(t)}{\mathrm{d}t}\right| \leqslant 2(c - \lambda_1 + 1)|\alpha_i(t)|, \quad \forall t \geqslant t_0. \tag{3.18}$$

If $e(x_0) = 0$, then from Lemma 3.1, we know either $x_0 = 0$ or x_0 is an eigenvector of A with $\|x_0\| = 1$. If $x_0 = 0$, from (3.14) we have $\alpha(t_0) = 0$. Therefore, (3.18) and Lemma 3.3 imply $\alpha(t) \equiv 0$, $\forall t \geqslant t_0$. Thus, $x(t) \equiv 0$, $\forall t \geqslant t_0$. If x_0 is an eigenvector of A with $\|x_0\| = 1$, let

$$Ax_0 = \lambda_k x_0.$$

Then from (3.14), $\alpha_i(t_0) = 0$ if $\lambda_i \neq \lambda_k$. Then from (3.18) and Lemma 3.3, we have

$$\alpha_i(t) \equiv 0, \quad \forall t \geqslant t_0, \quad \forall i \text{ with } \lambda_i \neq \lambda_k.$$

Therefore, $\|\alpha(t) - \nabla g(\alpha)\| = (2c + 1 - 2\lambda_k)\|\alpha(t)\|$. Since $\|\alpha(t_0)\| = 1$, $2c + 1 - 2\lambda_k > 1$, and $\alpha(t) \in \Omega\ \forall t$, (3.17) indicates that $\exists$ a $\bar{t} > t_0$ such that

$$\frac{\mathrm{d}\alpha_j(t)}{\mathrm{d}t} \equiv 0, \quad \forall t \in [t_0, \bar{t}], \quad \forall j \text{ with } \lambda_j = \lambda_k.$$

Therefore, $\alpha(t) \equiv \alpha(t_0)$ and $\|\alpha(t)\| = 1$, $\forall t \in [t_0, \bar{t}]$. This process can be repeated until $\bar{t} \to +\infty$. Therefore, $\alpha(t) \equiv \alpha(t_0)$, $\forall t \geqslant t_0$. Thus, $x(t) \equiv x_0$, $\forall t \geqslant t_0$.

(ii) Since $x_0 \in \Omega$, Theorem 3.2 ensures that $x(t) \in \Omega$, $\forall t \geqslant t_0$.

Since Ω is a closed convex set, from inequality (4) in [5], we have

$$[y - P_\Omega(y)]^{\mathrm{T}}[x - P_\Omega(y)] \leqslant 0, \quad \forall x \in \Omega, \quad \forall y \in R^n. \tag{3.19}$$

Taking $y = x - \nabla f(x)$ in (3.19), we have

$$[e(x) - \nabla f(x)]^{\mathrm{T}} e(x) \leqslant 0. \tag{3.20}$$

From (3.1)–(3.3) and (3.20), we have

$$\frac{\mathrm{d} f(x)}{\mathrm{d}t} = -[\nabla f(x)]^{\mathrm{T}} e(x) \leqslant -\|e(x)\|^2 \leqslant 0, \quad \forall t \geqslant t_0. \tag{3.21}$$

From the LaSalle invariant set theorem (Theorem 3.4 in [11]) and (3.21), we know

$$\lim_{t \to +\infty} e(x(t)) = 0.$$

This completes the proof. □

Theorem 3.4 is a little bit short of proving the convergence of $x(t)$ which is much more desirable. This result is summarized in the following theorem.

Theorem 3.5. *For any $x_0 \in \Omega$, let $x(t)$ be the solution of* (3.2) *with $x(t = t_0) = x_0$. Then $x(t)$ is convergent, i.e. there exists an $x^* \in \Omega$ such that* $\lim_{t \to +\infty} x(t) = x^*$. *In addition, if $x_0 \neq 0$, $\lim_{t \to +\infty} x(t)^{\mathrm{T}} A x(t) = \lambda_k$, where $k = \min\left\{i \mid x_0^{\mathrm{T}} u_i \neq 0, i = 1, \ldots, n\right\}$.*

Proof. Obviously, if $x_0 = 0$, then $x(t) \equiv 0$, $\forall t \geqslant t_0$ from the proof of Theorem 3.4. Therefore $x(t)$ is convergent. So we assume $x_0 \neq 0$ in the rest of proof.

From the proof of Theorem 3.2, we know $x(t) \in \Omega$ for all t's. The boundedness of Ω implies that there exists at least one limit point for $\{x(t)\}$. Let $\bar{x}$ be any limit point of $\{x(t)\}$. Then $e(\bar{x}) = 0$ from (ii) of Theorem 3.4 and $\bar{x} \neq 0$ from $x_0 \neq 0$ and (3.21). From Lemma 3.1, we know that $\bar{x}$ is an eigenvector of A with $\|\bar{x}\| = 1$. Therefore, we have $\lim_{t \to \infty} \|x(t)\| = 1$.

From $x \in \Omega \iff \alpha \in \Omega$ and $\|x(t)\| = \|\alpha(t)\|$, there exists a $t^* > t_0$ such that if $t > t^*$, $\|\alpha(t)\| > \frac{1}{2}$. Thus if $t > t^*$, we have

$$\|\alpha(t) - \nabla g(\alpha)\|^2 = \sum_{i=1}^{n} \alpha_i^2(t)[1 + 2(c - \lambda_i)]^2 \geqslant 9\|\alpha(t)\|^2 > 1.$$

This and (3.14)–(3.16) imply that for any $i = 1, \ldots, n$, if $t > t^*$, then

$$\begin{aligned} \frac{\mathrm{d}\alpha_i(t)}{\mathrm{d}t} &= -\alpha_i(t) + \frac{\alpha_i(t) - 2\alpha_i(t)(\lambda_i - c)}{\|\alpha(t) - \nabla g(\alpha)\|} \\ &= \frac{\alpha_i(t)[1 + 2(c - \lambda_i) - \|\alpha(t) - \nabla g(\alpha(t))\|]}{\|\alpha(t) - \nabla g(\alpha(t))\|}. \end{aligned} \tag{3.22}$$

From $x \in \Omega \iff \alpha \in \Omega$, we have

$$\|\alpha(t) - \nabla g(\alpha(t))\|^2 = \sum_{i=1}^{n} \alpha_i^2(t)[1 + 2(c - \lambda_i)]^2$$

$$\leqslant \sum_{i=1}^{n} \alpha_i^2(t)[1 + 2(c - \lambda_1)]^2$$
$$\leqslant [1 + 2(c - \lambda_1)]^2. \tag{3.23}$$

Obviously, from (3.16), we have

$$\frac{\mathrm{d}\alpha_i(t)}{\mathrm{d}t} = 2\alpha_i(c - \lambda_i), \quad \text{if } \|\alpha(t) - \nabla g(\alpha)\| \leqslant 1. \tag{3.24}$$

Eqs. (3.22)–(3.24) indicate that there exists an $M > 0$ such that

$$\left|\frac{d\alpha_i(t)}{dt}\right| \leqslant M|\alpha_i(t)|, \quad \forall t \geqslant t_0. \tag{3.25}$$

From the definition of k and Lemma 3.3, we know $x_i(t) = \alpha_i(t) = 0, i = 1, \ldots, k - 1, \forall t \geqslant t_0$. Therefore, (3.23) becomes

$$\|\alpha(t) - \nabla g(\alpha(t))\|^2 = \sum_{i=k}^{n} \alpha_i^2(t)[1 + 2(c - \lambda_i)]^2 \leqslant [1 + 2(c - \lambda_k)]^2. \tag{3.26}$$

On the other hand, (3.22), (3.26) and our assumption indicate that if $t > t^*$, we have

$$\frac{\mathrm{d}\alpha_k(t)}{\mathrm{d}t} \begin{cases} \geqslant 0, & \text{if } \alpha_k(t) > 0, \\ \leqslant 0, & \text{if } \alpha_k(t) < 0. \end{cases} \tag{3.27}$$

Eq. (3.27) is very important. Basically, it tells that when $t > t^*$

- if $\alpha_k(t_0) > 0$, $\alpha_k(t)$ will be monotonically nondecreasing in t but always stays in the interval $[\alpha_k(t_0), 1]$;
- if $\alpha_k(t_0) < 0$, $\alpha_k(t)$ will be monotonically nonincreasing in t but always stays in the interval $[-1, \alpha_k(t_0)]$.

Therefore, $\lim_{t\to+\infty} \alpha_k(t)$ exists and is nonzero since $\alpha_k(t_0) \neq 0$. Let $\alpha_k^* = \lim_{t\to+\infty} \alpha_k(t) \neq 0$. Similarly we can prove that $\alpha_i(t)$ is convergent if $\lambda_i = \lambda_k$, for any $i \geqslant k + 1$. If $\lambda_n = \lambda_k$, then our proof is finished. Otherwise, we prove

$$\lim_{t\to+\infty} \alpha_i(t) = 0, \quad i = j, \ldots, n, \tag{3.28}$$

where $j = \min\{i \mid \lambda_k < \lambda_i, i = k + 1, \ldots, n\}$.

Since j and n are finite and $\|\alpha(t)\| \leqslant 1$, then there exists a sequence of t_l with $t_l \to +\infty$ as $l \to +\infty$ such that $\lim_{l\to+\infty} \alpha_i(t_l)$, $i = j, \ldots, n$ exist. Let $\alpha_i^* = \lim_{l\to+\infty} \alpha_i(t_l), i = j, \ldots, n$.

Now we show $\alpha_i^* = 0, i = j, \ldots, n$. Suppose not, then there must exist an $\alpha_i^* \neq 0$ for some $i = j, \ldots, n$, without loss of generality, let $\alpha_j^* \neq 0$. Then $\alpha(t_l)$ and $x(t_l)$ are convergent as $l \to +\infty$. But from our earlier discussion, the limit of $\{x(t_l)\}$, say x^* is an eigenvector of A with $\|x^*\| = 1$. Let λ be the corresponding eigenvalue. Then we have

$$x^* = \sum_{i=1}^{n} \alpha_i^* u_i \quad \text{and} \quad Ax^* = \lambda x^*, \tag{3.29}$$

where $\alpha_k^* \neq 0$ and $\alpha_j^* \neq 0$.

From (3.29), we have

$$\sum_{i=1}^{n} \alpha_i^* \lambda_i u_i = \sum_{i=1}^{n} \alpha_i^* \lambda u_i.$$

This implies $\lambda = \lambda_k = \lambda_j$ which contradicts with $\lambda_k < \lambda_j$. Therefore, (3.28) holds. Thus $x(t)$ is convergent as $t \to +\infty$ and $\lim_{t\to+\infty} x(t)^{\mathrm{T}} Ax(t) = \lambda_k$. $\square$

It is worth of mentioning that if $x^* \neq 0$, from Lemma 3.1 and Theorem 3.4, we know that the x^* obtained in Theorem 3.5 is an eigenvector of A. Another important observation is that if $[x(t) - \nabla f(x(t))] \in \Omega$, the dynamical system (3.2) would become

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -2(A - cI)x.$$

In this case, the solution $x(t)$ can be viewed as a continuous variant of the power method whose convergence is well established.

Even though we have proved that for any starting point, the ODE solution would converge to an eigenvector of the matrix, yet this eigenvector will not correspond to the minimum eigenvalue if the projection of the initial point in the eigenspace corresponding to the minimum eigenvalue is zero. In other words, we can't say that for any starting point, the limit of the ODE solution is the eigenvector corresponding to the minimum eigenvalue. From the optimality conditions for problem (2.5) and Lemma 3.1, we know that the followings are equivalent:

- $e(x) = 0$ but $x \neq 0$.
- x is an eigenvector of A with $\|x\|_2 = 1$.
- x satisfies the first-order necessary conditions for problem (2.5).

The last result indicates that it would be quite difficult to move away from x if x is an eigenvector of A corresponding to some λ_i with $\lambda_i > \lambda_1$. In this case, one remedy is to move away from x along a direction $d \neq 0$ satisfying $x^{\mathrm{T}} d = 0$. Then, we can re-solve the dynamical system (3.2) with this new starting point.

3.2. *A continuous method for interior eigenvalue problem*

Now we focus on problem (2.15). Following the same procedure as the previous subsection, we have our continuous method for problem (2.15):

Merit function:

$$f(x) = x^{\mathrm{T}}(A - aI_n)(A - bI_n)x - cx^{\mathrm{T}}x. \tag{3.30}$$

Dynamical system:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -[x - P_\Omega(x - \nabla f(x))], \tag{3.31}$$

where $\Omega = \{x \in R^n \mid x^{\mathrm{T}}x \leqslant 1\}$ and $P_\Omega(\cdot)$ is the projection onto Ω. To simplify the following discussion, we define

$$e(x) = x - P_\Omega(x - \nabla f(x)). \tag{3.32}$$

Since our merit function (3.30) and dynamical system (3.31) are almost identical to the corresponding ones in the previous subsection, we can easily establish the following results.

Lemma 3.6. *$e(x) = 0$ with $x \neq 0 \iff x$ is an eigenvector of $(A - aI_n)(A - bI_n)$ with $\|x\| = 1$.*

Proof. See the proof of Lemma 3.1. □

Theorem 3.7. *For any $x_0 \in R^n$, there exists a unique solution $x(t)$ of the dynamical system (3.31) with $x(t = t_0) = x_0$ in $[t_0, +\infty)$.*

Proof. See the proof of Theorem 3.2. □

Theorem 3.8. *For any $x_0 \in \Omega$, let $x(t)$ be the solution of (3.31) with $x(t = t_0) = x_0$. Then (i) if $e(x_0) = 0$, $x(t) \equiv x_0$, $\forall t \geqslant t_0$; (ii) if $e(x_0) \neq 0$, then $\lim_{t\to+\infty} e(x(t)) = 0$.*

Proof. See the proof of Theorem 3.8. □

Theorem 3.9. *For any $x_0 \in \Omega$, let $x(t)$ be the solution of (3.31) with $x(t = t_0) = x_0$. Then $x(t)$ is convergent, i.e. there exists an $x^* \in \Omega$ such that $\lim_{t\to+\infty} x(t) = x^*$. In addition, if $x_0 \neq 0$, $\lim_{t\to+\infty} x(t)^{\mathrm{T}}(A - aI_n)(A - bI_n)x(t) = \mu_l$, where μ_i, $i = 1, \ldots, n$ are eigenvalues of $(A - aI_n)(A - bI_n)$ in the increasing order and $l = \min\left\{i \mid x_0^{\mathrm{T}}u_i \neq 0,\ i = 1, \ldots, n\right\}$.*

Proof. See the proof of Theorem 3.9. □

From Theorems 3.8, 3.9, and Lemma 3.6, we know

$$(x^*)^{\mathrm{T}}(A - aI_n)(A - bI_n)x^* = \mu_l \quad \text{and} \quad \|x^*\| = 1. \tag{3.33}$$

From (2.2), we let α^* be the corresponding vector to x^*. Then (3.33) is equivalent to

$$\sum_{i=1}^{n}(\alpha_i^*)^2(\lambda_i - a)(\lambda_i - b) = \mu_l \quad \text{and} \quad \|\alpha^*\| = 1. \tag{3.34}$$

But the proof of Theorem 3.5 indicates that

$$\alpha_i^* = 0, \quad \forall i \in \left\{ j \mid (\lambda_j - a)(\lambda_j - b) \neq \mu_l,\ j = 1, \ldots, n \right\} \quad \text{and} \quad \alpha_l \neq 0. \tag{3.35}$$

Eq. (3.34) and (3.35) imply that

$$(\lambda_l - a)(\lambda_l - b) = \mu_l. \tag{3.36}$$

If $\mu_l > 0$, then the λ_l defined in (3.36) is not in the interval $[a, b]$. If $\mu_l \leqslant 0$, there are two λ_l's available from (3.36). By checking $\|Ax^* - \lambda_l x^*\|$, we can determine which λ_l is an eigenvalue of A in the desired interval. We summarize the computational steps in the following.

Computational steps

(i) For given x_0, obtain x^* and u_l as defined in Theorem 3.9. If $u_l > 0$, then there is no eigenvalue of A in the interval $[a, b]$. Otherwise, go to (ii).
(ii) Find at most two λ_l's from (3.36).
(iii) If one of $\|Ax^* - \lambda_l x^*\|$'s is very small, stop. An eigenvalue of A in $[a, b]$ is found. Otherwise, a new starting point has to be selected.

Our final note of this section is on the selection of c in (2.15). From (2.11), we have

$$\kappa \leqslant \max_{1 \leqslant i \leqslant n} (|\lambda_i| + |a|)(|\lambda_i| + |b|) \leqslant (\|A\|_1 + |a|)(\|A\|_1 + |b|).$$

Therefore, we can choose

$$c = (\|A\|_1 + |a|)(\|A\|_1 + |b|) + 1.$$

We will adopt this formula for c as the default value in our numerical simulation.

4. Numerical results

In this section, we test our continuous methods on two examples. Since the focus of this paper is to introduce the new continuous method, therefore, we will not compare the numerical results of our methods with the existing ones. However, we will perform many tests to explore various properties of our continuous methods. Our simulation will stop whenever the following condition is satisfied:

$$\|e(x(t))\|_\infty \leqslant \delta,$$

where δ is a preset value. We use $\delta = 10^{-6}$ in all our test. All of our tests are run in Matlab platform on a PC with 2 Intel Xeon Processors at 2.8 GHz. But only one CPU is used in all runs. The ODE solver used is **ODE45** which is a nonstiff medium order method. We set **RelTol** $= 10^{-6}$ and **AbsTol** $= 10^{-9}$ in all our runs.

Our examples are constructed in the following ways.

Example 1. We construct the example in the following steps:
1. Select $\Lambda = \text{diag}(-1e-4, -1e-4, 0, 0, 1, \ldots, 1) \in R^{n\times n}$.
2. Let $B = \text{rand}(n, n)$ and $[Q, R] = qr(B)$.
3. Define $A = Q^{\mathrm{T}}\Lambda Q$.

Example 2. This example is similar to Example 1 except $\Lambda = \text{diag}(-1, -1, 0, 0, 1, \ldots, 1) \in R^{n\times n}$. It is worth of mentioning that the eigenvalues of A in both examples are clustered into 3 groups. In Example 1, the minimum eigenvalue -10^{-4} is relatively close to the nearby eigenvalue 0. While in this example, the minimum eigenvalue -1 is not close to the nearby eigenvalue 0. The time of constructing the matrix A is not included in the following CPU times.

The two starting points used are $x_0 = (1, \ldots, 1)^{\mathrm{T}}$ and $-x_0$.

4.1. *Extreme eigenvalue model*

Our numerical tests are aimed at the following three targets.

4.1.1. *Target one: sensitivity to the initial point*

In this group of tests, we fix $n = 5000$ and c as defined in Section 3 (default values). In fact, $c = 5.32$ for Example 1 and $c = 8.04$ for Example 2. Our results are summarized in Table 1.

The results in Table 1 indicate that (i) our continuous method for extreme eigenvalue problems is not very sensitive to the initial point; and (ii) for any starting point x, it is normally more attractive to use $P_\Omega(x)$ as the initial starting point than x. Therefore, in the remaining numerical tests, we will adopt this policy.

4.1.2. *Target two: sensitivity to c*

In this group of tests, we fix $n = 5000$ and compare the effect of c on the convergence. The default values of c (as defined in Section 3) are 5.32 and 8.04 for Examples 1 and 2, respectively.

Table 1
Numerical results for extreme eigenvalue problems—I

	Example 1		Example 2	
	CPU (s)	$\lambda^{a} + 10^{-4}$	CPU (s)	$\lambda^{a} + 1$
x_0	469.1	3.0×10^{-5}	475.6	-6.5×10^{-6}
$P_\Omega(x_0)$	298.6	3.0×10^{-5}	306.3	-5.8×10^{-6}
$-x_0$	469.0	3.0×10^{-5}	474.9	-6.5×10^{-6}
$P_\Omega(-x_0)$	301.3	3.0×10^{-5}	305.6	-5.8×10^{-6}

[a] $\lambda = (x^*)^{\mathrm{T}}Ax^*$ is the computed eigenvalue.

Table 2
Numerical results for extreme eigenvalue problems—II

		Example 1		Example 2	
		CPU (s)	$\lambda^{a} + 10^{-4}$	CPU (s)	$\lambda^{a} + 1$
$P_{\Omega}(x_0)$	$c = 10$	423.5	3.0×10^{-5}	361.8	3.8×10^{-7}
	$c =$ default	298.6	3.0×10^{-5}	306.3	-5.8×10^{-6}
	$c = 2$	253.5	3.0×10^{-5}	258.3	-4.0×10^{-9}
$P_{\Omega}(-x_0)$	$c = 10$	422.4	3.0×10^{-5}	362.0	3.8×10^{-7}
	$c =$ default	301.3	3.0×10^{-5}	305.6	-5.8×10^{-6}
	$c = 2$	252.6	3.0×10^{-5}	258.7	-4.0×10^{-9}

[a] $\lambda = (x^*)^{\mathrm{T}} A x^*$ is the computed eigenvalue.

The results in Table 2 clearly demonstrate that the smaller value of c, the faster convergence. Even the default value always works in computation, yet it would be more attractive to use a smaller value.

4.1.3. *Target three: computational cost*

In this group of tests, we fix the value of c at 2 (Table 3) and default value (Table 4) to see the change of CPU times versus the problem size n.

It should be mentioned that the slow convergence in Table 3 for Example 1 with $n \leqslant 2500$ is entirely due to the ODE solver (oscillation was observed). It is beyond the scope of this paper to investigate the proper ODE solver and/or the corresponding tolerance values.

It is very interesting to observe from Tables 3 and 4 that excluding those slow convergence cases, the CPU time grows at a rate of $n^{2+\epsilon}$ where $\epsilon > 0$. In addition, this rate seems to be independent of the choice of c value.

Table 3
Numerical results for extreme eigenvalue problems—III

$c = 2$		Example 1		Example 2	
		CPU (s)	$\lambda^{a} + 10^{-4}$	CPU (s)	$\lambda^{a} + 1$
$n = 1000$	$P_{\Omega}(x_0)$	13,253	7.4×10^{-6}	12.4	-1.8×10^{-9}
	$P_{\Omega}(-x_0)$	13,290	7.4×10^{-6}	12.3	-1.8×10^{-9}
$n = 2500$	$P_{\Omega}(x_0)$	10,694	1.7×10^{-5}	68.4	1.7×10^{-9}
	$P_{\Omega}(-x_0)$	10,729	1.7×10^{-5}	69.3	1.7×10^{-9}
$n = 5000$	$P_{\Omega}(x_0)$	253.5	3.0×10^{-5}	258.3	-4.0×10^{-9}
	$P_{\Omega}(-x_0)$	252.6	3.0×10^{-5}	258.7	-4.0×10^{-9}
$n = 7500$	$P_{\Omega}(x_0)$	857.5	7.1×10^{-5}	951.5	6.7×10^{-9}
	$P_{\Omega}(-x_0)$	861.8	7.1×10^{-5}	953.8	6.7×10^{-9}

[a] $\lambda = (x^*)^{\mathrm{T}} A x^*$ is the computed eigenvalue.

Table 4
Numerical results for extreme eigenvalue problems—IV

c = default		Example 1		Example 2	
		CPU (s)	$\lambda^{a} + 10^{-4}$	CPU (s)	$\lambda^{a} + 1$
$n = 1000$	$P_\Omega(x_0)$	13.8	7.3×10^{-5}	15.6	-4.2×10^{-6}
	$P_\Omega(-x_0)$	13.6	7.3×10^{-5}	15.5	-4.2×10^{-6}
$n = 2500$	$P_\Omega(x_0)$	85.3	3.1×10^{-5}	88.6	1.1×10^{-6}
	$P_\Omega(-x_0)$	85.6	3.1×10^{-5}	88.9	1.1×10^{-6}
$n = 5000$	$P_\Omega(x_0)$	298.6	3.0×10^{-5}	306.3	-5.8×10^{-6}
	$P_\Omega(-x_0)$	301.3	3.0×10^{-5}	305.6	-5.8×10^{-6}
$n = 7500$	$P_\Omega(x_0)$	1127	7.1×10^{-5}	1208	-4.9×10^{-6}
	$P_\Omega(-x_0)$	1136	7.1×10^{-5}	1209	-4.9×10^{-6}

[a] $\lambda = (x^*)^{\mathrm{T}} A x^*$ is the computed eigenvalue.

4.2. Interior eigenvalue model

Since our continuous model for interior eigenvalue problems is mainly based on the continuous model for extreme eigenvalue problems. To avoid the repeated tests, our numerical experiment will focus on the following three targets with $n = 5000$ in all tests.

4.2.1. Target one: no eigenvalue in the defined interval

We select $[a, b] = [-3 \times 10^{-4}, -2 \times 10^{-4}]$. Our numerical results are summarized in Table 5.

Table 5
Numerical results for interior eigenvalue problems—I

	$P_\Omega(x_0)$		$P_\Omega(-x_0)$	
	c = default	$c = 2$	c = default	$c = 2$
Example 1				
CPU (s)	280.6	105.2	279.5	107.7
$\lambda = (x^*)^{\mathrm{T}} A x^*$	-6.9×10^{-5}	-7.0×10^{-5}	-6.9×10^{-5}	-7.0×10^{-5}
$\\|Ax^* - \lambda x^*\\|_\infty$	2.0×10^{-5}	4.0×10^{-6}	2.0×10^{-5}	4.0×10^{-6}
μ_l	1.4×10^{-6}	5.1×10^{-8}	1.4×10^{-6}	5.1×10^{-8}
Example 2				
CPU (s)	538.3	111.0	534.4	106.0
$\lambda = (x^*)^{\mathrm{T}} A x^*$	1.2×10^{-5}	1.9×10^{-8}	1.2×10^{-5}	1.9×10^{-8}
$\\|Ax^* - \lambda x^*\\|_\infty$	5.9×10^{-5}	2.4×10^{-6}	5.9×10^{-5}	2.4×10^{-6}
μ_l	1.2×10^{-5}	7.9×10^{-8}	1.2×10^{-5}	7.9×10^{-8}

Table 6
Numerical results for interior eigenvalue problems—II

	$P_\Omega(x_0)$		$P_\Omega(-x_0)$	
	$c =$ default	$c = 2$	$c =$ default	$c = 2$
Example 1				
CPU (s)	85.5	45.6	105.6	30.6
$\lambda = (x^*)^{\mathrm{T}}Ax^*$	$1 - 1 \times 10^{-6}$	$1 + 4\times10^{-9}$	$1 - 1 \times 10^{-6}$	$1 + 4\times10^{-9}$
$\|Ax^* - \lambda x^*\|_\infty$	2.9×10^{-5}	2.4×10^{-6}	2.9×10^{-5}	2.4×10^{-6}
μ_l	-0.01	-0.01	-0.01	-0.01
Example 2				
CPU (s)	147.0	65.7	142.4	66.3
$\lambda = (x^*)^{\mathrm{T}}Ax^*$	$1 - 2 \times 10^{-6}$	$1 + 6 \times 10^{-8}$	$1 - 2 \times 10^{-6}$	$1 + 6 \times 10^{-8}$
$\|Ax^* - \lambda x^*\|_\infty$	6.4×10^{-5}	2.3×10^{-6}	6.4×10^{-5}	2.3×10^{-6}
μ_l	-0.01	-0.01	-0.01	-0.01

Since for every case in Table 5, $u_l > 0$, our results in Section 3.2 guarantee that there is no eigenvalue of A in the defined interval $[a, b]$. However since some μ_l's are very small, to be safe, we need to check if a or b is an eigenvalue of A. This case can be ruled out by checking if a or b is an eigenvalue.

4.2.2. *Target two: one eigenvalue in the defined interval*

We select $[a, b] = [0.9, 1.1]$. Our numerical results are summarized in Table 6.

From the results in Table 6, we can see that $u_l < 0$ in all cases. The theory in Section 3.2 ensures that there exists at least one eigenvalue of A in the interval $[a, b]$. The computed eigenvalues are all very close to the true one $\lambda^* = 1$ in all cases.

Table 7
Numerical results for interior eigenvalue problems—III

	$P_\Omega(x_0)$		$P_\Omega(-x_0)$	
	$c =$ default	$c = 2$	$c =$ default	$c = 2$
Example 1				
CPU (s)	135.9	37.3	133.7	39.5
$\lambda = (x^*)^{\mathrm{T}}Ax^*$	$1 - 1 \times 10^{-6}$	$1 + 1 \times 10^{-8}$	$1 - 1 \times 10^{-6}$	$1 + 1 \times 10^{-8}$
$\|Ax^* - \lambda x^*\|_\infty$	2.9×10^{-5}	2.7×10^{-6}	2.9×10^{-5}	2.7×10^{-5}
μ_l	-1	-1	-1	-1
Example 2				
CPU (s)	192.3	63.0	195.0	65.7
$\lambda = (x^*)^{\mathrm{T}}Ax^*$	$1 - 2 \times 10^{-6}$	$1 + 2 \times 10^{-8}$	$1 - 2 \times 10^{-6}$	$1 + 2 \times 10^{-8}$
$\|Ax^* - \lambda x^*\|_\infty$	6.4×10^{-5}	3.1×10^{-6}	6.4×10^{-5}	3.1×10^{-6}
μ_l	-1	-1	-1	-1

4.2.3. Target three: two eigenvalues in the defined interval

We select $[a, b] = [0, 2]$. Our numerical results are summarized in Table 7.

Once again, since $\mu_l < 0$ in all cases, there exists at least one eigenvalue of A in the interval $[a, b]$. From Table 7, we can see that the computed eigenvalues are very close to the true one $\lambda^* = 1$ in all cases.

5. Conclusions

In this paper, two new continuous methods are proposed for symmetric eigenvalue problems, one for extreme eigenvalue and one for interior eigenvalue problems. Our approach is different from the existing ones in that a continuous path (or trajectory) of the targeted eigenvalue is achieved. This is represented by a dynamical system (or ODE) for each eigenvalue problem. Strong convergence results of our two continuous methods are obtained. Our simulation results clearly indicate that our new methods are very effective and attractive.

Currently, the authors are applying the same approach for other linear algebra problems. Since there are many issues need to be solved and clarified, in addition, the convergence speed of our continuous methods is pretty much dependent on the ODE solver used, at this moment, the investigation on the continuous method is still not enough for a side-by-side comparison with the existing numerical linear algebra methods. The authors will continue to explore the continuous method before a full comparison is conducted.

Our final remark is on dynamical systems (3.2) and (3.31) since the success of our methods relies on the solutions of these systems. Notice that both (3.2) and (3.31) are autonomous systems and their right-hand-sides are relatively simple. Therefore, it is anticipated that by using some matrix-free ODE solvers, (3.2) and (3.31) could be solved for large-scale systems.

Acknowledgements

The authors would like to thank Dr. Gao Xingbao for many fruitful discussions. The authors are also very grateful for the constructive comments and suggestions from Guest Editor Dr. Henk van der Vorst and three anonymous referees.

References

[1] M.T. Chu, On the continuous realization of iterative processes, SIAM Rev. 30 (1988) 375–387.
[2] M.T. Chu, Matrix differential equations: a continuous realization process for linear algebra problems, Nonlinear Anal. Theory Methods Appl. 18 (1992) 1125–1146.
[3] A. Cichocki, R. Unbehauen, Neural networks for computing eigenvalues and eigenvectors, Biolog. Cybernet. 68 (1992) 155–164.

[4] G.H. Golub, C.F. Van Loan, Matrix Computation, third ed., The John Hopkins University Press, 1996.
[5] B.S. He, Inexact implicit methods for monotone general variational inequalities, Math. Prog. 86 (1999) 199–217.
[6] J.J. Hopfield, Neural networks and physical systems with emergent collective computational ability, Proc. Natl. Acad. Sci. 79 (1982) 2554–2558.
[7] J.J. Hopfield, Neurons with graded response have collective computational properties like those of two-state neurons, Proc. Natl. Acad. Sci. 81 (1984) 3088–3092.
[8] J.J. Hopfield, D.W. Tank, Neural computation of decisions in optimization problems, Biolog. Cybernet. 52 (1985) 141–152.
[9] L.-Z. Liao, H.D. Qi, L.Q. Qi, Neurodynamical optimization, J. Global Optim. 28 (2004) 175–195.
[10] W. Rudin, Principles of Mathematical Analysis, third ed., McGraw-Hill Book Company, 1976.
[11] J.-J.E. Slotine, W. Li, Applied Nonlinear Control, Prentice Hall, New Jersey, 1991.
[12] H.A. van der Vorst, G.H. Golub, 150 years old and still alive: eigenproblems, in: I.S. Duff (Ed.), The State of the Art in Numerical Analysis, 1997, pp. 93–119.