Note

Constrained partitioning problems

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Abstract

Burkard and Yao studied matroid-constrained optimal partitioning problems and developed an algorithm for determining optimal partitions. Crucial to their development is the property that pairwise consecutiveness implies consecutiveness, and they observed that this implication holds under the assumption that no loops are present in the underlying matroids. Here, we provide an example that demonstrates that the implication need not hold in general. We also present an alternative sufficient condition under which the implication is valid regardless of the presence of loops.

The usual partitioning problem concerns the partitioning of a given set of linearly ordered elements into subsets with the goal of minimizing some cost function. Throughout we denote the set that is to be partitioned by \( E \), its cardinality by \( n \), and the number of sets in a potential partition by \( p \). A partition is called consecutive (or ordered) if every subset consists of elements that are consecutive in the linear order. If the cost function guarantees the existence of a consecutive optimal partition, then dynamic programming can be used to find an optimal partition in \( O(n^2) \) time (see [4]), or in \( O(pn^2) \) time (see [3]).

Burkard and Yao [1] extended the general partitioning problem to cases where the potential partitions are restricted by allowing only certain subsets of the underlying

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(linearly ordered) set \( E \) in a partition. In particular, they considered the case where a partition \( \{S_1, S_2, \ldots, S_p\} \) is feasible only if for each \( i = 1, 2, \ldots, m \), the set \( S_i \) is a basis of a given matroid \( M_i \) which is defined on the set \( E \). To emphasize the dependence of such a partition on the underlying matroid we sometimes refer to it as an \((M_1, \ldots, M_p)\)-constrained partition. Also, the problem of determining an optimal partition of this type will be called a matroid-constrained partitioning problem.

Consider a matroid-constrained partitioning problem with given matroids \( M_1, \ldots, M_p \) and let \( \{S_1, S_2, \ldots, S_p\} \) be an \((M_1, \ldots, M_p)\)-constrained partition. We say that \( x \in S_i \) and \( y \in S_j \) are interchangeable for \( S_i \) and \( S_j \) if \( (S_i \setminus \{x\}) \cup \{y\} \) and \( (S_j \setminus \{y\}) \cup \{x\} \) are also bases of \( M_i \) and \( M_j \), respectively, and in this case we say that \( x \) and \( y \) form an interchangeable pair for \( S_i \) and \( S_j \). We say that such an interchangeable pair is in favor of \( j \) over \( i \) if \( x < y \). We say that \( S_i \) and \( S_j \) are consecutive if there do not exist two interchangeable pairs, one in favor of \( j \) over \( i \) and the other in favor of \( i \) over \( j \). We say that \( \{S_1, S_2, \ldots, S_p\} \) is pairwise consecutive if any two of its sets are consecutive. We say that \( \{S_1, S_2, \ldots, S_p\} \) is consecutive if there is a permutation \( \pi \) of \( \{1, \ldots, p\} \) such that \( \pi(i) < \pi(j) \) implies that there does not exist an interchangeable pair for \( S_i \) and \( S_j \) which favors \( i \) over \( j \). Of course, every consecutive partition is pairwise consecutive.

Burkard and Yao [1] study matroid-constrained partitioning problems and establish a framework for proving the existence of consecutive optimal partitions. In particular, they develop an algorithm which determines optimal partitions in such cases by consecutively interchanging pairs of elements. Crucial to their development is the property that pairwise consecutiveness implies consecutiveness. The following example demonstrates that this implication need not hold in general.

**Example 1.** Consider a constrained partitioning problem with underlying set \( E \equiv \{a, b, c, d, e, f\} \) and linear order \( < \) for which \( a < b < c < d < e < f \). Let \( M_1, M_2 \) and \( M_3 \) be the matroids that define the constraints of the partitions where the bases of \( M_1 \) are \( \{\{a, b\}, \{a, c\}, \{d, b\}, \{d, e\}\} \), the bases of \( M_2 \) are \( \{\{b, c\}, \{b, f\}, \{e, c\}, \{e, f\}\} \) and the bases of \( M_3 \) are \( \{\{c, a\}, \{c, d\}, \{f, a\}, \{f, d\}\} \). Then the partition \( \{S_1, S_2, S_3\} \) with \( S_1 = \{a, e\}, S_2 = \{b, f\} \) and \( S_3 = \{c, d\} \) is feasible. We next argue that it is pairwise consecutive. To verify this fact, note that \( S_1 \) and \( S_2 \) do not have an interchangeable pair in favor of \( S_2 \). Similarly, \( S_2 \) and \( S_3 \) do not have an interchangeable pair which is in favor of \( S_3 \), and \( S_1 \) and \( S_1 \) do not have an interchangeable pair which is in favor of \( S_1 \). So, indeed, the partition \( \{S_1, S_2, S_3\} \) is pairwise consecutive. We next observe that \( e \in S_1 \) and \( b \in S_2 \) form an interchangeable pair for \( S_1 \) and \( S_2 \) which is in favor of \( S_1 \), \( f \in S_2 \) and \( c \in S_3 \) form an interchangeable pair for \( S_2 \) and \( S_3 \) which is in favor of \( S_2 \), and \( d \in S_3 \) and \( a \in S_1 \) form an interchangeable pair for \( S_3 \) and \( S_1 \) which is in favor of \( S_3 \). So, for every \( i \in \{1, 2, 3\} \) there is an index \( j \) and an interchangeable pair for \( S_i \) and \( S_j \) which favors \( i \) over \( j \). Consequently, \( \{S_1, S_2, S_3\} \) is not consecutive.

Burkard and Yao [2] suggest a simple way of assuring that pairwise consecutiveness implies consecutiveness by requiring that the matroids do not contain loops. The verification of this condition requires a thorough search and analysis of the underlying matroids. We next describe an alternative condition under which pairwise consecutiveness implies consecutiveness. The condition concerns the underlying partition and allows for the existence of loops in the given matroids.
Consider a matroid-constrained partitioning problem over a set $E$ with linear order $<$ and underlying matroids $M_1, \ldots, M_p$, and let $\{S_1, \ldots, S_p\}$ be an $(M_1, \ldots, M_p)$-constrained partition of $E$. We say that the subset $F$ of $E$ is complete for $\{S_1, \ldots, S_p\}$ if each set $S_i$ contains exactly one element of $F$, and if each $f_i \in F \cap S_i$ and $f_j \in F \cap S_j$ are interchangeable for $S_i$ and $S_j$, where $i, j \in \{1, \ldots, p\}$.

**Theorem 2.** Suppose $\{S_1, \ldots, S_p\}$ is an $(M_1, \ldots, M_p)$-constrained partition of $E$ which is pairwise consecutive and has a complete subset $F$. Then $\{S_1, \ldots, S_p\}$ is consecutive.

**Proof.** Let $F$ be a subset of $E$ which is complete for $\{S_1, \ldots, S_p\}$. Then $F$ contains $p$ elements that can be enumerated $f_1, \ldots, f_p$ where $f_1 < \cdots < f_p$. Consider the permutation $\pi$ on $\{1, \ldots, p\}$ where for $i = 1, \ldots, p$, $\pi(i)$ is the index of the element of $F$ which is contained in $S_i$, i.e., $f_{\pi(i)} \in S_i$. As $F$ is complete for $\{S_1, \ldots, S_p\}$ we have that $f_{\pi(i)} \in F \cap S_i$ and $f_{\pi(j)} \in F \cap S_j$ form an interchangeable pair for $S_i$ and $S_j$ further, if $\pi(i) < \pi(j)$ then this interchangeable pair favors $j$ over $i$. We conclude from the assumption that $\{S_1, \ldots, S_p\}$ is pairwise consecutive, that if $i, j \in \{1, \ldots, p\}$ with $\pi(i) < \pi(j)$, then there is no interchangeable pair for $S_i$ and $S_j$ which favors $i$ over $j$. Thus, we have that the partition $\{S_1, \ldots, S_p\}$ is, indeed, consecutive. \qed

Consider a matroid-constrained partitioning problem over a set $E$ with linear order $<$ and underlying matroids $M_1, \ldots, M_p$. Let $F \equiv \{f_1, \ldots, f_p\}$ be a set that is disjoint from $E$ and extend the linear order $<$ from $E$ to $E' \equiv E \cup F$ by having $e < f_1 < \cdots < f_p$ for every $e \in E$. For $i = 1, \ldots, p$, let $M'_i$ be the matroid on $E'$ whose bases have the form $B \cup \{f\}$ for every basis $B$ of $M_i$ and $f \in F$.

**Corollary 3.** Every $(M'_1, \ldots, M'_p)$-constrained partition of $E'$ which is pairwise consecutive is consecutive.

**Proof.** The structure of the matroids $M'_1, \ldots, M'_p$ assures that the set $F$ is complete for every $(M'_1, \ldots, M'_p)$-constrained partition of $E'$. Hence, Theorem 2 implies if such a partition is pairwise consecutive then it is consecutive. \qed

We say that an $(M_1, \ldots, M_p)$-constrained partition $\{S_1, \ldots, S_p\}$ of $E$ has a pairwise consecutive extension if there exists a pairwise consecutive $(M'_1, \ldots, M'_p)$-constrained partition $\{S'_1, \ldots, S'_p\}$ of $E'$ with $S'_i$ containing $S_i$ for each $i = 1, \ldots, p$.

**Corollary 4.** Let $\{S_1, \ldots, S_p\}$ be an $(M_1, \ldots, M_p)$-constrained partition of $E$ which has a pairwise consecutive extension. Then $\{S_1, \ldots, S_p\}$ is consecutive.

**Proof.** Let $\{S'_1, \ldots, S'_p\}$ be a pairwise consecutive $(M'_1, \ldots, M'_p)$-constrained partition of $E'$ where $S'_i$ contains $S_i$ for each $i = 1, \ldots, p$. By Corollary 3, $\{S'_1, \ldots, S'_p\}$ is consecutive. It immediately follows that so is the $(M_1, \ldots, M_p)$-partition of $E$ given by $\{S_1 = S'_1 \cap E, \ldots, S_p = S'_p \cap E\}$. \qed
References