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On symmetries and cohomological invariants of equations possessing flat representations

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Abstract

We study the equation \mathcal{E}_{fc} of flat connections in a given fiber bundle and discover a specific geometric structure on it, which we call a *flat representation*. We generalize this notion to arbitrary PDE and prove that flat representations of an equation \mathcal{E} are in 1–1 correspondence with morphisms $\varphi : \mathcal{E} \rightarrow \mathcal{E}_{fc}$, \mathcal{E} and \mathcal{E}_{fc} being treated as submanifolds of infinite jet spaces. We show that flat representations include several known types of zero-curvature formulations of PDEs. In particular, the Lax pairs of the self-dual Yang–Mills equations and their reductions are of this type. With each flat representation φ we associate a complex C_φ of vector-valued differential forms such that $H^1(C_\varphi)$ describes infinitesimal deformations of the flat structure, which are responsible, in particular, for parameters in Bäcklund transformations. In addition, each higher infinitesimal symmetry S of \mathcal{E} defines a 1-cocycle c_S of C_φ . Symmetries with exact c_S form a subalgebra reflecting some geometric properties of \mathcal{E} and φ . We show that the complex corresponding to \mathcal{E}_{fc} itself is 0-acyclic and 1-acyclic (independently of the bundle topology), which means that higher symmetries of \mathcal{E}_{fc} are exhausted by generalized gauge ones, and compute the bracket on 0-cochains induced by commutation of symmetries.

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Introduction

Nonlinear PDE in more than two independent variables with nontrivial invariance properties is a sort of challenge for those who are interested in geometry of differential equations and related topics. Evidently, chosen “at random”, such an equation will be of no interest, but *natural* (whatever it means) equations arising in physics and geometry may provide the researcher with the desired results. Thinking of the second source (geometry), a first idea is to consider equations describing geometrical structures (e.g., integrable distributions, complex structures, etc.) and, by definition, possessing a rich symmetry group. One of the simplest equations of this type is the equation \mathcal{E}_{fc} describing flat connections in a certain locally trivial fiber bundle.

Let $\pi : E \rightarrow M$, $\dim M = n$, $\dim E = n + m$, be a smooth locally trivial fiber bundle over a smooth manifold M . A *connection* ∇ in the bundle π is a $C^\infty(M)$ -linear correspondence that takes any vector field X on M to a vector field ∇X on E in such a way that $\pi_*(\nabla X)_\theta = X_{\pi(\theta)}$ for any point $\theta \in E$. A connection is called *flat* if

$$\nabla[X, Y] = [\nabla X, \nabla Y]$$

for any two vector fields X and Y on M .

Let $\mathcal{U} \subset M$ be a local chart, such that π becomes trivial over \mathcal{U} , with local coordinates $x_1, \dots, x_n, v^1, \dots, v^m$ in $\pi^{-1}(\mathcal{U})$, x_1, \dots, x_n being coordinates in \mathcal{U} . Then ∇ is determined by nm arbitrary functions $v_i^\alpha = v_i^\alpha(x_1, \dots, x_n, v^1, \dots, v^m)$ by the relations

$$\nabla(\partial x_i) = \partial x_i + \sum_{\alpha=1}^m v_i^\alpha \partial v^\alpha, \quad i = 1, \dots, n,$$

where and below $\partial x_i = \partial/\partial x_i$, $\partial v^\alpha = \partial/\partial v^\alpha$, etc., while the condition of flatness is expressed in the form

$$[\nabla(\partial x_i), \nabla(\partial x_j)] \equiv \sum_{\alpha=1}^m \left(\frac{\partial v_j^\alpha}{\partial x_i} + \sum_{\beta=1}^m v_i^\beta \frac{\partial v_j^\alpha}{\partial v^\beta} \right) \partial v^\alpha - \sum_{\alpha=1}^m \left(\frac{\partial v_i^\alpha}{\partial x_j} + \sum_{\beta=1}^m v_j^\beta \frac{\partial v_i^\alpha}{\partial v^\beta} \right) \partial v^\alpha = 0,$$

$i, j = 1, \dots, n$. Thus, flat connections are distinguished by the following system of nonlinear equations

$$\frac{\partial v_j^\alpha}{\partial x_i} + \sum_{\beta=1}^m v_i^\beta \frac{\partial v_j^\alpha}{\partial v^\beta} = \frac{\partial v_i^\alpha}{\partial x_j} + \sum_{\beta=1}^m v_j^\beta \frac{\partial v_i^\alpha}{\partial v^\beta}, \quad (1)$$

$1 \leq i < j \leq n$, $\alpha = 1, \dots, m$. This is a system of $mn(n-1)/2$ equations imposed on mn unknown functions v_i^α .

A strong motivation to study this equation is the following. If one assumes that the coefficients v_i^α of a connection are expressed in terms of some functions f_k in a special way, then the flatness is equivalent to a system \mathcal{E} of differential equations on f_k such that each solution to \mathcal{E} gives a flat connection. It is well known that many integrable nonlinear PDEs arise exactly in this way. Treating the equations \mathcal{E} and \mathcal{E}_{fc} with their differential consequences as (infinite-dimensional) submanifolds of the corresponding infinite jet spaces, we obtain a smooth map

$$\varphi : \mathcal{E} \rightarrow \mathcal{E}_{fc}, \quad (2)$$

which preserves the Cartan distribution. Such a map generates on \mathcal{E} an additional geometric structure which we call a *flat representation*. In particular, the identical map $\text{id}: \mathcal{E}_{\text{fc}} \rightarrow \mathcal{E}_{\text{fc}}$ determines a canonical flat representation for \mathcal{E}_{fc} itself.

Thus \mathcal{E}_{fc} is a sort of universal space where integrable equations live (or at least a broad class of them). In particular, we show that coverings (generalized Lax pairs) of equations in two independent variables as well as the Lax pairs of the self-dual Yang–Mills equations and their reductions lead to flat representations. Therefore, studying the geometry of \mathcal{E}_{fc} may help to obtain new knowledge on integrable systems, and in this paper we begin the study. Despite the fact that locally any flat connection can be obtained from a trivial one by means of a local automorphism of the bundle, there is no natural way to parameterize all flat connections, and this makes the geometry of \mathcal{E}_{fc} nontrivial.

We show that each flat representation (2) leads to several complexes of vector-valued differential forms on \mathcal{E} constructed by means of the Nijenhuis bracket. One subcomplex denoted by C_φ is of particular interest, since its 1-cocycles are infinitesimal deformations of the flat representation, the exact cocycles being trivial deformations. As usual, higher cohomology contains obstructions to prolongation of infinitesimal deformations to formal deformations [4]. In particular, zero-curvature representations and Bäcklund transformations dependent on a parameter provide an example of a deformed flat representation and determine, therefore, some 1-cohomology classes of C_φ .

In addition, each higher infinitesimal symmetry S of \mathcal{E} defines a formal deformation and, therefore, a 1-cocycle c_S in C_φ . Symmetries with exact cocycles c_S form a Lie subalgebra with respect to the commutator. If morphism (2) is an embedding then it is precisely the subalgebra of those symmetries which are restrictions of some symmetries of the equation of flat connections. If (2) is coming from a covering of \mathcal{E} then this subalgebra consists of symmetries which can be lifted to the covering equation.

The complex C_{id} corresponding to the identical flat representation $\text{id}: \mathcal{E}_{\text{fc}} \rightarrow \mathcal{E}_{\text{fc}}$ itself is a new canonical complex associated with the bundle $\pi: E \rightarrow M$. Symmetries of \mathcal{E}_{fc} are in one-to-one correspondence with 1-cocycles of this complex.

We prove that

$$H^0(C_{\text{id}}) = H^1(C_{\text{id}}) = 0 \quad (3)$$

independently of the topology of M and E . This implies that the higher symmetries of \mathcal{E}_{fc} are in one-to-one correspondence with 0-cochains, which are sections of some vector bundle over \mathcal{E}_{fc} of rank m . That is, symmetries are determined locally by m -tuples of smooth functions on \mathcal{E}_{fc} . This is similar to the case of an infinite jet space. Due to their coordinate expression, it is natural to call them *generalized gauge symmetries*. The commutator of symmetries induces a bracket on 0-cochains, and we write down this bracket in explicit terms.

Note that only for a few multidimensional equations the complete description of symmetries is known. We mention the paper [2], which proves that symmetries of the vacuum Einstein equations in four spacetime dimensions are also freely determined by 5-tuples of smooth functions on the equation.

The paper is organized as follows. In Section 1, we review some facts on the geometry of PDE and the Nijenhuis bracket. In Section 2, we introduce the equation of flat connections and complexes of vector-valued forms on it. We prove (3) and deduce the description of symmetries of \mathcal{E}_{fc} in Section 3. In Section 4, we introduce flat representations in general and prove the above facts on their deformations and relations with symmetries. The bracket on 0-cochains of C_{id} is computed in Section 4.4. Finally, in Section 5, we show that coverings of PDE and the Lax pair of the self-dual Yang–Mills equations lead to flat representations φ and study the map from symmetries to 1-cocycles of C_φ for these examples.

1. Basic facts

This is a brief review of the geometry of PDE. We refer to [3,9] for more details.

1.1. Notation agreements

Let M be a smooth manifold and π be a smooth fiber bundle over M . Throughout the paper we use the following notation:

- $D(M)$ denotes the $C^\infty(M)$ -module (and Lie algebra over \mathbb{R}) of vector fields on M . A *distribution* on M is a $C^\infty(M)$ -submodule of $D(M)$.
- More generally, if P is an arbitrary $C^\infty(M)$ -module, then $D(P)$ denotes the module of P -valued derivations:

$$D(P) = \{X \in \text{Hom}_{\mathbb{R}}(C^\infty(M), P) \mid X(fg) = fX(g) + gX(f), f, g \in C^\infty(M)\}.$$

- $\Lambda^i(M)$ is the module of i -differential forms on M while

$$\Lambda^*(M) = \bigoplus_{i=1}^{\dim M} \Lambda^i(M)$$

is the corresponding algebra (with respect to the wedge product \wedge).

- $\Gamma(\pi)$ (resp., $\Gamma_{\text{loc}}(\pi)$) is the set of all (resp., all local) sections of π . In particular, when π is a vector bundle, $\Gamma(\pi)$ is a $C^\infty(M)$ -module.

1.2. Jets and equations

Let $\pi : E \rightarrow M$ be a fiber bundle and $\theta \in E$, $\pi(\theta) = x \in M$. Consider a local section $f \in \Gamma_{\text{loc}}(\pi)$ whose graph passes through the point θ and the class $[f]_x^k$ of all local sections whose graphs are tangent to the graph of f at θ with order $\geq k$. The set

$$J^k(\pi) = \{[f]_x^k \mid f \in \Gamma_{\text{loc}}(\pi), x \in M\}$$

carries a natural structure of a smooth manifold and is called the *manifold of k -jets* of the bundle π . Moreover, the natural projections

$$\pi_k : J^k(\pi) \rightarrow M, \quad [f]_x^k \mapsto x, \quad \pi_{k,k-1} : J^k(\pi) \rightarrow J^{k-1}(\pi), \quad [f]_x^k \mapsto [f]_x^{k-1},$$

are locally trivial bundles. If π is a vector bundle, then π_k is a vector bundle as well while all $\pi_{k,k-1}$ carry a natural affine structure. So, we have an infinite sequence of bundles

$$\dots \rightarrow J^k(\pi) \xrightarrow{\pi_{k,k-1}} J^{k-1}(\pi) \rightarrow \dots \rightarrow J^1(\pi) \xrightarrow{\pi_1} J^0(\pi) = E \xrightarrow{\pi} M$$

and its inverse limit is called the *manifold of infinite jets* and denoted by $J^\infty(\pi)$. In an obvious way, one can consider the fiber bundles $\pi_\infty : J^\infty(\pi) \rightarrow M$ and $\pi_{\infty,k} : J^\infty(\pi) \rightarrow J^k(\pi)$. Using the notation similar to the above one, we can state that $J^\infty(\pi)$ consists of classes of the form $[f]_x^\infty$.

Let $f \in \Gamma_{\text{loc}}(\pi)$; then one can consider the sections $j_k(f) \in \Gamma_{\text{loc}}(\pi_k)$, $k = 0, 1, \dots, \infty$, defined by $j_k(f) : x \mapsto [f]_x^k$ and called the k -jet of f . Consider a point $\theta = [f]_x^\infty \in J^\infty(\pi)$ and the tangent plane

to the graph of $j_\infty(f)$ at θ . One can show that this plane does not depend on the choice of f and is determined by the point θ only. Denote this plane by \mathcal{C}_θ .

Let $X \in D(M)$. Then by the above construction there exists a unique vector field $\mathcal{C}X \in D(J^\infty(\pi))$ such that $\mathcal{C}X_\theta \in \mathcal{C}_\theta$ and $(\pi_\infty)_*\mathcal{C}X_\theta = X_x$ for any $\theta \in J^\infty(\pi)$ and $x = \pi_\infty(\theta)$. Thus we obtain a connection \mathcal{C} in the bundle π_∞ called the *Cartan connection*. The distribution $\mathcal{C}D \subset D(J^\infty(\pi))$ generated by the fields of the form $\mathcal{C}X$, $X \in D(M)$, is called the *Cartan distribution*.

Proposition 1. *The Cartan distribution is integrable in the formal Frobenius sense, i.e., $[\mathcal{C}D, \mathcal{C}D] \subset \mathcal{C}D$. The Cartan connection is flat, i.e., $[\mathcal{C}X, \mathcal{C}Y] = \mathcal{C}[X, Y]$ for any $X, Y \in D(M)$.*

Remark 1. Consider a point $\theta_k \in J^k(\pi)$, $k > 0$. Then it is completely determined by the point $\theta_{k-1} = \pi_{k,k-1}(\theta_k) = [f]_x^{k-1}$, $f \in \Gamma_{\text{loc}}(\pi)$, and the tangent plane to the graph of $j_{k-1}f$ at θ_{k-1} . Consequently, sections of the bundle $\pi_{k,k-1}: J^k(\pi) \rightarrow J^{k-1}(\pi)$ are identified with horizontal n -dimensional distributions on $J^{k-1}(\pi)$. In particular, sections of $\pi_{1,0}: J^1(\pi) \rightarrow E$ are just connections in the bundle π .

A k th order differential equation imposed on sections of the bundle π is a submanifold $\mathcal{E}^0 \subset J^k(\pi)$ of the manifold $J^k(\pi)$. Any equation may be represented in the form

$$\mathcal{E}^0 = \{ \theta \in J^k(\pi) \mid (\Delta_{\mathcal{E}})_\theta = 0, \Delta_{\mathcal{E}} \in \Gamma(\pi_k^*(\xi)) \},$$

where $\xi: E' \rightarrow M$ is a bundle and $\pi_k^*(\xi)$ is its pullback. The section $\Delta_{\mathcal{E}} = \Delta$ can be understood as a *nonlinear differential operator* acting from $\Gamma(\pi)$ to $\Gamma(\xi)$: $\Delta(f) = j_k(f)^*(\Delta_{\mathcal{E}}) \in \Gamma(\xi)$, $f \in \Gamma(\pi)$. Of course, neither ξ , nor Δ is unique.

The l th prolongation of \mathcal{E}^0 is the set

$$\mathcal{E}^l = \{ [f]_x^{k+l} \mid \text{the graph of } j_k(f) \text{ is tangent to } \mathcal{E}^0 \text{ with order } \geq l \text{ at } [f]_x^k \in \mathcal{E} \},$$

$l = 0, 1, \dots, \infty$. Restricting the maps π_{k+l} and $\pi_{k+l,k+l-1}$ to \mathcal{E}^l and preserving the same notation for these restrictions, we obtain the sequence of maps

$$\dots \rightarrow \mathcal{E}^l \xrightarrow{\pi_{k+l,k+l-1}} \mathcal{E}^{l-1} \rightarrow \dots \rightarrow \mathcal{E}^0 \xrightarrow{\pi_{k,0}} E \xrightarrow{\pi} M.$$

Without loss of generality, one can always assume π_k to be surjective. Imposing natural conditions of regularity, we shall also assume that all \mathcal{E}^l are smooth manifolds, while the maps $\pi_{k+l,k+l-1}: \mathcal{E}^l \rightarrow \mathcal{E}^{l-1}$ are smooth fiber bundles (such equations are called *formally integrable*). Everywhere below the *infinite prolongation* \mathcal{E}^∞ will be denoted by \mathcal{E} .

It can be shown that any vector field of the form $\mathcal{C}X$, $X \in D(M)$, is tangent to \mathcal{E} . This means that the Cartan connection restricts from the bundle $J^\infty(\pi) \rightarrow M$ to $\mathcal{E} \rightarrow M$ and determines on \mathcal{E} a horizontal n -dimensional integrable distribution, the *Cartan distribution* on \mathcal{E} . We denote by $\mathcal{C}D_{\mathcal{E}} \subset D(\mathcal{E})$ the submodule generated by the vector fields lying in the Cartan distribution on \mathcal{E} . Its maximal integral manifolds are the infinite jets of solutions of \mathcal{E} . A (*higher*) *symmetry* of \mathcal{E} is a π -vertical vector field on \mathcal{E} preserving the Cartan distribution, i.e., $S \in D(\mathcal{E})$ is a symmetry if and only if

- (a) $S(f) = 0$ for any function $f \in C^\infty(M) \subset C^\infty(\mathcal{E})$;
- (b) $[S, Y]$ lies in $\mathcal{C}D_{\mathcal{E}}$ whenever $Y \in D(\mathcal{E})$ lies in $\mathcal{C}D_{\mathcal{E}}$ (in fact, from (a) it follows that $[S, \mathcal{C}X] = 0$ for any $X \in D(M)$).

The set $\text{sym } \mathcal{E}$ of all symmetries forms a Lie \mathbb{R} -algebra with respect to the commutation of vector fields. In the trivial case $\mathcal{E} = J^\infty(\pi)$ we denote this algebra by $\text{sym } \pi$. A complete description of symmetries is given by the following

Theorem 1. *Let $\pi : E \rightarrow M$ be a locally trivial fiber bundle. Then there exists a one-to-one correspondence between $\text{sym } \pi$ and sections of the pullback $\pi_{\infty,0}^*(\pi^v)$, where $\pi^v : T^v E \rightarrow E$ is the vertical tangent bundle to E . If $\mathcal{E}^0 \subset J^k(\pi)$ is an equation given by a differential operator $\Delta : \Gamma(\pi) \rightarrow \Gamma(\xi)$ and satisfying the above formulated conditions, then $\text{sym } \mathcal{E}$ is in one-to-one correspondence with solutions of the equation*

$$\ell_{\mathcal{E}}(\varphi) = 0, \quad (4)$$

where φ is a section of the pullback $\pi_{\infty,0}^*(\pi^v)$ restricted to \mathcal{E} while $\ell_{\mathcal{E}}$ is the linearization of Δ also restricted to \mathcal{E} .

Remark 2. We call φ the *generating section* of a symmetry. It may be considered as an element of the module $\Gamma(\pi^v) \otimes_{C^\infty(E)} C^\infty(J^\infty(\pi))$. When π is a vector bundle, this module is isomorphic to $\Gamma(\pi_\infty^*(\pi)) = \Gamma(\pi) \otimes_{C^\infty(M)} C^\infty(J^\infty(\pi))$.

1.2.1. Coordinates

Let $\mathcal{U} \subset M$ be a local chart with coordinates x_1, \dots, x_n and $\mathcal{U} \times \mathbb{R}^m$ be a trivialization of π over \mathcal{U} with coordinates v^1, \dots, v^m in \mathbb{R}^m . Then in $\pi_\infty^{-1}(\mathcal{U})$ the so-called *adapted coordinates* arise defined by

$$v_\sigma^\alpha([\!f\!]_x^l) = \frac{\partial^l f^\alpha}{\partial x_{i_1} \dots \partial x_{i_l}} \Big|_x, \quad f \in \Gamma_{\text{loc}}(\pi), \quad x \in \mathcal{U}, \quad \alpha = 1, \dots, m, \quad (5)$$

where $\sigma = i_1 \dots i_l$ is a multi-index, $1 \leq i_\alpha \leq n$. If $|\sigma| = l \leq k$, then v_σ^α together with x_1, \dots, x_n constitute a local coordinate system in $\pi_k^{-1}(\mathcal{U}) \subset J^k(\pi)$.

In these coordinates, the Cartan connection is completely determined by its values on partial derivatives and the corresponding vector fields are the so-called *total derivatives*

$$\mathcal{C}(\partial x_i) \equiv D_{x_i} = \partial x_i + \sum_{\sigma, \alpha} v_{\sigma i}^\alpha \partial v_\sigma^\alpha, \quad (6)$$

$i = 1, \dots, n$. The Cartan distribution is given by the system of *Cartan forms*

$$\omega_\sigma^\alpha = dv_\sigma^\alpha - \sum_{i=1}^n v_{\sigma i}^\alpha dx_i, \quad (7)$$

while the correspondence between $\Gamma(\pi_\infty^*(\pi))$ and $\text{sym } \pi$ (see Theorem 1 and Remark 2) is expressed by the formula

$$\varphi = (\varphi^1, \dots, \varphi^m) \mapsto \mathfrak{D}_\varphi = \sum_{\sigma, \alpha} D_\sigma(\varphi^\alpha) \partial v_\sigma^\alpha, \quad (8)$$

where $D_\sigma = D_{x_{i_1}} \circ \dots \circ D_{x_{i_k}}$ and \mathfrak{D}_φ is called the *evolutionary vector field* with the *generating section* φ . Finally, if $\Delta : \Gamma(\pi) \rightarrow \Gamma(\xi)$ is locally given by the relations

$$u^s = F^s(x_1, \dots, x_n, \dots, v_\sigma^\alpha, \dots), \quad s = 1, \dots, m',$$

where $m' = \dim \xi$ and u^s are coordinates in the fibers of the bundle ξ over \mathcal{U} , then

$$\ell_\Delta = \left\| \sum_\sigma \frac{\partial F^s}{\partial v_\sigma^\alpha} D_\sigma \right\|, \quad \alpha = 1, \dots, m, \quad s = 1, \dots, m', \tag{9}$$

is its linearization. To get the expression for $\ell_\mathcal{E}$ it suffices to rewrite (9) in *internal coordinates* on \mathcal{E} .

1.3. Nijenhuis bracket

Let N be a smooth manifold and consider the \mathbb{Z} -graded module $D(\Lambda^*(N)) = \bigoplus_{i=0}^{\dim N} D(\Lambda^i(N))$ over $C^\infty(N)$ of $\Lambda^*(N)$ -valued derivations. First note that for any vector field $X \in D(N)$ and a derivation $\Omega : C^\infty(N) \rightarrow \Lambda^i(N)$ one can define a derivation $X \lrcorner \Omega : C^\infty(N) \rightarrow \Lambda^{i-1}(N)$ by

$$(X \lrcorner \Omega)(f) = X \lrcorner \Omega(f), \quad f \in C^\infty(N),$$

(the inner product, or contraction), where $X \lrcorner \Omega(f)$ is the usual inner product of a vector field and a differential form. Further, if $\omega \in \Lambda^j(N)$, then the inner product $\Omega \lrcorner \omega \in \Lambda^{i+j-1}(N)$ is defined by induction on $i + j$:

$$X \lrcorner (\Omega \lrcorner \omega) = (X \lrcorner \Omega) \lrcorner \omega - (-1)^i \Omega \lrcorner (X \lrcorner \omega)$$

with an obvious base. Note that $\Omega \lrcorner$ is an operation of grading $i - 1$.

Now, let us define the *Lie derivative* $L_\Omega : \Lambda^j(N) \rightarrow \Lambda^{j+i}(N)$ by

$$L_\Omega(\omega) = d(\Omega \lrcorner \omega) + (-1)^i \Omega \lrcorner (d\omega).$$

Proposition 2. *For any elements $\Omega \in D(\Lambda^i(N))$ and $\Omega' \in D(\Lambda^{i'}(N))$ there exists a uniquely defined element $[\Omega, \Omega'] \in D(\Lambda^{i+i'}(N))$ satisfying the identity*

$$[L_\Omega, L_{\Omega'}] = L_{[\Omega, \Omega']},$$

where $[L_\Omega, L_{\Omega'}]$ denotes the graded commutator.

The element $[\Omega, \Omega']$ is called the *Nijenhuis* (or *Frölicher–Nijenhuis*) bracket of Ω and Ω' (see [7]). The module $D(\Lambda^*(N))$ is a graded Lie algebra with respect to this bracket, i.e.,

$$[\Omega, \Omega'] + (-1)^{i i'} [\Omega', \Omega] = 0, \tag{10}$$

$$[\Omega, [\Omega', \Omega'']] = [[\Omega, \Omega'], \Omega''] + (-1)^{i i'} [\Omega', [\Omega, \Omega'']], \tag{11}$$

where i, i' and i'' are the degrees of Ω, Ω' and Ω'' , respectively. Note that for “decomposable” elements of $D(\Lambda^*(N))$, i.e., for elements of the form $\Omega = \omega \otimes X, \omega \in \Lambda^*(N), X \in D(N)$, acting on $f \in C^\infty(M)$ by $(\omega \otimes X)(f) = X(f)\omega$, one has

$$\begin{aligned} [\omega \otimes X, \omega' \otimes X'] &= \omega \wedge \omega' \otimes [X, X'] + \omega \wedge X(\omega') \otimes X' - X'(\omega) \wedge \omega' \otimes X \\ &\quad + (-1)^i d\omega \wedge (X \lrcorner \omega') \otimes X' + (-1)^i (X' \lrcorner \omega) \wedge d\omega' \otimes X, \end{aligned} \tag{12}$$

where $X, X' \in D(N), \omega \in \Lambda^i(N), \omega' \in \Lambda^{i'}(N)$.

Let $U \in D(\Lambda^1(N))$. Then the operator $\partial_U = [U, \cdot] : D(\Lambda^i(N)) \rightarrow D(\Lambda^{i+1}(N))$ is defined. If $[U, U] = 0$, then, due to properties (10) and (11), the operator ∂_U satisfies the condition $\partial_U \circ \partial_U = 0$ and thus we

obtain the complex

$$0 \rightarrow D(N) \xrightarrow{\partial_U} D(\Lambda^1(N)) \rightarrow \cdots \rightarrow D(\Lambda^i(N)) \xrightarrow{\partial_U} D(\Lambda^{i+1}(N)) \rightarrow \cdots \quad (13)$$

with the corresponding cohomology groups $H_U^i(N)$.

1.4. Coverings

Let $\xi: N \rightarrow M$ and $\xi': N' \rightarrow M$ be two fiber bundles endowed with flat connections ∇ and ∇' respectively. Assume that N' is fibered over N by some smooth surjection $\tau: N' \rightarrow N$ and $\xi' = \xi \circ \tau$. We say that the pair (N', ∇') covers the pair (N, ∇) (or τ is a *covering*) if for any $x \in M$, $X \in D(M)$ and $\theta' \in N'$ one has $\tau_*(\nabla' X)_{\theta'} = (\nabla X)_{\tau(\theta')}$. A covering τ is called *linear* if

1. $\tau: N' \rightarrow N$ is a vector bundle;
2. vector fields of the form $C'X$, preserve the space $\text{Lin}(\tau) \subset C^\infty(N')$, where $\text{Lin}(\tau)$ consists of smooth functions on N' linear along the fibers of τ .

If $N = \mathcal{E}$, $\xi = \pi_\infty: \mathcal{E} \rightarrow M$ and $\nabla = \mathcal{C}$ is the Cartan connection on \mathcal{E} , we speak of a *covering over* \mathcal{E} . In local coordinates, any covering structure over \mathcal{E} is determined by a set of τ -vertical vector fields X_1, \dots, X_n on N' such that the conditions

$$[D_{x_i} + X_i, D_{x_j} + X_j] = 0$$

hold for all $1 \leq i < j \leq n$, where D_{x_α} are the total derivatives given by (6). In this case, if τ is finite-dimensional then the manifold N' is locally also isomorphic to an infinitely prolonged differential equation called the *covering equation* with the Cartan connection ∇' and the total derivatives $D_{x_i} + X_i$.

A detailed discussion of the theory of coverings over nonlinear PDE can be found in [3,11].

1.5. Horizontal cohomologies

The notion of the Cartan connection can be generalized. Namely, consider an infinitely prolonged equation $\pi_\infty: \mathcal{E} \rightarrow M$ and two vector bundles ξ, ξ' over M . Let $\Delta: \Gamma(\xi) \rightarrow \Gamma(\xi')$ be a linear differential operator. Consider the pullbacks $\pi_\infty^*(\xi), \pi_\infty^*(\xi')$ and denote the corresponding modules of sections by $\Gamma(\pi, \xi), \Gamma(\pi, \xi')$, respectively. Then there exists a unique linear differential operator $\mathcal{C}\Delta: \Gamma(\pi, \xi) \rightarrow \Gamma(\pi, \xi')$ satisfying the local condition

$$j_\infty(f)^* \mathcal{C}\Delta(\varphi) = \Delta(j_\infty(f)^*(\varphi))$$

for any $f \in \Gamma_{\text{loc}}(\pi)$ such that $\theta = [f]_x^\infty \in \mathcal{E}$ and any $\varphi \in \Gamma_{\text{loc}}(\pi, \xi)$ in the neighborhood of θ .

In particular, if $\xi = \xi': \bigwedge^i T^*M \rightarrow M$ is the i th external power of the cotangent bundle, elements of the module $\Gamma(\pi, \xi) \subset \Lambda^i(\mathcal{E})$ are called *horizontal forms* while the module itself is denoted by $\Lambda_h^i(\mathcal{E})$. Taking the de Rham differential $d: \Lambda^i(M) \rightarrow \Lambda^{i+1}(M)$ and introducing the notation $\mathcal{C}d = d_h$, we obtain the complex

$$0 \rightarrow C^\infty(\mathcal{E}) \xrightarrow{d_h} \Lambda_h^1(\mathcal{E}) \rightarrow \cdots \rightarrow \Lambda_h^i(\mathcal{E}) \xrightarrow{d_h} \Lambda_h^{i+1}(\mathcal{E}) \rightarrow \cdots \rightarrow \Lambda_h^n(\mathcal{E}),$$

the *horizontal de Rham complex* of \mathcal{E} . Its cohomology at the i th term is denoted by $H_h^i(\mathcal{E})$ and called the *horizontal cohomology*. In the adapted coordinates we have

$$d_h(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{i=1}^n D_{x_i}(f) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad f \in C^\infty(\mathcal{E}). \tag{14}$$

Proposition 3. *If $\mathcal{E} = J^\infty(\pi)$, $\pi : E \rightarrow M$, then $H_h^i(\mathcal{E})$ for $i < n$ is isomorphic to the i th de Rham cohomology of the manifold E .*

Remark 3. Let τ be a linear covering over \mathcal{E} and $P = \Gamma(\tau)$. Then one can extend the horizontal differential to $d_h : \Lambda_h^i(\mathcal{E}) \otimes P \rightarrow \Lambda_h^{i+1}(\mathcal{E}) \otimes P$ and thus obtain horizontal cohomologies with coefficients in P .

For more details on horizontal cohomology see [10].

2. The equation of flat connections

2.1. More on connections

Let $\pi : E \rightarrow M$ be a fiber bundle. Let us consider the $C^\infty(E)$ -module $D(M, E)$ of $C^\infty(E)$ -valued derivations $C^\infty(M) \rightarrow C^\infty(E)$. Due to the embedding $\pi^* : C^\infty(M) \rightarrow C^\infty(E)$, for any vector field $X \in D(E)$ one can consider its restriction $X_M \in D(M, E)$ to M . A *connection* in the bundle π is a $C^\infty(M)$ -linear map $\nabla : D(M) \rightarrow D(E)$ such that $\nabla(X)_M = X$ for any $X \in D(M)$.

Remark 4. If M is finite-dimensional, this definition becomes equivalent to the following one (see [8]): a connection in a bundle π is a $C^\infty(E)$ -linear map $\nabla : D(M, E) \rightarrow D(E)$ such that $\nabla(X)_M = X$ for any $X \in D(M, E)$. In particular, we use this fact to define the connection form below.

Consider the element $\bar{U}_\nabla \in D(\Lambda^1(E))$ defined by

$$X \lrcorner \bar{U}_\nabla(f) = (\nabla X_M)(f),$$

where $X \in D(E)$, $f \in C^\infty(E)$. The element $U_\nabla \in D(\Lambda^1(E))$ defined by

$$X \lrcorner U_\nabla(f) = X(f) - (\nabla X_M)(f)$$

is called the *connection form* of ∇ . Obviously, $\bar{U}_\nabla + U_\nabla = d$, where $d : C^\infty(E) \rightarrow \Lambda^1(E)$ is the de Rham differential.

Recall that a connection ∇ is called *flat*, if $[\nabla X, \nabla Y] = \nabla[X, Y]$ for any $X, Y \in C^\infty(M)$.

Proposition 4. *For any $\Omega \in D(\Lambda^*(E))$ one has $[[U_\nabla, \Omega]] = -[[\bar{U}_\nabla, \Omega]]$, and ∇ is flat if and only if*

$$[[\bar{U}_\nabla, \bar{U}_\nabla]] = [[U_\nabla, U_\nabla]] = 0.$$

From Proposition 4 and Section 1.3 it follows that to any flat connection ∇ we can associate the complex

$$0 \rightarrow D(E) \xrightarrow{\partial_{\nabla}} D(\Lambda^1(E)) \rightarrow \dots \rightarrow D(\Lambda^i(E)) \xrightarrow{\partial_{\nabla}} D(\Lambda^{i+1}(E)) \rightarrow \dots, \quad (15)$$

where $\partial_{\nabla} = \llbracket \bar{U}_{\nabla}, \cdot \rrbracket = -\llbracket U_{\nabla}, \cdot \rrbracket$, cf. (13), and the corresponding cohomology groups $H_{\nabla}^i(E)$.

Remark 5. In any fiber bundle $\pi : E \rightarrow M$ one can consider π -vertical vector fields defined by

$$D^v(E) = \{X \in D(E) \mid X_M = 0\}.$$

Dually, horizontal q -forms can be introduced as

$$\Lambda_h^q(E) = \{\omega \in \Lambda^q(E) \mid X_1 \lrcorner \dots \lrcorner X_q \lrcorner \omega = 0, X_1, \dots, X_q \in D^v(E)\}.$$

Given a flat connection ∇ , one can split the module $D(E)$ into the direct sum $D(E) = D^v(E) \oplus \nabla D(E)$, where $\nabla D(E) \subset D(E)$ is the submodule generated by the vector fields ∇X , $X \in D(M)$. This splitting induces the dual one: $\Lambda^1(E) = \Lambda_h^1(E) \oplus \Lambda_{\nabla}^1(E)$, where $\Lambda_{\nabla}^1(E)$ is the annihilator of $\nabla D(E)$. Finally we obtain the splitting $\Lambda^i(E) = \bigoplus_{i=p+q} \Lambda^{p,q}(E)$, where $\Lambda^{p,q}(E) = \Lambda_{\nabla}^p(E) \otimes \Lambda_h^q(E)$ and $\Lambda_{\nabla}^p(E)$ is defined in an obvious way.

Proposition 5. Let ∂_{∇} be the differential from (15). Then $\partial_{\nabla}(D(\Lambda^{p,q}(E))) \subset D(\Lambda^{p,q+1}(E))$ and thus the complexes

$$0 \rightarrow D(\Lambda^p(E)) \xrightarrow{\partial_{\nabla}} D(\Lambda^{p+1}(E)) \rightarrow \dots \rightarrow D(\Lambda^{p,q}(E)) \xrightarrow{\partial_{\nabla}} D(\Lambda^{p,q+1}(E)) \rightarrow \dots \quad (16)$$

are defined.

The corresponding cohomology is denoted by $H_{\nabla}^{p,q}(E)$.

Remark 6. In the sequel, we shall also deal with subcomplexes of complexes (15) and (16). As an example, let us indicate the complex

$$0 \rightarrow D^v(E) \xrightarrow{\partial_{\nabla}} D^v(\Lambda^1(E)) \rightarrow \dots \rightarrow D^v(\Lambda^i(E)) \xrightarrow{\partial_{\nabla}} D^v(\Lambda^{i+1}(E)) \rightarrow \dots \quad (17)$$

of vertical $\Lambda^*(E)$ -valued derivations.

2.1.1. Local coordinates

Let $\mathcal{U} \subset M$ be a local chart with the coordinates $x_1, \dots, x_n, v^1, \dots, v^m$ in $\pi^{-1}(\mathcal{U}) \subset E$ (the case $m = \infty$ is included). Then, as it was noted above, a connection ∇ in π is given by the correspondence

$$\partial x_i \mapsto \nabla_{x_i} = \nabla(\partial x_i) = \partial x_i + \sum_{\alpha=1}^m v_i^{\alpha} \partial v^{\alpha}, \quad i = 1, \dots, n, v_i^{\alpha} \in C^{\infty}(E).$$

By definition, one has

$$\bar{U}_{\nabla} = \sum_{i=1}^n dx_i \otimes \nabla_{x_i}, \quad U_{\nabla} = \sum_{\alpha=1}^m \left(dv^{\alpha} - \sum_{i=1}^n v_i^{\alpha} dx_i \right) \otimes \partial v^{\alpha}. \quad (18)$$

Restricting ourselves to vertical complex (17), from (12) and (18) we get

$$\partial_{\nabla} \left(\sum_{\alpha} \theta^{\alpha} \otimes \partial v^{\alpha} \right) = \sum_{i=1}^n dx_i \wedge \sum_{\alpha=1}^m \left(\nabla_{x_i}(\theta^{\alpha}) - \sum_{\beta=1}^m \frac{\partial v_i^{\alpha}}{\partial v^{\beta}} \theta^{\beta} \right) \otimes \partial v^{\alpha}, \quad (19)$$

where $\theta^{\alpha} \in \Lambda^*(E)$.

2.2. A coordinate-free formulation

Fix a finite-dimensional bundle $\pi : E \rightarrow M$ and recall that connections (see Remark 1) in this bundle are identified with the sections of the bundle $\pi_{1,0} : J^1(\pi) \rightarrow E$. Consider a point $\theta_1 \in J^1(\pi_{1,0})$. Then it can be represented as a class $[\nabla]_{\theta}^1$, $\theta = (\pi_{1,0})(\theta_1)$, of local sections of $\pi_{1,0}$, or local connections in the bundle π over a neighborhood of the point $\pi_1(\theta) \in M$.

Clearly, the value of $[\overline{U}_{\nabla}, \overline{U}_{\nabla}]$ at θ is independent of the choice of a representative ∇ in the class $\theta_1 = [\nabla]_{\theta}^1$ and is determined by the point $\theta_1 \in J^1(\pi_{1,0})$ only. The equation of flat connections is defined to be the submanifold

$$\mathcal{E} = \{ \theta_1 \in J^1(\pi_{1,0}) \mid [\overline{U}_{\nabla}, \overline{U}_{\nabla}]_{\pi_{1,0}(\theta_1)} = 0, \theta_1 = [\nabla]_{\theta}^1 \}.$$

Indeed, by Proposition 4, a section s of $\pi_{1,0}$ determines a flat connection if and only if $s(E) \subset \mathcal{E}$.

In adapted coordinates this equation can be rewritten as

$$\left[\sum_i dx_i \otimes \left(D_{x_i} + \sum_{\alpha} v_i^{\alpha} D_{v^{\alpha}} \right), \sum_i dx_i \otimes \left(D_{x_i} + \sum_{\alpha} v_i^{\alpha} D_{v^{\alpha}} \right) \right] = 0, \tag{20}$$

or, equivalently,

$$\left[D_{x_i} + \sum_{\alpha} v_i^{\alpha} D_{v^{\alpha}}, D_{x_j} + \sum_{\alpha} v_j^{\alpha} D_{v^{\alpha}} \right] = 0, \quad 1 \leq i < j \leq n,$$

where D_{x_i} , $i = 1, \dots, n$, $D_{v^{\alpha}}$, $\alpha = 1, \dots, m$, are the total derivatives corresponding to ∂x_i and ∂v^{α} , respectively, by the Cartan connection \mathcal{C} in the fiber bundle $(\pi_{1,0})_{\infty} : J^{\infty}(\pi_{1,0}) \rightarrow E$.

We shall now describe the element $\sum_i (D_{x_i} + \sum_{\alpha} v_i^{\alpha} D_{v^{\alpha}}) \otimes dx_i$ appearing in Eq. (20) in a new, invariant way. To this end, consider the Cartan connection \mathcal{C}^{π} in the bundle $\pi_{\infty} : J^{\infty}(\pi) \rightarrow M$. This connection takes any vector field $X \in M$ to the vector field $\mathcal{C}^{\pi} X$ on $J^{\infty}(\pi)$. Due to the projection $\pi_{\infty,0} : J^{\infty}(\pi) \rightarrow E$, we can restrict $\mathcal{C}^{\pi} X$ to E (more precisely, we restrict the map $\mathcal{C}^{\pi} X : C^{\infty}(J^{\infty}(\pi)) \rightarrow C^{\infty}(J^{\infty}(\pi))$ to the subalgebra $C^{\infty}(E) \subset C^{\infty}(J^{\infty}(\pi))$). This restriction, $(\mathcal{C}^{\pi} X)_0$, is an element of the module $D(E, J^1(\pi))$, and, by Remark 4, since $\dim E < \infty$, we can apply the Cartan connection \mathcal{C} thus obtaining a vector field on $J^{\infty}(\pi_{1,0})$. The sequence of maps

$$D(M) \xrightarrow{\mathcal{C}^{\pi}} D(J^{\infty}(\pi)) \xrightarrow{\text{restriction to } E} D(E, J^1(\pi)) \xrightarrow{\mathcal{C}} D(J^{\infty}(\pi_{1,0}))$$

results in a new connection $\mathcal{C}_{fc} : D(M) \rightarrow D(J^{\infty}(\pi_{1,0}))$ in the bundle

$$\tilde{\pi} = \pi \circ (\pi_{1,0})_{\infty} : J^{\infty}(\pi_{1,0}) \rightarrow M.$$

When dealing with this connection, we shall use the notation $\overline{U}_{fc} = \overline{U}_{\mathcal{C}_{fc}}$ and $\partial_{fc} = \partial_{\mathcal{C}_{fc}}$.

It is easily checked that in local coordinates one has

$$\overline{U}_{fc} = \sum_i dx_i \otimes \left(D_{x_i} + \sum_{\alpha} v_i^{\alpha} D_{v^{\alpha}} \right)$$

and we have the following

Proposition 6. *The equation of flat connections $\mathcal{E} \subset J^1(\pi_{1,0})$ is distinguished by the condition $[\overline{U}_{fc}, \overline{U}_{fc}]_{\theta_1} = 0$, $\theta_1 \in J^1(\pi_{1,0})$. All vector fields of the form $\mathcal{C}_{fc} X$, $X \in D(M)$, are tangent to its infinite prolongation $\mathcal{E}_{fc} \subset J^{\infty}(\pi_{1,0})$ and thus the restriction of \mathcal{C}_{fc} to the subbundle $\tilde{\pi} : \mathcal{E}_{fc} \rightarrow M$ of the bundle $\pi \circ (\pi_{1,0})_{\infty} : J^{\infty}(\pi_{1,0}) \rightarrow M$ is flat.*

3. Symmetries of the equation of flat connections

From the last proposition it follows that the connection \mathcal{C}_{fc} determines on \mathcal{E}_{fc} the complexes similar to (16). We are especially interested in the case $p = 0$ and thus obtain

$$0 \rightarrow \mathcal{D}(\mathcal{E}_{\text{fc}}) \xrightarrow{\partial_{\text{fc}}} \mathcal{D}(\Lambda_h^1(\mathcal{E}_{\text{fc}})) \xrightarrow{\partial_{\text{fc}}} \cdots \rightarrow \mathcal{D}(\Lambda_h^q(\mathcal{E}_{\text{fc}})) \xrightarrow{\partial_{\text{fc}}} \mathcal{D}(\Lambda_h^{q+1}(E)) \rightarrow \cdots, \quad (21)$$

where $\partial_{\text{fc}} = \partial_{\mathcal{C}_{\text{fc}}}$.

Consider now in $\mathcal{CD}(\mathcal{E}_{\text{fc}})$ the submodule of $\tilde{\pi}$ -vertical vector fields. In coordinates, it is spanned by the vector fields D_{v^α} , $\alpha = 1, \dots, m$. Set also $\mathcal{V}^q = \tilde{\pi}^*(\Lambda^q(M)) \otimes \mathcal{V} \subset \mathcal{D}(\Lambda_h^q(\mathcal{E}_{\text{fc}}))$.

Lemma 1. *The modules \mathcal{V}^q are preserved by the differential ∂_{fc} , i.e., $\partial_{\text{fc}}(\mathcal{V}^q) \subset \mathcal{V}^{q+1}$.*

Proof. This is easily seen from the definition of \mathcal{V}^q . By (19), in coordinates ∂_{fc} acts as follows

$$\begin{aligned} \partial_{\text{fc}}(f dx_{i_1} \wedge \cdots \wedge dx_{i_q} \otimes D_{v^\alpha}) &= \sum_{i=1}^n D_i(f) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \otimes D_{v^\alpha} \\ &\quad - (-1)^q \sum_{i=1}^n \sum_{\beta=1}^m D_{v^\alpha}(v_i^\beta) f dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q} \otimes D_{v^\beta}, \end{aligned} \quad (22)$$

where $f \in C^\infty(\mathcal{E}_{\text{fc}})$ and, as everywhere below, $D_i = D_{x_i} + \sum_{\beta} v_i^\beta D_{v^\beta} = \mathcal{C}_{\text{fc}} \partial x_i$ is the total derivative with respect to the connection \mathcal{C}_{fc} . \square

Consequently, we obtain the complex

$$0 \rightarrow \mathcal{V}^0 = \mathcal{V} \xrightarrow{\partial_{\text{fc}}} \mathcal{V}^1 \xrightarrow{\partial_{\text{fc}}} \cdots \rightarrow \mathcal{V}^q \xrightarrow{\partial_{\text{fc}}} \mathcal{V}^{q+1} \rightarrow \cdots. \quad (23)$$

Let us now return back to the equation of flat connections written in the form (20). Since this equation is bilinear, its linearization in local coordinates is of the form

$$\left[\overline{U}_{\text{fc}}, \sum_{i,\alpha} \varphi_i^\alpha dx_i \otimes D_{v^\alpha} \right] + \left[\sum_{i,\alpha} \varphi_i^\alpha dx_i \otimes D_{v^\alpha}, \overline{U}_{\text{fc}} \right] = 2 \left[\overline{U}_{\text{fc}}, \sum_{i,\alpha} \varphi_i^\alpha dx_i \otimes D_{v^\alpha} \right] = 0,$$

where $\varphi = \sum_{i,\alpha} \varphi_i^\alpha dx_i \otimes D_{v^\alpha}$ is the generating section of a symmetry, $\varphi_i^\alpha \in C^\infty(\mathcal{E}_{\text{fc}})$, or

$$\ell_{\mathcal{E}_{\text{fc}}} \varphi \equiv 2 \partial_{\text{fc}} \varphi = 0, \quad \varphi \in \mathcal{V}^1. \quad (24)$$

Proposition 7. *The map*

$$\text{sym } \mathcal{E}_{\text{fc}} \rightarrow \mathcal{V}^1, \quad S \mapsto \llbracket \overline{U}_{\text{fc}}, S \rrbracket \quad (25)$$

is an isomorphism of the space of symmetries onto the space of 1-cocycles of complex (23).

Proof. Let $S \in \text{sym } \mathcal{E}_{\text{fc}}$. Since S commutes with all total derivatives, we have $\llbracket \overline{U}_{\text{fc}}, S \rrbracket \in \mathcal{V}^1$. Moreover, $\llbracket \overline{U}_{\text{fc}}, S \rrbracket$ is a cocycle, since $\llbracket \overline{U}_{\text{fc}}, \llbracket \overline{U}_{\text{fc}}, \cdot \rrbracket \rrbracket = 0$ (since $S \notin \mathcal{V}$, this cocycle may be not exact). The fact that (25) is an isomorphism onto the kernel of ∂_{fc} follows from the above coordinate consideration, since $\llbracket \overline{U}_{\text{fc}}, S \rrbracket = - \sum_{i,\alpha} S(v_i^\alpha) dx_i \otimes D_{v^\alpha}$ and $\varphi_i^\alpha = S(v_i^\alpha)$ is nothing but the generating section of the symmetry S . \square

Remark 7. It is not surprising that in (24) the symmetry generating section is an element of the module \mathcal{V}^1 . Indeed, in our case the bundle $\xi = \pi_{1,0} : J^1(\pi) \rightarrow E$ is affine and $\pi_{1,0}^v$ (see Remark 2) is isomorphic to the pullback $\pi_1^*(\pi \otimes \tau^*)$, where $\tau^* : T^*M \rightarrow M$ is the cotangent bundle. Consequently, the module of generating sections is isomorphic to \mathcal{V}^1 .

Theorem 2. Complex (23) is 0-acyclic, i.e.,

$$\ker(\partial_{fc} : \mathcal{V} \rightarrow \mathcal{V}^1) = 0, \tag{26}$$

and 1-acyclic, i.e.,

$$\partial_{fc}(\mathcal{V}) = \ker(\partial_{fc} : \mathcal{V}^1 \rightarrow \mathcal{V}^2). \tag{27}$$

Corollary 1 (Description of symmetries). Any symmetry (φ_i^α) , $i = 1, \dots, n$, $\alpha = 1, \dots, m$, of Eq. (1) is of the form

$$\varphi_i^\alpha = D_i(f^\alpha) - \sum_{\beta=1}^m v_i^{\alpha,\beta} f^\beta \tag{28}$$

for arbitrary functions $f^\alpha \in C^\infty(\mathcal{E}_{fc})$ uniquely determined by (φ_i^α) , where $v_i^{\alpha,\beta} = D_{v^\beta}(v_i^\alpha)$.

In Section 4.4, we shall describe a Lie algebra structure of symmetries in terms of functions f^α .

Proof of Corollary 1. The result immediately follows from Theorem 2 in combination with (22) and (24). \square

To prove Theorem 2, we shall need some auxiliary constructions. Consider the following smooth functions on \mathcal{E}_{fc}

$$v_I^{\alpha,A} = D_{v^A} D_I(v^\alpha), \quad |I| > 0, \tag{29}$$

where $D_{v^A} = D_{v^{\alpha_1}} \circ \dots \circ D_{v^{\alpha_l}}$, $D_I = D_{i_1} \circ \dots \circ D_{i_k}$ for the multi-indices $A = \alpha_1 \dots \alpha_l$ and $I = i_1 \dots i_k$, respectively. In particular, $v_{\emptyset}^{\alpha,\emptyset} = v^\alpha$ and $v_i^{\alpha,\emptyset} = v_i^\alpha$. It is easily seen that the functions $x_i, v_I^{\alpha,A}$ form a system of coordinates on \mathcal{E}_{fc} . Let $\mathcal{F}_r \subset C^\infty(\mathcal{E}_{fc})$ be the subalgebra of functions dependent on the variables

$$x_i, \quad v_I^{\alpha,A}, \quad |I| \leq r,$$

only. Then we have

$$\mathcal{F}_r \subset \mathcal{F}_{r+1}, \quad \bigcup_{r \geq 0} \mathcal{F}_r = C^\infty(\mathcal{E}_{fc}).$$

Evidently, for $f \in \mathcal{F}_r$ one obtains

$$D_i(f) \equiv \sum_{\alpha,A,I} v_{Ii}^{\alpha,A} \frac{\partial f}{\partial v_I^{\alpha,A}} \pmod{\mathcal{F}_r}. \tag{30}$$

We have also the filtration $\mathcal{V}_r^q \subset \mathcal{V}_{r+1}^q$ in the modules \mathcal{V}^q , where $\mathcal{V}_r^q \subset \mathcal{V}^q$ consists of forms with coefficients in \mathcal{F}_r . From (22) and (30) for $r > 0$ we get

$$\partial_{fc}(f dx_{l_1} \wedge \dots \wedge dx_{l_q} \otimes D_{v^\alpha}) \equiv \sum_{i=1}^n \sum_{I, \beta, A} v_i^{\beta, A} \frac{\partial f}{\partial v_I^{\beta, A}} dx_i \wedge dx_{l_1} \wedge \dots \wedge dx_{l_q} \otimes D_{v^\alpha} \pmod{\mathcal{V}_r^{q+1}}. \tag{31}$$

On the other hand, consider the infinite-dimensional vector bundle

$$\pi' : \mathbb{R}^\infty \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

with the coordinates $u^{\alpha, A}$, $\alpha = 1, \dots, m$, $A = \alpha_1 \dots \alpha_l$ where $\alpha_j \geq 0$, along fibers and coordinates y_i , $i = 1, \dots, n$, in the base. The infinite jet bundle $J^\infty(\pi') \rightarrow \mathbb{R}^n$ of π' has the adapted coordinates

$$u_J^{\alpha, A} = D_{y_I}(u^{\alpha, A})$$

along the fibers, where, as above, D_{y_I} denotes the composition $D_{y_{i_1}} \circ \dots \circ D_{y_{i_k}}$ of total derivatives.

To any element

$$s = \sum_{i, \alpha} f_i^\alpha dx_i \otimes D_{v^\alpha} \in \mathcal{V}^1$$

and an integer $a \geq 0$ we associate an m -tuple $\omega(s, a)^1, \dots, \omega(s, a)^m$ of parameter-dependent horizontal 1-forms on $J^\infty(\pi')$ as follows.

Let $r(s)$ be the minimal integer such that all the coefficients f_i^α of s belong to $\mathcal{F}_{r(s)}$. Replace the variables $v_J^{\beta, A}$ with $|I| \geq r(s) - a$ in f_i^α by $u_I^{\alpha, A}$ and denote thus obtained functions by \tilde{f}_i^α . We treat \tilde{f}_i^α as functions on $J^\infty(\pi')$ dependent on the parameters x_1, \dots, x_n and $v_J^{\beta, A}$ with $|I| < r(s) - a$. Set

$$\omega(s, a)^\alpha = \sum_i \tilde{f}_i^\alpha dy_i.$$

We say that an element $s \in \mathcal{V}^1$ is *semilinear* if the coefficients of s are linear with respect to the highest order coordinates $v_J^{\alpha, A}$, $|I| = r(s)$.

Lemma 2. *The following facts are valid:*

1. *If $s \in \mathcal{V}^1$ and $\partial_{fc}(s) = 0$ then $d_h(\omega(s, 0)^\alpha) = 0$ for each α and for all values of the parameters, where d_h is the horizontal de Rham differential on $J^\infty(\pi')$.*
2. *If $s \in \mathcal{V}^1$ is semilinear, $r(s) \geq 2$, and $\partial_{fc}(s) = 0$, then*

$$d_h(\omega(s, 1)^\alpha) = 0.$$

Proof. (1) By (31) and (14), equations $d_h(\omega(s, 0)^\alpha) = 0$ for all $\alpha = 1, \dots, m$ are equivalent to

$$\partial_{fc}(s) = 0 \pmod{\mathcal{V}_{r(s)}^2}.$$

(2) Let $\tilde{\mathcal{F}}_r \subset \mathcal{F}_r$ be the \mathcal{F}_{r-1} -submodule generated by $v_J^{\alpha, A}$, $|I| = r$, and, respectively, let $\tilde{\mathcal{V}}_r^1 \subset \mathcal{V}_r^1$ be the subset of forms with coefficients in $\tilde{\mathcal{F}}_r$. Then from the identity

$$[D_i, D_{v^\alpha}] = - \sum_{\beta} v_i^{\beta, \alpha} D_{v^\beta} \tag{32}$$

we obtain

$$D_i(v_I^{\alpha,A}) \equiv v_{I_i}^{\alpha,A} \pmod{\tilde{\mathcal{F}}_{|I|}}. \tag{33}$$

By (22) and (33), the equation

$$\partial_{fc}(s) = 0 \pmod{\mathcal{V}_{r(s)-1}^2 + \tilde{\mathcal{V}}_{r(s)}^2}$$

for semilinear s implies $d_h(\omega(s, 1)^\alpha) = 0$. \square

Lemma 3. Any cocycle $s \in \mathcal{V}^1$ with $r(s) \geq 1$ is semilinear.

Proof. If there is a nonlinear coefficient, we can fix the values of $x_i, v_I^{\alpha,A}, |I| < r(s)$, such that some form $\omega(s, 0)^\alpha$ is not linear with respect to $u_I^{\alpha,A}, |I| = r(s)$, for these values of the parameters. Proposition 3 implies

$$H_h^1(J^\infty(\pi')) = 0. \tag{34}$$

By Lemma 2(1) and (34), we have $\omega(s, 0)^\alpha = d_h(w')$ for some function dependent only on $u_I^{\alpha,A}$ with $|I| = r(s) - 1$. Taking into account (14) and (6), we see that $\omega(s, 0)^\alpha$ is linear. \square

Lemma 4. For each cocycle $s \in \mathcal{V}^1$ there exists a cohomological cocycle with coefficients in \mathcal{F}_1 .

Proof. Assume that $r(s) \geq 2$. By Lemma 3, Lemma 2(2), and (34), we have

$$\omega(s, 1)^\alpha = d_h(w_\alpha) \tag{35}$$

for some functions $w_\alpha \in C^\infty(J^\infty(\pi'))$ dependent also on the parameters $x_i, v_I^{\alpha,A}$ with $|I| \leq r(s) - 2$. Moreover, we can assume that w_α depend only on the coordinates $u_I^{\alpha,A}$ with $|I| \leq r(s) - 1$. Replace the variables $u_I^{\alpha,A}$ in w_α by $v_I^{\alpha,A}$ and denote the obtained functions by \tilde{w}_α . Consider the vector field

$$s' = \sum_{\alpha} \tilde{w}_\alpha D_{v^\alpha} \in \mathcal{V}_{r(s)-1}^0.$$

Then from (35) and (31) we have $r(s - \partial_{fc}(s')) \leq r(s) - 1$. Proceeding in this way, by induction on $r(s)$ one completes the proof. \square

Proof of Theorem 2. Let us first prove (26). Let $q \in \mathcal{V}$. If $r(q) > 0$ then $\partial_{fc}(q) \neq 0$ by (31). If $r(q) = 0$, i.e.,

$$q = \sum_{\alpha} f\alpha(x_1, \dots, x_n, v^1, \dots, v^m) D_{v^\alpha},$$

then $\partial_{fc}(q) \neq 0$ is proved by direct computation.

Statement (26) means that it is sufficient to prove (27) locally.

Let $s \in \mathcal{V}^1$ be a cocycle. By Lemmas 3 and 4, there is a cohomological cocycle

$$s' = \sum_{i,\beta} f_i^\beta dx_i \otimes D_{v^\beta} \in \mathcal{V}^1$$

such that the functions f_i^β belong to \mathcal{F}_1 and are linear with respect to the coordinates $v_I^{\alpha,A}$, $|I| = 1$. From $d_h\omega(s', 0) = 0$ we obtain

$$f_i^\beta = f_{i,\beta}(x_1, \dots, x_n, v^1, \dots, v^m) + \sum_{\gamma,A} f_\beta^{\gamma,A}(x_1, \dots, x_n, v^1, \dots, v^m) v_i^{\gamma,A}.$$

By (22), the equation $\partial_{fc}(s') = 0$ is equivalent to

$$D_i(f_j^\beta) - \sum_{\gamma=1}^m v_i^{\beta,\gamma} f_j^\gamma = D_j(f_i^\beta) - \sum_{\gamma=1}^m v_j^{\beta,\gamma} f_i^\gamma \tag{36}$$

for all β, i and j . Comparing Eqs. (1) and (36), one shows that the functions $f_\beta^{\gamma,A}$ vanish for $|A| > 1$. Then a straightforward computation implies that $s' = \partial_{fc}(F)$ for some $F \in \mathcal{V}_0^0$. \square

4. Flat representations

4.1. Definition and the main property

Consider a formally integrable infinitely prolonged differential equation $\xi : \mathcal{E} \rightarrow M$ with the Cartan distribution $CD \subset D(\mathcal{E})$, where M is the space of independent variables. A *flat representation* of \mathcal{E} is given by a fibering $\pi : M \rightarrow N$ and a flat connection \mathcal{F} in the composition bundle $\eta = \pi \circ \xi : \mathcal{E} \rightarrow N$ such that for each $X \in D(N)$

$$\mathcal{F}(X) \in CD. \tag{37}$$

Let $x_1, \dots, x_{n'}$ be local coordinates in N and y^1, \dots, y^m be coordinates along the fibers of the bundle π . Then any flat representation is determined by the correspondence

$$\partial x_i \mapsto D_{x_i} + \sum_{\alpha=1}^m a_i^\alpha D_{y^\alpha}, \quad i = 1, \dots, n', \tag{38}$$

where D_{x_i}, D_{y^α} are the total derivatives on \mathcal{E} and the functions $a_i^j \in C^\infty(\mathcal{E})$ satisfy the condition

$$\left[D_{x_i} + \sum_{\alpha=1}^m a_i^\alpha D_{y^\alpha}, D_{x_j} + \sum_{\alpha=1}^m a_j^\alpha D_{y^\alpha} \right] = 0, \quad 1 \leq i < j \leq n'. \tag{39}$$

Let \mathcal{E}_{fc} be the equation of flat connections in the bundle π and consider a morphism $\varphi : \mathcal{E} \rightarrow \mathcal{E}_{fc}$ such that the diagram

$$\begin{array}{ccc}
 \mathcal{E} & \xrightarrow{\varphi} & \mathcal{E}_{fc} \\
 & \searrow \xi & \swarrow \\
 & & M
 \end{array} \tag{40}$$

is commutative. Take a point $\theta \in \mathcal{E}$. Then $\varphi(\theta)$ is the infinite jet of some flat connection ∇_θ in the bundle π . This means that any tangent vector to N at the point $\eta(\theta)$ can be lifted by ∇ to M and then by the

Cartan connection in \mathcal{E} to a tangent vector to \mathcal{E} at the point θ . Evidently, this procedure is independent of the choice of ∇ and we obtain a flat representation for \mathcal{E} .

The converse construction is also valid, i.e., any flat representation determines a morphism φ satisfying the commutativity condition (40). To establish this fact let us consider a point $\theta \in \mathcal{E}$ of the equation \mathcal{E} and a tangent vector w to N at the point $\eta(\theta)$. Then we can construct a vector $\theta_{\mathcal{F}}(w) \in T_{\xi(\theta)}M$ by

$$\theta_{\mathcal{F}}(w) = \xi_*|_{\theta}(\mathcal{F}(w)) \in T_{\xi(\theta)}M, \tag{41}$$

and one has $\pi_*(\theta_{\mathcal{F}}(w)) = w$. Assume now that $\mathcal{E} \subset J^\infty(\zeta)$ for some bundle ζ over M and the connection \mathcal{F} can be extended to some connection $\tilde{\mathcal{F}}$ in the bundle $\pi \circ \zeta_\infty$ in such a way that $\tilde{\mathcal{F}}$ also possesses property (37) ($\tilde{\mathcal{F}}$ may not be flat). This can be always done. For a point $\theta \in \mathcal{E}$, consider the points $x = \xi(\theta) \in M$, $x' = \pi(x) \in N$ and a neighborhood $\mathcal{U}' \subset N$ of x' in N . Set $\mathcal{U} = \pi^{-1}(\mathcal{U}') \subset M$. Let $\theta = [f]_x^\infty$, $f \in \Gamma_{\text{loc}}(\zeta)$ and $\mathcal{U}_f = j_\infty(f)(\mathcal{U}) \subset J^\infty(\zeta)$. Then, using (41), for any point $\tilde{x} \in \mathcal{U}$ and a vector field X on \mathcal{U}' we set

$$(\nabla_\theta X)_{\tilde{x}} = \tilde{\theta}_{\tilde{\mathcal{F}}}(X_{\tilde{x}}),$$

where $\tilde{\theta} = j_\infty(f)(\tilde{x})$ and $\tilde{x}' = \pi(\tilde{x})$. It is straightforward to check that ∇_θ is a local connection in π , its infinite jet at x depends on θ only and lies in \mathcal{E}_{fc} . The map $\varphi: \theta \mapsto [\nabla_\theta]_x^\infty$ is a morphism of \mathcal{E} to \mathcal{E}_{fc} and the corresponding flat representation coincides with \mathcal{F} . Thus we obtain

Theorem 3. *Let $\xi: \mathcal{E} \rightarrow M$ be an infinitely prolonged equation and $\pi: M \rightarrow N$ be a fiber bundle. Then flat representations of \mathcal{E} in the bundle π are in one-to-one correspondence with the morphisms of \mathcal{E} to the equation of flat connections for π satisfying commutativity condition (40).*

Remark 8. In local coordinates the construction of the morphism φ looks quite simple. Namely, let a flat representation be given by the functions a_i^α , see (38). Then we set $\varphi^*(v_i^\alpha) = a_i^\alpha$ and take all differential consequences of these relations, i.e.,

$$\varphi^*(v_{i,\sigma}^\alpha) = D_\sigma(a_i^\alpha).$$

Due to relations (39), these formulas are consistent. Here v_i^α are the coordinates introduced at the end of Section 2.1.

4.2. Deformations of a flat representation

Integrable equations possess parametric families of flat representations. Let us study this structure.

A smooth family of flat connections $\mathcal{F}(\varepsilon)$ in the bundle η is called a *deformation* of a flat representation \mathcal{F}^0 if $\mathcal{F}(\varepsilon)$ meets (37) and $\mathcal{F}(0) = \mathcal{F}^0$. Below we also consider *formal deformations*, which are not smooth families, but formal power series in ε .

Set $U = \bar{U}_{\mathcal{F}^0} \in D(\Lambda^1(\mathcal{E}))$. Consider the submodule $\mathcal{V}_\mathcal{E} \subset \mathcal{CD}$ of η -vertical Cartan fields and set $\mathcal{V}_\mathcal{E}^q = \eta^*(\Lambda^q(N)) \otimes \mathcal{V} \subset D(\Lambda_h^q(\mathcal{E}))$.

Let $\mathcal{F}(\varepsilon)$ be a (formal) deformation of $\mathcal{F}^0 = \mathcal{F}(0)$. From (37) it follows that $\bar{U}_{\mathcal{F}(\varepsilon)} = U + U_1\varepsilon + \mathcal{O}(\varepsilon^2)$, where $U_1 \in \mathcal{V}_\mathcal{E}^1$. By Proposition 4, we have

$$\llbracket U + U_1\varepsilon + \mathcal{O}(\varepsilon^2), U + U_1\varepsilon + \mathcal{O}(\varepsilon^2) \rrbracket = 2\llbracket U, U_1 \rrbracket \varepsilon + \mathcal{O}(\varepsilon^2) = 0.$$

Therefore, the *infinitesimal part* $U_1 \in \mathcal{V}_\mathcal{E}^1$ of any deformation satisfies the identity

$$[[U, U_1]] = 0. \quad (42)$$

For any vector field $V \in \mathcal{V}_\mathcal{E}$ we can consider the formal deformation

$$e^{\varepsilon \operatorname{ad} V}(U) = U + [[V, U]]\varepsilon + \frac{1}{2}[[V, [[V, U]]]\varepsilon^2 + \dots \quad (43)$$

with the infinitesimal part

$$[[V, U]]. \quad (44)$$

Since such deformations exist independently of the equation structure, it is natural to call them, as well as their infinitesimal parts $[[V, U]]$, $V \in \mathcal{V}_\mathcal{E}$, *trivial*.

Formulas (42) and (44) prompt the following construction. It is easily seen that $\partial_U(\mathcal{V}_\mathcal{E}^q) \subset \mathcal{V}_\mathcal{E}^{q+1}$, and, therefore, we obtain a complex

$$0 \rightarrow \mathcal{V}_\mathcal{E}^0 = \mathcal{V}_\mathcal{E} \xrightarrow{\partial_U} \mathcal{V}_\mathcal{E}^1 \xrightarrow{\partial_U} \dots \rightarrow \mathcal{V}_\mathcal{E}^q \xrightarrow{\partial_U} \mathcal{V}_\mathcal{E}^{q+1} \rightarrow \dots \quad (45)$$

By the construction and formulas (42), (44), its 1-cocycles are infinitesimal deformations, and exact 1-cocycles are trivial infinitesimal deformations. From the general theory of deformations [4], we obtain that obstructions to prolongation of infinitesimal deformations to formal ones belong to higher cohomology groups of (45).

Remark 9. Deformations of *zero-curvature representations* (finite-dimensional linear coverings) of PDE were studied analogously in [13]. In fact, the *gauge complex* of a zero-curvature representation introduced in [12,13] is a particular case of the above construction, since according to Section 5.1 any finite-dimensional covering determines a flat representations.

4.3. Symmetries and cocycles

Let $S \in \mathcal{D}(\mathcal{E})$ be a symmetry of \mathcal{E} . Since S commutes with the fields of the form $\mathcal{C}(X)$, $X \in \mathcal{D}(M)$, the power series $e^{\varepsilon \operatorname{ad} S}(U)$ is a formal deformation of \mathcal{F} , and its infinitesimal part $[[S, U]] \in \mathcal{V}_\mathcal{E}^1$ is a 1-cocycle of complex (45). Let us set $c_S = [[U, S]]$. Note that this cocycle is not generally exact, since $S \notin \mathcal{V}_\mathcal{E}$.

Theorem 4. *Symmetries S with exact cocycles c_S form a Lie subalgebra in $\operatorname{sym} \mathcal{E}$.*

Proof. Let S_1, S_2 be two symmetries such that

$$[[U, S_1]] = [[U, V_1]], \quad [[U, S_2]] = [[U, V_2]], \quad V_1, V_2 \in \mathcal{V}_\mathcal{E}. \quad (46)$$

Using this and properties (10), (11) of the Nijenhuis bracket, we easily obtain

$$c_{[S_1, S_2]} = [[U, [S_1, S_2]]] = [[U, [V_1, V_2] + [V_1, S_2] + [S_1, V_2]]]. \quad (47)$$

Note that $[\mathcal{V}_\mathcal{E}, \mathcal{V}_\mathcal{E}] \subset \mathcal{V}_\mathcal{E}$ and for each symmetry S one has $[S, \mathcal{V}_\mathcal{E}] \subset \mathcal{V}_\mathcal{E}$. Therefore, $[V_1, V_2] + [V_1, S_2] + [S_1, V_2] \in \mathcal{V}_\mathcal{E}$ and the cocycle $c_{[S_1, S_2]}$ is exact. \square

Theorem 5. Suppose that the morphism $\varphi: \mathcal{E} \rightarrow \mathcal{E}_{fc}$ corresponding to the flat representation \mathcal{F} is an embedding such that $\varphi(\mathcal{E})$ is a submanifold of \mathcal{E}_{fc} . Then the cocycle c_S is exact if and only if there is a symmetry $S' \in D(\mathcal{E}_{fc})$ of \mathcal{E}_{fc} such that

$$\varphi_*(S) = S'|_{\varphi(\mathcal{E})}. \tag{48}$$

Remark 10. Recall that \mathcal{E} is a submanifold of some infinite jet space $J^\infty(\zeta)$. It is well known [3] that under rather weak regularity assumptions each symmetry of \mathcal{E} is the restriction of some symmetry of $J^\infty(\zeta)$. The above theorem provides a criterion for the similar property in the situation when \mathcal{E}_{fc} is considered instead of $J^\infty(\zeta)$. In the next section we give examples of both types of symmetries: with exact and non-exact c_S . Therefore, generally not every symmetry is the restriction of some symmetry of \mathcal{E}_{fc} .

Proof. First suppose that (48) holds. By Theorem 2, there is $V \in \mathcal{V}$ such that

$$[[U, S']] = [[U, V]]. \tag{49}$$

Since V belongs to the Cartan distribution and φ is a morphism of differential equations, the vector field V is tangent to the submanifold $\varphi(\mathcal{E})$. Denote by $V' \in \mathcal{V}_{\mathcal{E}}$ the preimage $\varphi_*^{-1}(V)$. Then from (48) and (49) it follows that

$$c_S = [[U, V']], \tag{50}$$

i.e., the cocycle c_S is exact.

Conversely, if c_S is exact then (50) holds for some $V' \in \mathcal{V}_{\mathcal{E}}$. Extend $\varphi_*(V') \in D(\varphi(\mathcal{E}))$ to a vector field $V \in \mathcal{V}$. Then the symmetry S' of \mathcal{E}_{fc} with the generating section $[[U, V]]$ satisfies (48). \square

4.4. A bracket on the equation of flat connections

The identity mapping

$$\text{id}: \mathcal{E}_{fc} \rightarrow \mathcal{E}_{fc}$$

is a flat representation, and complex (23) coincides with (45) for this flat representation. From the results of Section 3 it follows that $\text{sym } \mathcal{E}_{fc}$ is isomorphic to \mathcal{V} . Therefore, the commutator of symmetries induces a bracket on \mathcal{V} . By formulas (46) and (47), it is given by

$$\{V_1, V_2\} = [V_1, V_2] + [V_1, S_2] + [S_1, V_2], \quad V_1, V_2 \in \mathcal{V}, \tag{51}$$

where S_i is the symmetry with the generating section $[[\bar{U}_{fc}, V_i]]$. Let us compute this bracket in coordinates.

In adapted coordinates, any element of \mathcal{V} is of the form

$$V_f = \sum_{i=1}^n f^i D_{v^i}, \quad f^i \in C^\infty(\mathcal{E}_{fc}).$$

Hence, locally, the bracket $\{V_f, V_g\}$ generates a bracket $\{f, g\}$ on vector-functions defined by $\{V_f, V_g\} = V_{\{f, g\}}$. Using (51), we compute $\{f, g\}$.

Let

$$S_f = \sum_{|I|>0} S_I^{\alpha,A} \partial v_I^{\alpha,A}$$

be the symmetry of \mathcal{E}_{fc} with the generating section $[[\bar{U}_{fc}, V_f]]$. Then by direct computations we obtain that

$$S_I^{\alpha,A} = D_{v^A}(S_I^{\alpha,\emptyset}), \tag{52}$$

where D_{v^A} is the composition of the total derivatives D_{v^β} corresponding to the multi-index A , while

$$S_{Ii}^{\alpha,\emptyset} = D_i(S_I^{\alpha,\emptyset}) + \sum_{\beta=1}^m v_i^{\alpha,\beta} \varphi_i^\beta, \tag{53}$$

where $\varphi_i^\beta = S_i^{\beta,\emptyset}$, and, as we already know (see (28)),

$$\varphi_i^\beta = D_i(f^\beta) - \sum_{\gamma=1}^m v_i^{\beta,\gamma} f^\gamma.$$

Substituting these expressions to (51), we obtain the following

Proposition 8. *The commutator of symmetries determines the following bracket on the module of smooth m -dimensional vector-functions on \mathcal{E}*

$$\{f, g\}^\alpha = S_f g^\alpha - S_g f^\alpha + V_f g^\alpha - V_g f^\alpha. \tag{54}$$

For any vector-function $h = (h^1, \dots, h^m)$ the coefficients of the vector field $S_h = \sum S_I^{\alpha,A} \partial v_I^{\alpha,A}$ in special coordinates $v_I^{\alpha,A}$ can be computed by the formulas

$$S_I^{\alpha,A} = D_{v^A}(S_I^{\alpha,\emptyset}), \quad S_{Ii}^{\alpha,\emptyset} = D_i(S_I^{\alpha,\emptyset}) + \sum_{\beta=1}^m v_i^{\alpha,\beta} \zeta_i^\beta,$$

where $\zeta_i^\beta = S_i^{\beta,\emptyset}$ and $\zeta_i^\beta = D_i(h^\beta) - \sum_{\gamma=1}^m v_i^{\beta,\gamma} h^\gamma$, while $V_h = \sum_\alpha h^\alpha D_{v^\alpha}$.

5. Examples of flat representations

5.1. Coverings as flat representations and lifting of symmetries

Consider an infinitely prolonged differential equation $\xi : \mathcal{E} \rightarrow M$ with the Cartan connection

$$\mathcal{C} : D(M) \rightarrow D(\mathcal{E}).$$

Let $\pi : E \rightarrow M$ be a fiber bundle of finite rank and consider its pullback with respect to ξ

$$\begin{array}{ccc}
 \mathcal{E}' & \xrightarrow{\xi'} & E \\
 \xi^*(\pi) \downarrow & & \downarrow \pi \\
 \mathcal{E} & \xrightarrow{\xi} & M
 \end{array} \tag{55}$$

Suppose that the bundle $\tau = \xi^*(\pi) : \mathcal{E}' \rightarrow \mathcal{E}$ is endowed with a covering structure given by a flat connection $\mathcal{F} : D(M) \rightarrow D(\mathcal{E}')$, $\tau_* \circ \mathcal{F} = \mathcal{C}$. This structure determines a flat representation not of the equation \mathcal{E} itself, but of some trivial extension of \mathcal{E} .

Namely, consider the distribution $\mathcal{CD}_1 = \tau_*^{-1}(\mathcal{CD}_{\mathcal{E}}) \subset D(\mathcal{E}')$. Clearly, the differential ξ'_* projects \mathcal{CD}_1 isomorphically onto $D(E)$. Denote by $\mathcal{C}_1 : D(E) \rightarrow D(\mathcal{E}')$ the arising connection in the bundle $\xi' : \mathcal{E}' \rightarrow E$. It is easily seen that this connection is flat, and, moreover, the pair $(\mathcal{E}', \mathcal{C}_1)$ is isomorphic to an infinitely prolonged equation with E as the space of independent variables and \mathcal{C}_1 as the Cartan connection.

Remark 11. In coordinates, this construction looks as follows. Locally the bundle τ is trivial $\tau : M \times W \rightarrow M$. Let x_i, w_k be coordinates in M and W respectively. Then the equation $(\mathcal{E}', \mathcal{C}_1)$ is obtained from $(\mathcal{E}, \mathcal{C})$ as follows. We assume that the dependent variables u^j in \mathcal{E} are functions of not only x_i , but also w_k , and add the equations $\partial u^j / \partial w_k = 0$.

The bundle $\pi : E \rightarrow M$ and the connection \mathcal{F} constitute a flat representation of the equation $(\mathcal{E}', \mathcal{C}_1)$.

Consider complex (45) corresponding to this flat representation. By construction, the module $\mathcal{V}_{\mathcal{E}}$ consists of τ -vertical vector fields on \mathcal{E}' and (45) in this case is the horizontal de Rham complex of the covering equation $(\mathcal{E}', \mathcal{F})$ with coefficients in $\mathcal{V}_{\mathcal{E}}$ (see Section 3). For arbitrary coverings this complex was studied in [6] in relation with parametric families of Bäcklund transformations.

By construction (55), each ξ -vertical vector field $Y \in D(\mathcal{E})$ is canonically lifted to \mathcal{E}' as a ξ' -vertical vector field denoted by $\widehat{Y} \in D(\mathcal{E}')$. Evidently, for each symmetry S of \mathcal{E} the vector field \widehat{S} is a symmetry of the equation $(\mathcal{E}', \mathcal{C}_1)$.

Theorem 6. Let S be a symmetry of \mathcal{E} and consider complex (45) corresponding to the flat representation \mathcal{F} of $(\mathcal{E}', \mathcal{C}_1)$. The cocycle $c_{\widehat{S}}$ is exact if and only if there is a symmetry $S' \in D(\mathcal{E}')$ of the covering equation $(\mathcal{E}', \mathcal{F})$ such that $\tau_*(S') = S$.

Proof. If $c_{\widehat{S}} = [\overline{U}_{\mathcal{F}}, \widehat{S}] = [\overline{U}_{\mathcal{F}}, V]$ for some $V \in \mathcal{V}_{\mathcal{E}}$ then

$$[\overline{U}_{\mathcal{F}}, \widehat{S} - V] = 0,$$

i.e., the vector field $S' = \widehat{S} - V$ is a symmetry of $(\mathcal{E}', \mathcal{F})$. Clearly, $\tau_*(S') = S$.

Conversely, if S' is a symmetry of $(\mathcal{E}', \mathcal{F})$ and $\tau_*(S') = S$ then $c_{\widehat{S}} = [\overline{U}_{\mathcal{F}}, V]$ for $V = \widehat{S} - S' \in \mathcal{V}_{\mathcal{E}}$. \square

5.2. Flat representations and symmetries of the KdV equation

Let us illustrate the above construction by the following classical example of covering.

Consider the well-known system

$$v_x = \lambda + u + v^2, \tag{56}$$

$$v_t = u_{xx} + 2u^2 - 2\lambda u - 4\lambda^2 + 2u_x v + v^2(2u - 4\lambda), \quad \lambda \in \mathbb{R}, \tag{57}$$

associated with the Miura transformation. Its compatibility condition is equivalent to the KdV equation

$$u_t = u_{xxx} + 6uu_x. \tag{58}$$

The infinite prolongation \mathcal{E} of (58) has the natural coordinates

$$x, t, u_k = \frac{\partial^k u}{\partial x^k}, \quad k = 0, 1, 2, \dots \quad (59)$$

The total derivatives restricted to \mathcal{E} are written in these coordinates as follows

$$D_x = \partial x + \sum_{k \geq 0} u_{k+1} \partial u_k, \quad (60)$$

$$D_t = \partial t + \sum_{k \geq 0} D_x^k (u_3 + 6uu_1) \partial u_k. \quad (61)$$

Evidently, functions (59) and v form a system of coordinates for the infinite prolongation $\mathcal{E}' = \mathcal{E} \times \mathbb{R}$ of system (56), (57). In these coordinates, the total derivatives restricted to \mathcal{E}' are $\tilde{D}_x = D_x + A\partial/\partial v$, $\tilde{D}_t = D_t + B\partial/\partial v$, where $A, B \in C^\infty(\mathcal{E}')$ are the right-hand sides of (56) and (57) respectively. We have the trivial bundle $\tau: \mathcal{E}' \rightarrow \mathcal{E}$, $(x, t, u_k, v) \mapsto (x, t, u_k)$, equipped with the covering structure $\tau_*(\tilde{D}_x) = D_x$, $\tau_*(\tilde{D}_t) = D_t$.

By the construction of the previous subsection, this covering determines the following flat representation of the KdV equation.

We assume that u in (58) is a function of x, t, v and add the condition $\partial u/\partial v = 0$. The infinite prolongation of this trivially extended KdV equation is the manifold $\mathcal{E} \times \mathbb{R}$ with coordinates (59) and v . The Cartan distribution \mathcal{CD}_1 on $\mathcal{E} \times \mathbb{R}$ is 3-dimensional. The total derivatives with respect to x, t are given by (60) and (61), while the total derivative with respect to v is just ∂v . Then a flat representations dependent on parameter λ is given by the flat connection

$$\partial x \mapsto D_x + A\partial v \in \mathcal{CD}_1, \quad \partial t \mapsto D_t + B\partial v \in \mathcal{CD}$$

in the bundle $\mathcal{E} \times \mathbb{R} \rightarrow \mathbb{R}^2$.

The infinitesimal part of this parametric family is

$$\frac{\partial A}{\partial \lambda} dx + \frac{\partial B}{\partial \lambda} dt = dx \otimes \partial v - (2u + 8\lambda + 4v^2) dt \otimes \partial v.$$

This 1-cocycle is not exact, which reflects the well-known fact that the parameter here is essential.

According to Theorem 6, the cocycle c_S corresponding to a symmetry S of (58) is exact if and only if the symmetry is lifted to a symmetry of system (56), (57). It is well known which symmetries of (58) can be lifted. Namely, all (x, t) -independent symmetries (including the symmetries corresponding to the higher KdV equations) are lifted, while the Galilean symmetry is not. The scaling symmetry can be lifted only in the case $\lambda = 0$.

Remark 12. For an equation \mathcal{E} in two independent variables existence of a nontrivial covering is an important indication of integrability and often leads to Bäcklund transformations and the inverse scattering method [11]. For non-overdetermined equations in more than two independent variables there are no nontrivial finite-dimensional coverings (this was proved in [12] for linear coverings and in [5] for arbitrary coverings). The notion of flat representation may serve as a good substitute in this case, see a typical example in the next subsection.

5.3. Flat representations of the self-dual Yang–Mills equations

Consider four smooth vector-functions $A_i : \mathbb{R}^4 \rightarrow \mathfrak{g}$, $i = 1, 2, 3, 4$, where \mathfrak{g} is a finite-dimensional Lie algebra. The self-dual Yang–Mills equations are

$$[\partial x_1 + A_1 + \lambda(\partial x_3 + A_3), \partial x_2 + A_2 + \lambda(\partial x_4 + A_4)] = 0, \tag{62}$$

which must hold for all values of the parameter λ (see, for example, [15]). Here x_i are coordinates in \mathbb{R}^4 . Denote by \mathcal{E} the infinite prolongation of (62) and by \mathcal{C} the Cartan connection in the bundle $\xi : \mathcal{E} \rightarrow \mathbb{R}^4$.

Fix an action $\sigma : \mathfrak{g} \rightarrow D(W)$ of \mathfrak{g} on a finite-dimensional manifold W . Similarly to Section 5.1, we construct a trivial extension $\mathcal{E}' = \mathcal{E} \times W$ of \mathcal{E} as follows. Consider the flat connection

$$C_1 = \mathcal{C} \oplus \text{id} : D(\mathbb{R}^4 \times W) \rightarrow D(\mathcal{E}')$$

in the bundle $\xi_1 = \xi \times \text{id} : \mathcal{E}' \rightarrow \mathbb{R}^4 \times W$.

Evidently, the pair (\mathcal{E}', C_1) is isomorphic to an infinitely prolonged equation with $\mathbb{R}^4 \times W$ as the space of independent variables and C_1 as the Cartan connection. Consider the bundle

$$\begin{aligned} \pi : \mathbb{R}^4 \times W &\rightarrow \mathbb{R}^2, & (a, w) &\mapsto (x_1, x_2) \in \mathbb{R}^2, \\ a &= (x_1, x_2, x_3, x_4) \in \mathbb{R}^4, & w &\in W. \end{aligned}$$

The bundle π and the flat connection \mathcal{F} in the bundle $\pi \circ \xi_1$ given by

$$\mathcal{F}(\partial x_i) = C_1(\partial x_i + \sigma(A_i) + \lambda(\partial x_{i+2} + \sigma(A_{i+2}))), \quad i = 1, 2, \tag{63}$$

constitute a λ -dependent family of flat representations for the (trivially extended) self-dual Yang–Mills equations (62).

The 1-cocycle $C_1(\partial x_3 + \sigma(A_3)) dx^1 + C_1(\partial x_4 + \sigma(A_4)) dx^2$ corresponding to the family of flat representations (63) is not exact, which says that the parameter in (63) is essential.

Let $H : \mathcal{E} \rightarrow \mathfrak{g}$ be a smooth function; then the formula

$$G_H(A_i) = D_{x_i}(H) - [H, A_i] \tag{64}$$

defines a higher symmetry of \mathcal{E} . Since in the case when H depends on x_i only (does not depend on A_i and their derivatives) this is a classical gauge symmetry, it is natural to call (64) the *generalized gauge symmetry* corresponding to H .

By the definition in Section 4.3, the 1-cocycle corresponding to the symmetry G_H is

$$\begin{aligned} [\overline{U}_{\mathcal{F}}, G_H] &= \sum_{i=1,2} [D_{x_i} + \sigma(A_i) + \lambda(D_{x_{i+2}} + \sigma(A_{i+2})), G_H] dx^i \\ &= -\sigma(G_H(A_1 + \lambda A_3)) dx^1 - \sigma(G_H(A_2 + \lambda A_4)) dx^2 = -[\overline{U}_{\mathcal{F}}, \sigma(H)]. \end{aligned} \tag{65}$$

The last equality in (65) says that this cocycle is exact.

On the other hand, it can be shown that cocycles corresponding to classical conformal symmetries of (62) (see, for example, [14]) are not exact.

Remark 13. Some other symmetries of (62) (see [14] and references therein) have been extensively studied, but they are *nonlocal* and, therefore, lie out of scope of the present article.

Remark 14. Various integrable reductions of the self-dual Yang–Mills equations possess similar to (62) Lax pairs (see [1,15] and references therein) and, therefore, admit flat representations.

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