On the dimension of global sections of adjoint bundles for polarized 3-folds and 4-folds

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Abstract

Let \((X, L)\) be a polarized manifold of dimension \(n\) defined over the field of complex numbers. In this paper, we treat the case where \(n = 3\) and \(n = 4\). First we study the case of \(n = 3\) and we give an explicit lower bound for \(h^0(K_X + L)\) if \(\kappa(X) \geq 0\). Moreover, we show the following: if \(\kappa(K_X + L) \geq 0\), then \(h^0(K_X + L) > 0\) unless \(\kappa(X) = -\infty\) and \(h^1(O_X) = 0\). This gives us a partial answer of Effective Non-vanishing Conjecture for polarized 3-folds. Next for \(n = 4\) we investigate the dimension of \(H^0(K_X + mL)\) for \(m \geq 2\). If \(n = 4\) and \(\kappa(X) \geq 0\), then a lower bound for \(h^0(K_X + mL)\) is obtained. We also consider a conjecture of Beltrametti–Sommese for 4-folds and we can prove that this conjecture is true unless \(\kappa(X) = -\infty\) and \(h^1(O_X) = 0\). Furthermore we prove the following: if \((X, L)\) is a polarized 4-fold with \(\kappa(X) \geq 0\) and \(h^1(O_X) > 0\), then \(h^0(K_X + L) > 0\).

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1. Introduction

Let \(X\) be a projective variety of dimension \(n\) defined over the field of complex numbers, and let \(L\) be an ample line bundle on \(X\). Then \((X, L)\) is called a polarized variety. If \(X\) is smooth, then we say that \((X, L)\) is a polarized manifold.

When \(X\) is smooth, adjoint bundles \(K_X + mL\) of \((X, L)\) play an important role for investigating this \((X, L)\) (for example, see [3, Chapter 7.9, and 11]), where \(K_X\) is the canonical line bundle of \(X\). In particular, it is important to get an explicit lower bound for \(h^0(K_X + mL)\).

The motivation of this research is to investigate a conjecture of Beltrametti–Sommese. In [3, Conjecture 7.2.7], Beltrametti and Sommese gave the following conjecture.

Conjecture 1 (Beltrametti–Sommese). Let \((X, L)\) be an \(n\)-dimensional polarized manifold with \(n \geq 3\). Assume that \(K_X + (n - 1)L\) is nef. Then \(h^0(K_X + (n - 1)L) > 0\).
For this conjecture, the following results have been obtained.

(1.a) In [25, Theorem 4.1] and [9, Theorem 3.5], it was proved that Conjecture 1 is true if $\dim \text{Bs}|L| \leq 0$, where $\text{Bs}|L|$ is the base locus of the complete linear system $|L|$.

(1.b) In [16] we investigated Conjecture 1, and we proved that Conjecture 1 is true for $n = 3$ and we gave an explicit lower bound for $h^0(K_X + 2L)$. Moreover we obtained a lower bound for $h^0(K_X + mL)$ for $m \geq 3$. We also obtained a classification of $(X, L)$ such that $K_X + 2L$ is nef and $h^0(K_X + 2L) = 1$.

Furthermore there is the following conjecture [1, Section 4], [21, Conjecture 2.1].

**Conjecture 2 (Ambro).** Let $X$ be a complex normal variety, $B$ an effective $\mathbb{R}$-divisor on $X$ such that the pair $(X, B)$ is KLT, and $D$ a Cartier divisor on $X$. Assume that $D$ is nef, and that $D - (K_X + B)$ is nef and big. Then $h^0(D) > 0$.

For Conjecture 2, the following results have been obtained.

(2.a) If $\dim X = 2$, then Conjecture 2 is true (see [21, Theorem 3.1]).

(2.b) Let $X$ be a 3-dimensional projective variety with at most canonical singularities such that $K_X$ is nef, and let $D$ be a Cartier divisor such that $D - K_X$ is nef and big. Then $h^0(D) > 0$ (see [21, Proposition 4.1]).

(2.c) Let $X$ be a 4-dimensional projective variety with at most Gorenstein canonical singularities. Assume that $D \sim -K_X$ is ample. Then $h^0(D) > 0$ (see [21, Theorem 5.2]).

(2.d) Let $X$ be a smooth projective variety of dimension 3 with $h^1(O_X) > 0$, and $L$ a nef and big Cartier divisor on $X$ such that $K_X + L$ is nef. Then $h^0(K_X + L) > 0$ (see [5, Theorem 4.2]).

In [4], J.A. Chen, M. Chen and Zhang proposed another effective nonvanishing problem.

In this paper, we consider the positivity of $h^0(K_X + mL)$ when $X$ is smooth with dim $X = 3$ or 4. As corollaries, we can give a partial answer for Conjectures 1 and 2 when $X$ is smooth. Here we use a method similar to that in [16], that is, we use properties of the sectional geometric genus and the sectional $H$-arithmetic genus of $(X, L)$. The method of this paper seems to be very useful because not only can we prove that $h^0(K_X + mL) > 0$ but also we can classify $(X, L)$ by the value of $h^0(K_X + mL)$.

Here we explain the $i$th sectional geometric genus (or the $i$th sectional $H$-arithmetic genus) of $(X, L)$. In [10], in order to study polarized varieties more deeply, the author introduced the notion of the $i$th sectional geometric genus $g_i(X, L)$ of $(X, L)$ for every integer $i$ with $0 \leq i \leq n$ (see Definition 2.2 below). This is a generalization of the degree $L^n$ and the sectional genus $g(L)$ of $(X, L)$. Namely $g_0(X, L) = L^n$ and $g_1(X, L) = g(L)$.

Here we recall the reason why this invariant is called the $i$th sectional geometric genus. Let $(X, L)$ be a polarized manifold of dimension $n \geq 2$ with $\text{Bs}|L| = \emptyset$. Let $i$ be an integer with $1 \leq i \leq n$. Let $X_{n-i}$ be the transversal intersection of general $n - i$ members of $|L|$. In this case $X_{n-i}$ is a smooth projective variety of dimension $i$. Then we can prove that $g_i(X, L) = h^0(O_{X_{n-i}})$, that is, $g_i(X, L)$ is the geometric genus of $X_{n-i}$.

Hence we can expect that $g_i(X, L)$ has analogous properties of the geometric genus of $i$-dimensional varieties.

In [12,15], we defined the $i$-th sectional $H$-arithmetic genus $\chi^H_i(X, L)$ of $(X, L)$ (see Definition 2.2 below). By definition we can prove that if $\text{Bs}|L| = \emptyset$, then $\chi^H_i(X, L) = \chi(O_{X_{n-i}})$, where $X_{n-i}$ is the transversal intersection of general $n - i$ members of $|L|$. Namely $\chi^H_i(X, L)$ is the Euler–Poincaré characteristic of the structure sheaf of $X_{n-i}$.

We note that $\chi(O_{X_{n-i}})$ is called the arithmetic genus of $X_{n-i}$ in the sense of Hirzebruch (see [18, 15.5 (13), Section 15, Chapter IV]). We also call $\chi(O_{X_{n-i}})$ the $H$-arithmetic genus of $X_{n-i}$.

Hence we can also expect that $\chi^H_i(X, L)$ has analogous properties of the $H$-arithmetic genus of $i$-dimensional varieties. In particular, if $i = 2$, then we can expect that $\chi^H_2(X, L)$ has analogous properties of the $H$-arithmetic genus of surfaces.

Let $S$ be a smooth projective surface. Then Castelnuovo proved that $\chi(O_S) \geq 0$ (resp. $> 0$) if $\kappa(S) \geq 0$ (resp. $= 2$).

In [12] or [15] we proposed a polarized version of this result.

**Conjecture 3 (See [12, Section 3]).** Let $X$ be a smooth projective variety of dimension $n \geq 3$, and let $L$ be an ample line bundle on $X$. Then

1. $\chi^H_2(X, L) > 0$ if $\kappa(K_X + (n-2)L) \geq 2$.
2. $\chi^H_2(X, L) \geq 0$ if $0 \leq \kappa(K_X + (n-2)L) \leq 1$. 
Here we note that $\chi_2^H(X, L) > 0$ is equivalent to $g_2(X, L) \geq h^1(O_X)$ (see Remark 2.1(5) below).

For Conjecture 3(2), we obtained the following more stronger result [15, Theorem 3.2.1]: Let $X$ be a smooth projective variety of dimension $n \geq 3$, and let $L$ be an ample line bundle on $X$. If $0 \leq \kappa(K_X + (n-2)L) \leq 1$, then $\chi_2^H(X, L) > 0$.

Moreover Conjecture 3(1) is true if $(X, L)$ is one of the following:

(3.a) The case where $n = 3$ (see [15, Theorem 3.3.1(2)]).
(3.b) The case where $n \geq 4$ and $\kappa(X) \geq 0$ (see [11, Corollary 3.5.2(1)] or [14, Theorem 2.3.2]).

In fact we use this result in order to investigate a lower bound for $h^0(K_X + mL)$.

The contents of this paper are the following: In Section 2, we state some results which are used later. In Section 3, first, for any polarized manifolds $(X, L)$ with dim $X \geq 3$ and $\kappa(X) \geq 0$ we investigate a lower bound for $\chi_2^H(X, L) - \chi_3^H(X, L)$ (see Theorem 3.1). By using this lower bound, some results for dim $X = 3$ or 4 are obtained.

If dim $X = 3$ and $\kappa(X) \geq 0$, then we get a lower bound for $h^0(K_X + L)$ (see Theorem 3.2 below). We note that in [5, Theorem 4.2] Chen and Hacon have obtained that $h^0(K_X + L) > 0$ if dim $X = 3$, $h^1(O_X) > 0$ and $K_X + L$ is nef. But in Theorem 3.2, we get an explicit lower bound for $h^0(K_X + L)$ by using intersection numbers $L^3$ and $K_X L^2$ if $\kappa(X) \geq 0$. We also note that in Theorem 3.2 we do NOT assume that $K_X + L$ is nef. Furthermore the lower bound in Theorem 3.2 is important and useful because this makes us possible to classify $(X, L)$ by the value of $h^0(K_X + L)$. In this paper, we study $(X, L)$ with dim $X = 3$, $\kappa(X) \geq 0$ and $h^0(K_X + L) = 1$ (see Proposition 3.1 and Example 3.1). And we can also show that $h^0(K_X + L) > 0$ unless $\kappa(X) = -\infty$ and $h^1(O_X) = 0$ if $\kappa(K_X + L) \geq 0$ (see Theorem 3.3). This gives us a partial answer of Conjecture 2 (see Corollary 3.1).

For dim $X = 4$, we get an explicit lower bound for $h^0(K_X + mL)$ under the assumption that $\kappa(X) \geq 0$ and $m \geq 2$ (see Theorem 3.4). Moreover we prove that $h^0(K_X + mL) > 0$ for every integer $m$ with $m \geq 2$ unless $\kappa(X) = -\infty$ and $h^1(O_X) = 0$ if $\kappa(K_X + mL) \geq 0$ (see Theorem 3.6), which gives us a partial answer of Conjecture 1 (see Corollary 3.3 below). We also prove that $h^0(K_X + L) > 0$ if $(X, L)$ is a polarized 4-fold with $\kappa(X) \geq 0$ and $h^1(O_X) > 0$ (see Theorem 3.7).

Notation and conventions

We say that $X$ is a variety if $X$ is an integral separated scheme of finite type. In particular $X$ is irreducible and reduced if $X$ is a variety. Varieties are always assumed to be defined over the field of complex numbers. In this article, we shall study mainly a smooth projective variety. The words “line bundles” and “Cartier divisors” are used interchangeably. The tensor products of line bundles are denoted additively.

$\mathcal{O}(D)$: invertible sheaf associated with a Cartier divisor $D$ on $X$.
$\mathcal{O}_X$: the structure sheaf of $X$.
$\chi(F)$: the Euler–Poincaré characteristic of a coherent sheaf $F$.
$h^i(F) := \dim H^i(X, F)$ for a coherent sheaf $F$ on $X$.
$h^i(D) := h^i(\mathcal{O}(D))$ for a Cartier divisor $D$.
$q(X) := h^1(O_X)$: the irregularity of $X$.
$|D|$: the complete linear system associated with a divisor $D$.
$K_X$: the canonical divisor of $X$.
$\kappa(D)$: the Iitaka dimension of a Cartier divisor $D$ on $X$.
$\kappa(X)$: the Kodaira dimension of $X$.
$\mathbb{P}^n$: the projective space of dimension $n$.
$\mathbb{Q}^n$: a hyperquadric surface in $\mathbb{P}^{n+1}$.
$\sim$ (or $=)$: linear equivalence.
$\equiv$: numerical equivalence.

For a real number $m$ and a non-negative integer $n$, let

$$[m]^n := \begin{cases} m(m+1) \cdots (m+n-1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0, \end{cases}$$

$$[m]_n := \begin{cases} m(m-1) \cdots (m-n+1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Then for $n$ fixed, $[m]^n$ and $[m]_n$ are polynomials in $m$ whose degree are $n$. 
For any non-negative integer \( n \),
\[
\begin{align*}
\lfloor n \rfloor & := \begin{cases} 
\lfloor n \rfloor & \text{if } n \geq 1, \\
1 & \text{if } n = 0.
\end{cases}
\end{align*}
\]
Assume that \( m \) and \( n \) are integers with \( n \geq 0 \). Then we put
\[
\binom{m}{n} := \frac{\lfloor m \rfloor}{n!}.
\]
We note that \( \binom{m}{n} = 0 \) if \( m \) and \( n \) are integers with \( 0 \leq m < n \), and \( \binom{m}{0} = 1 \).

2. Preliminaries

**Definition 2.1.** Let \( X \) be a projective variety of dimension \( n \). Then \( \chi(O_X) \) is called the arithmetic genus in the sense of Hirzebruch (see [18, 15.5(13), Section 15, Chapter IV]). We also call this \( \chi(O_X) \) the \( H \)-arithmetic genus of \( X \).

**Notation 2.1.** Let \( X \) be a projective variety of dimension \( n \) and let \( L \) be a line bundle on \( X \). Then we put
\[
\chi(tL) = \sum_{j=0}^{n} \chi_j(X, L) \frac{[t]^j}{j!}.
\]

**Definition 2.2 (10, Definition 2.1) and [15, Definition 2.1].** Let \( X \) be a projective variety of dimension \( n \) and let \( L \) be a line bundle on \( X \).

1. For every integer \( i \) with \( 0 \leq i \leq n \), the \( i \)-th sectional geometric genus \( g_i(X, L) \) of \((X, L)\) is defined by the following:
   \[
   g_i(X, L) := (-1)^i (\chi_{n-i}(X, L) - \chi(O_X)) + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(O_X).
   \]

2. For every integer \( i \) with \( 0 \leq i \leq n \), the \( i \)-th sectional \( H \)-arithmetic genus \( \chi^H_i(X, L) \) of \((X, L)\) is defined by the following:
   \[
   \chi^H_i(X, L) := \chi_{n-i}(X, L).
   \]

**Remark 2.1.**

1. Since \( \chi_{n-i}(X, L) \in \mathbb{Z} \), \( \chi^H_i(X, L) \) and \( g_i(X, L) \) are integers by definition.
2. If \( i = \dim X = n \), then \( g_n(X, L) = h^n(O_X) \) and \( \chi^H_n(X, L) = \chi(O_X) \).
3. If \( i = 0 \), then \( g_0(X, L) = L^n \) and \( \chi^H_0(X, L) = L^n \).
4. If \( i = 1 \), then \( g_1(X, L) = g(L) \), where \( g(L) \) is the sectional genus of \((X, L)\).
5. In general for every integer \( i \) with \( 1 \leq i \leq n \) we get
   \[
   \chi^H_i(X, L) = 1 - h^1(O_X) + \cdots + (-1)^{i-1} h^{i-1}(O_X) + (-1)^i g_1(X, L).
   \]

**Definition 2.3.** A line bundle \( L \) on a variety \( V \) is said to be \( k \)-big if \( \kappa(L) \geq \dim V - k \). Here we note the following:

(a) \( L \) is big if and only if \( L \) is 0-big.
(b) If \( L \) is \( k \)-big, then \( L \) is \((k+1)\)-big.

**Theorem 2.1.** Let \( X \) be a smooth projective variety of dimension \( n \geq 3 \), and let \( H_1, \ldots, H_{n-2} \) be ample Cartier divisors on \( X \). Let \( B \) be an ample \( \mathbb{Q} \)-Cartier divisor on \( X \), and let \( E \) be a vector bundle of rank \( r \) on \( X \) such that \( E \) is generically \( B \)-semipositive and \( c_1(E) + r B \) is nef and \((n-2)\)-big, where \( B := \{ H_1, \ldots, H_{n-2} \} \).

Then
\[
c_2(E) H_1 \cdots H_{n-2} \geq -(r-1)c_1(E) BH_1 \cdots H_{n-2} - \binom{r}{2} B^2 H_1 \cdots H_{n-2}.
\]

**Proof.** See [14, Theorem 2.1]. □
Theorem 2.2. Let $X$ be a smooth projective variety of dimension $n \geq 3$, and let $H_1, \ldots, H_{n-2}$ be ample Cartier divisors on $X$. Assume that $X$ is not uniruled. Then $\Omega^1_X$ is generically $B$-semipositive, where $B := \{H_1, \ldots, H_{n-2}\}$.

Proof. See [22, Corollary 6.4]. □

Theorem 2.3. Let $(X, L)$ be a polarized manifold of dimension $n$. Then for every integer $i$ with $0 \leq i \leq n - 1$

$$g_i(X, L) = \sum_{j=0}^{n-i-1} (-1)^j \binom{n-i}{j} h^0(K_X + (n-i-j)L) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(O_X).$$

Proof. See [10, Theorem 2.3]. □

Proposition 2.1. Let $(X, L)$ be a polarized manifold of dimension $n \geq 3$. Then

$$g_2(X, L) = \frac{(n-2)(3n-5)}{24} L^n + \frac{n-2}{4} K_X L^{n-1} + \frac{1}{12} (K_X^2 + c_2(X)) L^{n-2} - 1 + h^1(O_X)$$

and

$$g_3(X, L) = \frac{(n-2)(n-3)^2}{48} L^n + \frac{(n-3)(3n-8)}{48} K_X L^{n-1}$$

$$+ \frac{n-3}{24} (K_X^2 + c_2(X)) L^{n-2} + \frac{1}{24} K_X c_2(X) L^{n-3} + 1 - h^1(O_X) + h^2(O_X).$$

Proof. See [13, (2.2.A) and (2.2.B)]. □

Proposition 2.2. Let $(X, L)$ be a polarized manifold of dimension $n \geq 3$. Then

$$\chi^H_2(X, L) = \frac{(n-2)(3n-5)}{24} L^n + \frac{n-2}{4} K_X L^{n-1} + \frac{1}{12} (K_X^2 + c_2(X)) L^{n-2}$$

and

$$\chi^H_3(X, L) = -\frac{(n-2)(n-3)^2}{48} L^n - \frac{(n-3)(3n-8)}{48} K_X L^{n-1}$$

$$- \frac{n-3}{24} (K_X^2 + c_2(X)) L^{n-2} - \frac{1}{24} K_X c_2(X) L^{n-3}.$$ 

Proof. Since $\chi^H_2(X, L) = 1 - h^1(O_X) + g_2(X, L)$ and $\chi^H_3(X, L) = 1 - h^1(O_X) + h^2(O_X) - g_3(X, L)$, we get the assertion by Proposition 2.1. □

Definition 2.4. (1) Let $X$ (resp. $Y$) be an $n$-dimensional projective manifold, and let $L$ (resp. $A$) be an ample line bundle on $X$ (resp. $Y$). Then $(X, L)$ is called a simple blowing up of $(Y, A)$ if there exists a birational morphism $\pi : X \to Y$ such that $\pi$ is a blowing up at a point of $Y$ and $L = \pi^*(A) - E$, where $E$ is the $\pi$-exceptional reduced divisor.

(2) Let $X$ (resp. $X'$) be an $n$-dimensional projective manifold, and let $L$ (resp. $L'$) be an ample line bundle on $X$ (resp. $X'$). Then we say that $(X', L')$ is a reduction of $(X, L)$ if $(X, L)$ is obtained by a composite of simple blowing ups of $(X', L')$, and $(X', L')$ is not obtained by a simple blowing up of any polarized manifold. The morphism $\mu : X \to X'$ is called the reduction map.

Remark 2.2. (1) Let $(X, L)$ be a polarized manifold of dimension $n$ and $(X', L')$ a reduction of $(X, L)$. Then we obtain $g_i(X, L) = g_i(X', L')$ and $\chi^H_i(X, L) = \chi^H_i(X', L')$ for any integer $i$ with $1 \leq i \leq n$ (see [10, Proposition 2.6] and [15, Remark 2.1 (5)]).

(2) Let $(X, L)$ be a polarized manifold. If $(X, L)$ is not obtained by a simple blowing up of another polarized manifold, then $(X, L)$ is a reduction of itself.

(3) For any polarized manifold $(X, L)$, there exists a reduction of $(X, L)$. (See [7, (11.11), Chapter II].)
Theorem 2.4. Let \((X, L)\) be a polarized manifold of dimension \(n \geq 3\). Then \((X, L)\) is one of the following types:

1. \((\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)).\)
2. \((\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)).\)
3. A scroll over a smooth curve.
4. \(K_X \sim -(n - 1)L\), that is, \((X, L)\) is a Del Pezzo manifold.
5. A quadric fibration over a smooth curve \(C\).
6. A scroll over a smooth surface \(S\).

(7) Let \((X', L')\) be a reduction of \((X, L)\).

(7.1) \(n = 4\), \((X', L') = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)).\)

(7.2) \(n = 3\), \((X', L') = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)).\)

(7.3) \(n = 3\), \((X', L') = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)).\)

(7.4) \(n = 3\), \(X'\) is a \(\mathbb{P}^2\)-bundle over a smooth curve \(C\) such that \((F', L'|_{F'}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))\) for any fiber \(F'\) of it.

(7.5) \(K_{X'} \sim -(n - 2)L'\), that is, \((X', L')\) is a Mukai manifold.

(7.6) \((X', L')\) is a Del Pezzo fibration over a smooth curve.

(7.7) \((X', L')\) is a quadric fibration over a normal surface.

(7.8) \(n \geq 4\), and there exist a normal projective variety \(W\) with \(\dim W = 3\) and a fiber space \(\Phi : X' \to W\) such that \((F', L'|_{F'}) \cong (\mathbb{P}^{n-3}, \mathcal{O}_{\mathbb{P}^{n-3}}(1))\) for a general fiber \(F'\) of \(\Phi\).

(7.9) \(K_{X'} + (n - 2)L'\) is nef and big.

Proof. See [3, Proposition 7.2.2, Theorems 7.2.4, 7.3.2, 7.3.4, and 7.5.3]. See also [7, Chapter II, (11.2), (11.7), and (11.8)], or [20, Theorem]. □

Remark 2.3. Let \((X, L)\) be a polarized manifold of dimension \(n \geq 3\) and let \((Y, A)\) be a reduction of \((X, L)\). Assume that \(\kappa(X) \geq 0\). Then by Theorem 2.4 we see that \(K_X + (n - 1)L\) is ample and \(K_Y + (n - 2)L\) is nef and big.

Theorem 2.5. Let \((X, L)\) be a polarized manifold of dimension \(n \geq 3\). Assume that \(\kappa(X) \geq 0\). Then \(g_2(X, L) \geq h^1(\mathcal{O}_X)\).

Proof. See [11, Corollary 3.5.2(1)] or [14, Theorem 2.3.2]. □

Theorem 2.6. Let \((X, L)\) be a polarized manifold of dimension \(n \geq 3\). For every integer \(m\) with \(2 \leq m\), we get the following equality:

\[
h^0(K_X + mL) - h^0(K_X + (m - 1)L) = \sum_{s=0}^{n-1} \binom{m - 1}{n - s - 1} g_s(X, L) - \sum_{s=0}^{n-2} \binom{m - 2}{n - s - 2} h^s(\mathcal{O}_X).
\]

Proof. See [16, Theorem 2.1]. □

Theorem 2.7. Let \((X, L)\) be a polarized manifold of dimension 3.

1. If \(K_X + 2L\) (resp. \(K_X + 3L\)) is nef, then \(h^0(K_X + 2L) > 0\) (resp. \(h^0(K_X + 3L) > 0\)).
2. Let \(m\) be an integer with \(m \geq 4\). Then

\[
h^0(K_X + mL) \geq \binom{m - 1}{3} > 0.
\]

Proof. See [16, Theorems 2.4 and 2.5]. □

Corollary 2.1. Let \((X, L)\) be a polarized manifold of dimension 3. Assume that \(\kappa(K_X + mL) \geq 0\) for an integer \(m\) with \(m \geq 2\). Then \(h^0(K_X + mL) > 0\).
**Proof.** First we consider the case where \( m = 2 \). If \( h^0(K_X + 2L) = 0 \), then \( K_X + 2L \) is not nef by Theorem 2.7(1). Hence by [3, Proposition 7.2.2 and Theorem 7.2.4], [7, (11.2) Theorem and (11.7) Theorem] or [20, Theorem] \((X, L)\) is \((\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))\), \((\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(1))\) or a scroll over a smooth curve. But in these cases, \( \kappa(K_X + 2L) = -\infty \) and this contradicts the assumption.

Next we consider the case where \( m = 3 \). If \( h^0(K_X + 3L) = 0 \), then \( K_X + 3L \) is not nef by Theorem 2.7(1). Hence by [3, Proposition 7.2.2], [7, (11.2) Theorem] or [20, Theorem] \((X, L)\) is \((\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))\). But in this case, \( \kappa(K_X + 3L) = -\infty \) and this contradicts the assumption.

For every integer \( m \) with \( m \geq 4 \), \( h^0(K_X + mL) > 0 \) by Theorem 2.7(2). Hence we get the assertion. \( \Box \)

**Remark 2.4.** If \((X, L)\) is a polarized manifold of dimension 3, then \( K_X + mL \) is always nef for every integer \( m \) with \( m \geq 4 \). Hence by the non-vanishing theorem [24] \( \kappa(K_X + mL) \geq 0 \) for every integer \( m \) with \( m \geq 4 \).

**Theorem 2.8.** Let \((X, L)\) be a polarized manifold of dimension \( n = 1 \) or 2. If \( \kappa(K_X + mL) \geq 0 \) for \( m \geq 1 \), then \( h^0(K_X + mL) > 0 \).

**Proof.** If \( n = 1 \), then by the Kodaira vanishing theorem and the Riemann–Roch theorem we have \( h^0(K_X + mL) = \deg L + h^1(O_X) - 1 \).

If \( h^1(O_X) > 0 \), then \( h^0(K_X + mL) > 0 \).

So we may assume that \( h^1(O_X) = 0 \). Then by assumption \( 0 \leq \deg(K_X + mL) = \deg L - 1 \). Hence \( h^0(K_X + mL) = \deg L - 1 \geq 1 \) and we get the assertion for \( n = 1 \).

If \( n = 2 \), then by the Kodaira vanishing theorem and the Riemann–Roch theorem we have

\[
h^0(K_X + mL) = \chi(O_X) + \frac{1}{2}(K_X + mL)(mL) - 1 = \chi(O_X) + g_1(X, L) + \frac{m - 1}{2}(K_X + (m + 1)L) - 1.
\]

Here we note that \( g_1(X, L) = 1 + \frac{1}{2}(K_X + L), \) which is the sectional genus of \((X, L)\) (see Remark 2.1(4)). We also note that \((K_X + mL)L \geq 0 \) by assumption.

If \( \kappa(X) \geq 0 \), then \( \chi(O_X) \geq 0 \) and \( g_1(X, L) \geq 2 \). Hence

\[
h^0(K_X + mL) = \chi(O_X) + g_1(X, L) + \frac{m - 1}{2}(K_X + (m + 1)L) - 1 = \chi(O_X) + g_1(X, L) - 1 + \frac{m - 1}{2}(K_X + mL) + \frac{m - 1}{2}L^2 > 0.
\]

If \( \kappa(X) = -\infty \), then \( \chi(O_X) = 1 - h^1(O_X) \) and by [8, Theorem 2.1] we have \( g_1(X, L) \geq h^1(O_X) \). Hence

\[
h^0(K_X + mL) = \chi(O_X) + g_1(X, L) + \frac{m - 1}{2}(K_X + (m + 1)L) - 1 \geq \frac{m - 1}{2}(K_X + (m + 1)L).
\]

If \( m \geq 2 \), then \( h^0(K_X + mL) > 0 \). So we assume that \( m = 1 \). Since \( L \) is ample, \( \kappa(X) = -\infty \) and \( \kappa(K_X + L) \geq 0 \) by assumption, we get \( g_1(X, L) = h^1(O_X) \) by [8, Theorem 3.1]. Therefore \( g_1(X, L) \geq h^1(O_X) + 1 \) and

\[
h^0(K_X + L) = \chi(O_X) + g_1(X, L) - 1 = g_1(X, L) - h^1(O_X) > 0.
\]

This completes the proof. \( \Box \)

3. Main results

**Theorem 3.1.** Let \((X, L)\) be a polarized \( n \)-fold with \( n \leq 3 \). Assume that \( \kappa(X) \geq 0 \). Let \((M, A)\) be a reduction of \((X, L)\). Then

\[
-\chi_3^H(X, L) + \chi_2^H(X, L) \geq \frac{(n - 1)(n - 2)(2n - 1)}{24n}A^n + \frac{2n - 3}{24}K_MA^{n-1} > 0.
\]
Proof. By Proposition 2.2 we get
\[-\chi_3^H(M, A) + \chi_2^H(M, A) = \frac{(n - 2)(n^2 - 1)}{48} A^n + \frac{n(3n - 5)}{48} K_M A^{n-1}\]
\[+ \frac{n - 1}{24} K_M^2 A^{n-2} + \frac{1}{24} c_2(M)K_M + (n - 1)A A^n - 3.\]  \(1\)

By Remark 2.2(1), we obtain \[-\chi_3^H(X, L) + \chi_2^H(X, L) = -\chi_3^H(M, A) + \chi_2^H(M, A).\] Here we note that \(K_M + (n - 1)A\) is ample and \(K_M + (n - 2)A\) is nef and big by Remark 2.3 and the assumption that \(\kappa(M) = \kappa(X) \geq 0\). In Theorem 2.1, we put \(H_j := K_M + (n - 1)A, H_j := A\) for \(j = 2, \ldots, n - 2, E := O_M, \) and \(B := (n - 2)A/n\). Then by Theorems 2.1 and 2.2, for every integer \(t \) with \(t \geq 1\) we get
\[c_2(M)(K_M + (n - 1)A)A^{n-3} \geq -\chi_3^H(M, A) + \chi_2^H(M, A)\]
\[= \frac{(n - 1)K_M^2 A^{n-2}}{24} - \frac{(n - 1)(n - 2)(3n - 4)}{2n} K_M A^{n-1}\]
\[- (n - 1)^2(n - 2)^2 A^n.\]

Hence we obtain
\[-\chi_3^H(X, L) + \chi_2^H(X, L) \geq \frac{(n - 2)(n^2 - 1)}{48} A^n + \frac{n(3n - 5)}{48} K_M A^{n-1} + \frac{n - 1}{24} K_M A^{n-2}\]
\[- \frac{1}{24} \frac{(n - 1)(n - 2)}{n} K_M^2 A^{n-2} + \frac{(n - 1)(n - 2)(3n - 4)}{2n} K_M A^{n-1} + \frac{(n - 1)^2(n - 2)^2}{2n} A^n\]
\[= \frac{(n - 1)(n - 2)(2n - 1)}{24n} A^n + \frac{n - 1}{12n} K_M (K_M + (n - 2)A)A^{n-2} + \frac{2n - 3}{24} K_M A^{n-1}.\]

Therefore
\[-\chi_3^H(X, L) + \chi_2^H(X, L) \geq \frac{(n - 1)(n - 2)(2n - 1)}{24n} A^n + \frac{2n - 3}{24} K_M A^{n-1} > 0\]
because \(n \geq 3, K_M + (n - 2)A\) is nef, and \(\kappa(M) = \kappa(X) \geq 0\). Hence we get the assertion. \(\square\)

First we study the case where \(\dim X = 3\) by using Theorem 3.1.

Remark 3.1. Let \((X, L)\) be a polarized manifold of dimension 3. Then by Theorem 2.3 and Remark 2.1(2) and (5), we get
\[h^0(K_X + L) = g_2(X, L) - h^2(O_X) + h^3(O_X)\]
\[= \chi_2^H(X, L) - \chi(O_X)\]
\[= \chi_2^H(X, L) - \chi_3^H(X, L).\]

Theorem 3.2. Let \((X, L)\) be a polarized manifold of dimension 3. Let \((M, A)\) be a reduction of \((X, L)\). Assume that \(\kappa(X) \geq 0\). Then
\[h^0(K_X + L) \geq \frac{5}{36} A^3 + \frac{1}{8} K_M A^2 > 0.\]

Proof. By Remark 3.1 we have
\[h^0(K_X + L) = \chi_2^H(X, L) - \chi_3^H(X, L).\]
By Theorem 3.1, we have
\[ \chi_2^H(X, L) - \chi_3^H(X, L) \geq \frac{5}{36} A^3 + \frac{1}{8} K_M A^2 > 0. \]
Therefore we get the assertion. \( \Box \)

**Remark 3.2.** Let \((X, L)\) be a polarized manifold of dimension 3. If \(\kappa(X) \geq 0\), then by Theorem 3.2 we get \(h^0(K_X + L) > 0\) without the assumption that \(K_X + L\) is nef.

**Proposition 3.1.** Let \((X, L)\) be a polarized manifold of dimension 3. Let \((M, A)\) be a reduction of \((X, L)\). Assume that \(\kappa(X) \geq 0\). If \(h^0(K_X + L) = 1\), then \(1 \leq A^3 \leq 6\). Furthermore if \(A^3 = 6\), then \(K_M \equiv 0\).

**Proof.** By Theorem 3.2 and \(\kappa(X) = \kappa(M) \geq 0\), we get \(1 \leq A^3 \leq 7\).
If \(A^3 = 7\), then \(K_M A^2 = 0\). Because \(A\) is ample and \(\kappa(M) \geq 0\), we have \(K_M \equiv 0\). In particular \(K_M\) is nef. Hence by a Miyaoka’s result [22, Theorem 6.6] we have \(c_2(M)(K_M + 2A) \geq 0\) because \(K_M + 2A\) is ample. Hence by Remark 3.1 and (1) in the proof of Theorem 3.1
\[ h^0(K_X + L) = \chi_2^H(X, L) - \chi_3^H(X, L) \]
\[ \geq \frac{5}{36} A^3 + \frac{1}{8} K_M A^2 \]
This contradicts the assumption. Therefore \(A^3 \leq 6\).
Assume that \(A^3 = 6\) and \(K_M A^2 > 0\). Then \(K_M A^2 \geq 2\) because \(K_M A^2\) is even. Hence
\[ h^0(K_X + L) = \chi_2^H(X, L) - \chi_3^H(X, L) \]
\[ \geq \frac{5}{36} A^3 + \frac{1}{8} K_M A^2 \]
\[ \geq \frac{13}{12} > 1. \]
Therefore if \(A^3 = 6\), then \(K_M A^2 = 0\). By the same argument as above, we get \(K_M \equiv 0\). \( \Box \)

**Example 3.1.** Here we give an example of a polarized 3-fold \((X, L)\) such that \((X, L)\) is a reduction of itself, \(\kappa(X) \geq 0\) and \(h^0(K_X + L) = 1\).

1. Let \(C\) be a smooth elliptic curve, and \(A\) an ample line bundle of \(\deg A = 1\) on \(C\). We set \(X := C \times C \times C\) and \(L := p_1^* (A) + p_2^* (A) + p_3^* (A)\), where \(p_i : X \to C\) is the \(i\)th projection for \(i = 1, 2, 3\). Then this \((X, L)\) is a reduction of itself because \(K_Y + 2L\) is ample. Moreover \(L^3 = 6\) and \(h^0(K_X + L) = 1\). In general if \((X, L)\) is a principally polarized Abelian 3-fold, then \((X, L)\) is a reduction of itself, \(h^0(K_X + L) = 1\), and \(L^3 = 6\).

2. Let \((S, A_1)\) be a polarized surface such that \(S\) is minimal, \(\varphi(A_1) = 2\), \(\varphi(S) = 1\), \(h^0(K_S) = 0\), and \((A_1)^2 = 1\). This \((S, A_1)\) exists (see [23, Theorems 2.1, 2.2 and Example 2.6]). Then \(h^0(K_S + A_1) = 1\). Let \((E, A_2)\) be a polarized curve such that \(\varphi(E) = 1\) and \(\deg A_2 = 1\). Then \(h^0(K_E + A_2) = 1\). We set \(X := S \times E\) and \(L := p_1^*(A_1) + p_2^*(A_2)\), where \(p_i\) is the \(i\)th projection for \(i = 1\) or 2. Then \((X, L)\) is a reduction of itself, \(h^0(K_X + L) = 1\), and \(L^3 = 3\).

3. In [2, Theorem 1.1], Beauville gave an example of a polarized Calabi–Yau 3-fold \((X, L)\) such that \(h^0(L) = 1\) and \(L^3 = 2\). Namely there exists an example of \((X, L)\) such that \(\dim X = 3\), \(\kappa(X) \geq 0\), \((X, L)\) is a reduction of itself, \(h^0(K_X + L) = 1\) and \(L^3 = 2\). For details, see [2, Theorem 1.1].

4. Let \(\mathbb{P}^4\) be the projective space of dimension 4 and let \((\xi_0 : \xi_1 : \xi_2 : \xi_3 : \xi_4)\) be the homogeneous coordinate of it. Let \(\rho := \exp(2\pi \sqrt{-1}/5)\) and we set \(G := \langle \rho \rangle\). Then we define an action of \(G\) on \(\mathbb{P}^4\) as follows:
\[ \rho^a \cdot (\xi_0 : \xi_1 : \xi_2 : \xi_3 : \xi_4) := (\xi_0 : \rho^a \xi_1 : \rho^{2a} \xi_2 : \rho^{3a} \xi_3 : \rho^{4a} \xi_4). \]
We set
\[ Y := \{ (\xi_0 : \xi_1 : \xi_2 : \xi_3 : \xi_4) \in \mathbb{P}^4 | \sum_{i=0}^4 \xi_i^5 = 0 \}. \]
Then \(Y \in |\mathcal{O}_{\mathbb{P}^4}(5)|\), \(Y\) is smooth, \(G\) acts on \(Y\), and \(G\) is fixed point free. Hence \(X := Y/G\) is smooth, \(\dim X = 3\) and \(\pi : Y \to X\) is an etale covering of degree 5. Here we note that \(\chi(\mathcal{O}_Y) = 0\). Since \(\pi\) is etale, we have
\( \chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) / \deg \pi = 0 \). We also note that \( h^i(\mathcal{O}_X) = 0 \) for \( i = 1, 2 \) because \( h^i(\pi_* (\mathcal{O}_Y)) = h^i(\mathcal{O}_Y) = 0 \) for \( i = 1, 2 \) and \( \pi_* (\mathcal{O}_Y) = \mathcal{O}_X \oplus \mathcal{E} \) for some vector bundle \( \mathcal{E} \) on \( X \). Hence \( h^0(K_X) = h^3(\mathcal{O}_X) = h^0(\mathcal{O}_X) = 1 \).

On the other hand since \( \mathcal{O}_Y = K_Y = \pi^*(K_X) \), we see that \( K_X = 0 \). Hence we get \( K_X = \mathcal{O}_X \). Here we take a hyperplane

\[ H := \{ (\xi_0 : \xi_1 : \xi_2 : \xi_3 : \xi_4) \in \mathbb{P}^4 \mid \xi_0 = 0 \}. \]

Then \( H \in |\mathcal{O}_{\mathbb{P}^4}(1)| \). Let \( A := H \cap Y \). Then \( G \) acts on \( A \) freely and we set \( L := A/G \). Then \( A \) (resp. \( L \)) is a divisor on \( Y \) (resp. \( X \)) such that \( L \) is ample and \( A = \pi^*(L) \). We note that \( A^3 = 5 \) and therefore \( L^3 = 1 \) because \( \deg \pi = 5 \). Since \( \pi \) is an etale covering, we get

\[ \chi(A) = (\deg \pi) \chi(L) = 5 \chi(L). \]

We note that \( h^0(A) = 0 \) for any \( t \) with \( 1 \leq t \leq 3 \). Hence we have \( h^0(L) = 0 \) for any \( t \) with \( 1 \leq t \leq 3 \) because \( h^0(\pi_* (A)) = h^0(A) \) and \( \pi_* (A) = \pi_* (\pi^* L) = L \oplus (\mathcal{E} \otimes L) \). Therefore we obtain \( \chi(A) = h^0(L) \) and

\[ \chi(L) = h^0(L). \]

On the other hand since \( h^0(A) = 5 \), we have \( h^0(L) = 1 \). This \( (X, L) \) is an example with \( h^0(K_X + L) = h^0(L) = 1 \) and \( L^3 = 1 \). Here we note that this \( (X, L) \) is a reduction of itself.

**Problem 3.1.** Find an example of \( (X, L) \) such that \( \dim X = 3, \kappa(X) \geq 0 \), \( (X, L) \) is a reduction of itself, \( h^0(K_X + L) = 1 \) and \( L^3 = 4 \) or 5.

**Theorem 3.3.** Let \( (X, L) \) be a polarized 3-fold. Assume that \( \kappa(K_X + L) \geq 0 \). Then \( h^0(K_X + L) > 0 \) unless \( \kappa(X) = -\infty \) and \( h^1(\mathcal{O}_X) = 0 \).

**Proof.** If \( \kappa(X) \geq 0 \), then \( h^0(K_X + L) > 0 \) by Theorem 3.2.

So we may assume that \( \kappa(X) = -\infty \). If \( h^1(\mathcal{O}_X) > 0 \), then \( X \) has the Albanese map \( \alpha : X \to A \) such that \( \dim \alpha(X) = 1, 2 \) or 3, where \( A \) is its Albanese variety. Then by [17, Corollary 10.7, Chapter III, Section 10], a general fiber \( F_0 \) of \( \alpha \) is the following type:

\[ F_0 = \bigcup_{j=1}^r F_j, \]

where \( F_j \) is a smooth subvariety for every integer \( j \) with \( 1 \leq j \leq r \), \( \dim F_k = \dim F_l \) and \( F_k \cap F_l = \emptyset \) for any \( k \neq l \).

Here we note that if \( \kappa(K_X + mL) \geq 0 \), then \( \kappa(K_{F_j} + mL_{F_j}) \geq 0 \) for every integer \( j \) with \( 1 \leq j \leq r \). We also note that \( 0 \leq \dim F_j \leq 2 \) for every \( j \).

If \( \dim F_j = 0 \), then \( h^0(K_{F_j} + mL_{F_j}) > 0 \) for every integer \( j \) and \( m \) with \( 1 \leq j \leq r \) and \( m \geq 1 \).

If \( \dim F_j = 1 \) or 2, then by Theorem 2.8 we have \( h^0(K_{F_j} + mL_{F_j}) > 0 \) for every integer \( j \) and \( m \) with \( 1 \leq j \leq r \) and \( m \geq 1 \). Hence by [5, Lemma 4.1], we get \( h^0(K_X + mL) > 0 \). Therefore we get the assertion.

We note that if \( K_X + L \) is nef, then by the non-vanishing theorem [24] we get \( \kappa(K_X + L) \geq 0 \). Hence we get the following by Theorem 3.3:

**Corollary 3.1.** Let \( (X, L) \) be a polarized 3-fold. Assume that \( K_X + L \) is nef. Then \( h^0(K_X + L) > 0 \) unless \( \kappa(X) = -\infty \) and \( h^1(\mathcal{O}_X) = 0 \).

**Remark 3.3.** As we said in the introduction, Chen and Hacon [5, Theorem 4.2] proved that \( h^0(K_X + L) > 0 \) if \( \dim X = 3 \), \( K_X + L \) is nef, and \( h^1(\mathcal{O}_X) > 0 \). So the essential part of Corollary 3.1 is the case when \( \kappa(X) \geq 0 \) and \( h^1(\mathcal{O}_X) = 0 \).

Next we investigate the case where \( \dim X = 4 \).

**Theorem 3.4.** Let \( (X, L) \) be a polarized 4-fold. Assume that \( \kappa(X) \geq 0 \). Then for every integer \( m \) with \( m \geq 2 \),

\[ h^0(K_X + mL) \geq \left( \frac{m + 2}{4} \right) > 0. \]
Proof. By Theorem 2.6, for every integer \( m \) with \( m \geq 2 \)
\[
h^0(K_X + mL) - h^0(K_X + (m-1)L) = \sum_{s=0}^{3} \binom{m-1}{3-s} g_s(X, L) - \sum_{s=0}^{2} \binom{m-2}{2-s} h^s(\mathcal{O}_X)
\]
\[
= \left( \binom{m-1}{3} \right) L^4 + \left( \binom{m-1}{2} \right) g_1(X, L) + (m-1)g_2(X, L) + g_3(X, L)
\]
\[
- \left( \binom{m-2}{2} \right) - (m-2)h^1(\mathcal{O}_X) - h^2(\mathcal{O}_X)
\]
\[
= \left( -\chi_3^H(X, L) + \chi_2^H(X, L) \right) + (m-2)\chi_2^H(X, L)
\]
\[
+ \left( \binom{m-1}{3} \right) L^4 + \left( \binom{m-1}{2} \right) g_1(X, L) - (m-2) - \left( \binom{m-2}{2} \right).
\]

Hence
\[
h^0(K_X + mL) = h^0(K_X + L) + (m-1) \left( -\chi_3^H(X, L) + \chi_2^H(X, L) \right) + \sum_{k=2}^{m} \binom{k-2}{2} \chi_2^H(X, L)
\]
\[
+ \sum_{k=2}^{m} \left( \binom{k-1}{3} L^4 + \binom{k-1}{2} g_1(X, L) - \sum_{k=2}^{m} \binom{k-2}{2} \right)
\]
\[
= h^0(K_X + L) + (m-1) \left( -\chi_3^H(X, L) + \chi_2^H(X, L) \right) + \left( \binom{m-1}{2} \right) g_1(X, L)
\]
\[
+ \left( \binom{m}{4} \right) L^4 + \left( \binom{m}{3} \right) g_1(X, L) - \left( \binom{m}{2} \right) - \left( \binom{m-1}{3} \right).
\]

Theorem 2.5 and Remark 2.1(5) show that \( \chi_2^H(X, L) \geq 1 \) because \( \kappa(X) \geq 0 \). Moreover by [6, (1.10) Theorem] and [7, (12.1) Theorem and (12.3) Theorem] or [20, Corollaries 8 and 9], we have \( g_1(X, L) \geq 3 \) since \( \kappa(X) \geq 0 \).

Hence by Theorem 3.1, we have
\[
h^0(K_X + mL) \geq (m-1) + \left( \binom{m-1}{2} \right) + \left( \binom{m}{4} \right) + 3 \left( \binom{m}{3} \right) - \left( \binom{m-1}{2} \right) - \left( \binom{m-1}{3} \right)
\]
\[
= \left( \binom{m}{4} \right) + 2 \left( \binom{m}{3} \right) + \left( \binom{m}{2} \right) + (m-1)
\]
\[
= \left( \binom{m+2}{4} \right).
\]

This completes the proof. \( \square \)

By the following theorem, we see that [9, Conjecture 3.8] is true if \( \kappa(X) \geq 0 \) and \( \dim X = 4 \).

**Theorem 3.5.** Let \((X, L)\) be a polarized manifold of dimension 4 with \( \kappa(X) \geq 0 \). Then \( h^0(K_X + 3L) \geq g_1(X, L) + 2 \).

**Proof.** By using Theorem 2.6 we get
\[
h^0(K_X + 3L) = h^0(K_X + L) + 2(-\chi_3^H(X, L) + \chi_2^H(X, L)) + g_2(X, L) + g_1(X, L) - h^1(\mathcal{O}_X).
\]

On the other hand, by Theorem 3.1 (resp. Theorem 2.5), \( -\chi_3^H(X, L) + \chi_2^H(X, L) > 0 \) (resp. \( g_2(X, L) \geq h^1(\mathcal{O}_X) \)) holds. Therefore
\[
h^0(K_X + 3L) \geq g_1(X, L) + 2.
\]

This completes the proof. \( \square \)

**Theorem 3.6.** Let \((X, L)\) be a polarized 4-fold and let \( m \) be an integer with \( m \geq 2 \). Assume that \( \kappa(K_X + mL) \geq 0 \). Then \( h^0(K_X + mL) > 0 \) unless \( \kappa(X) = -\infty \) and \( h^1(\mathcal{O}_X) = 0 \).
Proof. By using Corollary 2.1, Theorems 2.8 and 3.4, we can prove this theorem by the same argument as the proof of Theorem 3.3. □

We note that if $K_X + mL$ is nef, then by the non-vanishing theorem [24] we get $\kappa(K_X + mL) \geq 0$. Hence we get the following by Theorem 3.6:

Corollary 3.2. Let $(X, L)$ be a polarized 4-fold and let $m$ be an integer with $m \geq 2$. Assume that $K_X + mL$ is nef. Then $h^0(K_X + mL) > 0$ unless $\kappa(X) = -\infty$ and $h^1(O_X) = 0$.

Corollary 3.2 includes that the Beltrametti–Sommese conjecture (see Conjecture 1 in the introduction) is true unless $\kappa(X) = -\infty$ and $h^1(O_X) = 0$.

Corollary 3.3. Let $(X, L)$ be a polarized 4-fold. Assume that $K_X + 3L$ is nef. Then $h^0(K_X + 3L) > 0$ unless $\kappa(X) = -\infty$ and $h^1(O_X) = 0$.

Furthermore we can prove the following:

Theorem 3.7. Let $(X, L)$ be a polarized manifold of dimension 4. Assume that $\kappa(X) \geq 0$ and $h^1(O_X) > 0$. Then $h^0(K_X + L) > 0$.

Proof. Let $\alpha : X \to A$ be the Albanese map of $X$, where $A$ is its Albanese variety. We put $Y := \alpha(X)$. Then $1 \leq \dim Y \leq 4$ and by [17, Corollary 10.7, Chapter III, Section 10], a general fiber $F_\alpha$ of $\alpha$ is the following type:

$$F_\alpha = \bigcup_{j=1}^r F_j,$$

where $F_j$ is a smooth subvariety for every integer $j$ with $1 \leq j \leq r$, dim $F_k = \dim F_l$ and $F_k \cap F_l = \emptyset$ for any $k \neq l$.

Hence

$$h^0(K_{F_\alpha} + L_{F_\alpha}) = \sum_{j=1}^r h^0(K_{F_j} + L_{F_j}).$$

Claim 3.1. $h^0(K_{F_j} + L_{F_j}) > 0$ for every $j$.

Proof. Here we note that $\kappa(F_j) \geq 0$ by [19, Theorem 4] because $\kappa(X) \geq 0$.

(a) If $\dim Y = 4$, then $\dim F_j = 0$ and $h^0(K_{F_j} + L_{F_j}) > 0$ for every $j$.

(b) Next we consider the case where $\dim Y = 3$ (resp. 2). Then $\dim F_j = 1$ (resp. 2). Since $\kappa(F_j) \geq 0$ for every $j$, we see that $\kappa(K_{F_j} + L_{F_j}) \geq 0$. Hence by Theorem 2.8, we have $h^0(K_{F_j} + L_{F_j}) > 0$ for every $j$.

(c) If $\dim Y = 1$, then $\dim F_j = 3$. Since $\kappa(F_j) \geq 0$, we see that $h^0(K_{F_j} + L_{F_j}) > 0$ for every $j$ by Theorem 3.2.

Therefore we get the assertion of Claim 3.1. □

By Claim 3.1 we have $h^0(K_{F_\alpha} + L_{F_\alpha}) > 0$. Hence by [5, Lemma 4.1], we obtain $h^0(K_X + L) > 0$. □

Remark 3.4. In Theorem 3.7, we do NOT assume that $K_X + L$ is nef.

Finally we propose the following problem:

Problem 3.2. For any fixed positive integer $n$, determine the smallest positive integer $p$, which depends only on $n$, such that the following $(\ast)$ is satisfied:

$$(\ast) \ h^0(p(K_X + L)) > 0 \text{ for any polarized manifold } (X, L) \text{ of dimension } n \text{ with } \kappa(K_X + L) \geq 0.$$

Remark 3.5. By Theorem 2.8, we see that $p = 1$ if $X$ is a curve or surface.
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References