Holographic research on phase transitions for a five dimensional AdS black hole with conformally coupled scalar hair

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ABSTRACT

In the framework of holography, we survey the phase structure for a higher dimensional hairy black hole including the effects of the scalar field hair. It is worth emphasizing that, not only black hole entropy, but also entanglement entropy and two point correlation function exhibit the Van der Waals-like phase transition in a fixed scalar charge ensemble. Furthermore, by making use of numerical computation, we show that the Maxwell’s equal area law is valid for the first order phase transition. In addition, we also discuss how the hair parameter affects the black hole’s phase transition.

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1. Introduction

In recent years, in asymptotically Anti-de Sitter (AdS) spacetime, a black hole’s Van der Waals-like phase transition has attracted great attention. This phase transition was first observed by Andrew Chamblin et al. [1,2]. In their pioneering work, in the canonical ensemble, a “swallow tail” shape was exhibited in a free energy–temperature plane, and the deformation of a quartic potential gave the classic “cusp” catastrophe, which implied the critical behaviour — phase transition. According to the relation between temperature $T$ and entropy $S$, in the $T – S$ plane, they plotted isocharge lines below, equal to and above the critical charge, and the result showed that, for a fixed charge ensemble, charged black holes’ thermodynamic phase structure turns out to be completely similar to a well-defined Van der Waals phase transition of the ordinary liquid-gas thermodynamic system. Following that, the Van der Waals-like phase transition was promoted to the extended phase space, where a negative cosmological constant and its conjugate variable were taken as thermodynamics pressure $P$ and volume $V$ [3–8]. This extended phase space gave a clearer picture description of the Van der Waals-like phase transition. Now, in the $P – V$ plane in various of AdS spacetimes, the Van der Waals-like phase transition has been discussed in detail [9–19]. In addition, the Van der Waals-like phase transition has also been discovered in other planes such as $\Phi – q$ plane [20] and $T – \alpha$ plane [21].

It is worth noting that, entanglement entropy has become an important component of the AdS/CFT correspondence, and has also been used as a probe of different phases. Especially, making use of entanglement entropy, Johnson [22] proposed that the Van der Waals-like phase transition can be illustrated in the entanglement entropy–temperature plane. Now, the related work has been generalized to massive gravity [23] and Weyl gravity [24], as well as supergravity spacetime [25]. Furthermore, the equal area construction has also been verified in the entanglement entropy–temperature plane [26].

In this paper, we would like to investigate holographic Van der Waals-like phase transition of the five-dimensional AdS hairy black hole sourced by a conformally coupled scalar field. In asymptotically flat spacetime, exact hairy black hole solutions in General Relativity conformally coupled to a scalar field theory do not exist in higher dimensions [27,28]. Although hairy black holes with vanishing cosmological constant are known in four dimensions, the scalar field configuration of these black holes diverges at the horizon [29,30]. This may be seen as a natural consequence of the well-known no-hair theorems. However, by introducing a cosmological constant and a conformal coupling, no-hair theorems can be circumvented. By virtue of the existence of “hair”, the solutions of
hairy black holes become richer than ordinary ones in general relativity [31–33]. In particular, G. Giribet et al. [34,35] in 2014 proved the analytic solutions to higher dimensional hairy black holes do exist and the scalar configuration is regular everywhere outside and on the horizon. Further, the thermodynamics of higher dimensional hairy black holes were discussed in detail [19,35,36]. It is worthwhile to note that, these small hairy black holes with positive specific heat may exist at arbitrarily low temperature, and can yield a zero-temperature remnant with finite mass by Hawking radiation. The five-dimensional hairy black holes in AdS space exhibit many peculiar features, and have important advantages of being a simple and tractable model for investigating phase transition in higher dimensional AdS spacetime.

Our main motivation for this paper is to study, whether the Van der Waals-like phase transition in the framework of holography can be detected for the five-dimensional hairy black hole including the effects of the scalar field hair. In addition, we will also check the Maxwell’s equal area construction is valid or not in this background. In particular, in the light of holography, we investigate the hairy black hole’s phase structure employing not only black hole entropy, but also entanglement entropy and two point correlation function. Here, we will show that, all of them exhibit the Van der Waals-like phase transition in a fixed scalar charge ensemble. Moreover, the phase transition is associated with a hair parameter q. When the absolute values of scalar charge q and the critical charge qC satisfy |q| < |qC|, the scalar isocharge presents the van de Waals oscillation similar to the liquid-gas phase transition, that is, the scalar isocharge develops an unstable branch with a negative slope, which corresponds to an unstable hole interpolating between the small stable hole and large stable hole. According to Maxwell’s equal area construction, the oscillating portion should be replaced by an isotherm which obeys that the areas above and below the isotherm are equal to one another. This implies that the small stable hole undergoes a first order phase transition to the large stable black hole. As |q| increases to |qC|, the oscillating portion squeezes to an inflection point. Namely, the small and large black hole merge into one and coexist, which corresponds to the second order phase transition. For |q| > |qC|, the hairy black hole is always stable and no phase transition takes place.

The rest of this paper is organized as follows. In Sec. 2, we briefly review analytic black hole solutions with conformally coupled scalar hair and investigate the black hole entropy’s phase transition in Ts = SS plane dominated by the scalar hair. In Sec. 3 and Sec. 4, for a fixed scalar charge ensemble, the hairy black hole’s phase transition is investigated in the framework of holography by two point correlation function and entanglement entropy, which is completely similar to Van der Waals behaviour. By numerical calculation, we check the equal area law of a first order phase transition and get an analogous heat capacity’s critical exponent of the second order phase transition. Finally, our conclusion and discussion are presented in Sec. 5.

2. Van der Waals behaviour of thermodynamics entropy

Let us first review the hairy black holes sourced by a conformally coupled scalar field in five-dimensional asymptotically AdS space. Different from the cosmological Einstein terms, the theory contains a conformal field theory contribution. The corresponding action reads [34,35]

\[ I = \frac{1}{k} \int d^5 x \sqrt{-g} \left( R - 2\Lambda + k \mathcal{L}_m(\phi, \nabla \phi) \right), \tag{1} \]

where \( k = 16\pi G \), and the Lagrangian matter \( \mathcal{L}_m(\phi, \nabla \phi) \) takes the form

\[ \mathcal{L}_m(\phi, \nabla \phi) = \phi^{15} \left( b_0 \phi^{S(0)} + b_1 \phi^{-B} \phi^{S(1)} + b_2 \phi^{-16} \phi^{S(2)} \right), \tag{2} \]

with

\[ S^{(0)} = 1, \]
\[ S^{(1)} = S \equiv g^\mu \nu S_{\mu \nu} \equiv g^\mu \nu \delta^\sigma_\sigma S_{\mu \nu}, \]
\[ S^{(2)} = S_{\mu \nu \alpha \beta} S^{\mu \nu \alpha \beta} - 4 S_{\mu \nu} S_{\mu \nu} + S^2, \tag{3} \]

here \( b_0, b_1 \) and \( b_2 \) are the coupling constants of conformal field theory. The theory coming from above action can exhibit analytic black hole solutions with a scalar hair parameter. The metric of hairy black hole is given by [19,35]

\[ ds^2 = -N^2(r) f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_5^2, \tag{4} \]

with

\[ f(r) = 1 - \frac{m}{r^2} - q \frac{r}{r^3} \]
\[ N^2(r) = 1, \tag{5} \]

where \( d\Omega_5^2 \) is the line element of the unit 3-sphere, and \( l \equiv (-\Lambda/6)^{-1/2} \) is the AdS radius. Here \( m \) stands for mass parameter, and \( q \) can be interpreted as the black hole’s charge under a scalar field. The scalar hair configuration of the theory is

\[ \phi(r) = \frac{n}{r^{17/2}}, \tag{6} \]

where in (6) and q in (5) are given by the following relationship

\[ q = \frac{64\pi G}{5} b_1 n^9, \quad n = \left( -\frac{18}{5} \frac{b_1}{b_0} \right)^{1/6}, \tag{7} \]

with \( \varepsilon = -1, 0, 1 \). The scalar coupling constants satisfy the following constraint

\[ 10 b_0 b_2 = 9 b_1^2, \tag{8} \]

The mass and entropy of a hairy black hole are given by [35]

\[ M = \frac{3\pi m}{8} = \frac{3\pi (r_+^2 l^2 - q^2 l^2 + r_+^2)}{8\pi r_+}, \tag{9} \]
\[ S = \pi \left( \frac{r_+^3}{2} - \frac{5q}{4} \right) = \frac{A}{4} - \frac{5}{4} \pi^2 q, \tag{10} \]

where \( r_+ \) is the event horizon of a hair black hole (the largest root of \( f(r_+) = 0 \)), and the area is \( A = 2\pi r_+^2 \). On the right side of the expression (10), the first term \( A/4 \) is the Bekenstein–Hawking area law, the second term \( -5\pi^2 q/4 \) comes from the higher curvature terms coupling only an association with the hair parameter \( q \) in action (1).

The Hawking temperature of the black hole is

\[ T_h = \frac{1}{\pi Br_+^2} \left( \frac{q l^2}{4} + \frac{\phi_l^2 r_+^3}{2} + r_+^4 \right). \tag{11} \]

From Eqs. (9)-(11), it is obvious that the entropy, mass and Hawking temperature satisfy the first law of black hole thermodynamics [35]

\[ dM = T_h dS. \tag{12} \]

Employing the relation \( F = M - T_h S \), we have

\[ F = \frac{\pi}{16} \left[ 20 q r_+^2 - 2 r_+^3 + \phi_l^2 \left( 5q^2 l^2 + 2q r_+^2 + 2r_+^6 \right) \right]. \tag{13} \]
By using Eq. (10) and Eq. (11), we can eliminate $r_+$, and obtain the relationship between Hawking temperature $T_h$ and entropy $S$ of a five-dimensional AdS hairy black hole

$$T_h(S, q) = \frac{1}{2^{2\pi^2/3} 5^{5/3} (5\pi^2 q + 4S)^{1/3}} \left[ 6l^2 \pi^{10/3} q + 4l^2 \pi^{4/3} S 
+ 5\sqrt{2} \pi^2 q(5\pi^2 q + 4S)^{2/3} + 4\sqrt{2} \pi^2 S(5\pi^2 q + 4S)^{2/3} \right].$$

(14)

According to the function $T_h(S, q)$ in Eq. (14), for different hair parameter $q$, we can plot the scalar isocharge in the $T_h - S$ plane. In order to plot the scalar isocharge, we first find out the critical values of phase transition by calculating with the following relations

$$\left( \frac{\partial T_h}{\partial S} \right)_q = \left( \frac{\partial^2 T_h}{\partial S^2} \right)_q = 0.$$

(15)

Substituting Eq. (14) into Eq. (15), we obtain the hairy black hole's critical scalar charge $q_C$, critical entropy $S_C$ and critical temperature $T_C$

$$q_C = -\frac{3}{50} \sqrt{\frac{3}{10}} l^2,$$

$$S_C = \frac{9}{40} \sqrt{\frac{3}{10}} \pi^2 l^3,$$

$$T_C = 5\sqrt{\frac{3}{10}} \frac{1}{2\pi l}.$$

(16)

(17)

(18)

Heat capacity is given by

$$C_q = T_h \frac{\partial S}{\partial T_h} \bigg|_{q}.$$

(19)

It is particularly interesting in exhibiting the effect of the hairy parameter $q$ on the phase transition of a black hole. Since the solution in Eq. (4) with $q > 0$ leads to negative entropy, which is somehow pathological in physics, and $q = 0$ reduces to the five-dimensional AdS Schwarzschild black hole, in this paper we will focus on investigating a phase transition in case of $q < 0$. For convenience, we order $l = 1$. Here, we take $|q| = |q_C|, |q| = -0.05 > |q_C|$ and $|q| = -0.02 < |q_C|$ as examples. In order to understand clearly the hairy black hole's phase transition, now we will plot scalar isocharge curves on the $T_h - r_+$ plane and $T_h - M$ plane for given different scalar charges.

From Fig. 1 and Fig. 2, we can see that the number of the black hole solution is related to the value of a hair parameter $q$. For the case of $|q| < |q_C|$, within a certain range of temperature, there may exist three hairy black hole solutions with the same hair parameter $q$ and temperature $T_h$ for a certain range of temperature. The smallest and largest are stable, while the intermediate is unstable since the heat capacity is negative. For the case $|q| = |q_C|$, an inflection point is exhibited, and for the case $|q| > |q_C|$, there only exists one single hairy black hole solution for each temperature. Therefore, the analysis summarized in Figs. 1 and 2 might have implication for phase transition.

Then, we focus on illustrating the hairy black hole's phase structure in a temperature $T_h$ and entropy $S$ plane. According to the expressions in (13) and (14), we plot the related phase transition curves in Figs. 3–5.

As can be observed from Figs. 3–5, the phase structure of the hairy black hole is strikingly analogous to Van der Waals behaviour. Here in the $T_h - S$ plane, we survey hairy black hole's phase transition in the fixed scalar charge ensemble. A different scalar charge $q$ corresponds to different choice of the coupling constant. In Fig. 3, we see that, scalar isocharges present three different qualitative behaviours for three given scalar charges. In the $|q| > |q_C|$ case, temperature is monotonically larger as entropy increases, and the slope is positive. According to the definition of the heat capacity in Eq. (19), positive (negative) slope is equivalent to positive (negative) heat capacity. This means that, at each temperature only one stable black hole exists. In the $|q| < |q_C|$ case, a Van der Waals-like oscillation can be observed. Namely, there exists an interval of Hawking temperature. Within this interval, the
slope of the curve changes from positive to negative, then to positive again, where the oscillating portion with a negative slope is unstable in physics. In contrast to Figs. 1 and 2, we can infer that, there exists an unstable black hole with negative heat capacity interpolating between the stable (locally) small hole and stable (locally) large hole. This implies, at a subcritical temperature $T^*$, the small stable black hole undergoes a phase transition to the stable large black hole, which is strikingly similar to the Van der Waals phase transition of the liquid-gas system in ordinary thermodynamics physics. This transition is interpreted as a first order phase transition [2]. The oscillating portion should be eliminated by using an isotherm which obeys Maxwell’s equal area construction. In the $|q| = |q_c|$ case, the curve presents an inflection point, which implies the small and large black holes merge into one and coexist. As shown in Eqs. (15) and (19), it is obvious that the heat capacity is divergent in this case. Therefore, the corresponding phase transition is second order.

According to Figs. 2 and 3, we can infer that, when $T_h > T^*$, the black hole corresponds to a large one, and it corresponds to a small one when $T_h < T^*$. In Fig. 4, we can also see classic swallow-tail shape, which is responsible for the first order phase transition. The corresponding phase transition temperature $T^* = 0.4479$ is the longitudinal coordinate of the junction between the stable small hole and the large stable hole. When $T_h < T^*$, the small black hole is globally stable since it possesses the lowest free energy. On the contrary, when $T_h > T^*$, the large black hole corresponding to the lowest free energy is favoured by thermodynamics. When $T_h = T^*$, the small hole and large hole coexist due to having the same free energy, and can transit from one phase to another.

In Fig. 5, we find that an inflection point emerges, and it corresponds to the inflection point of the second order phase transition as is shown in the middle curve in Fig. 3. Furthermore, the longitudinal coordinate of an inflection point just agrees with the critical temperature $T_C$ in the expression (18). For further discussion, it is necessary to verify whether Maxwell equal area construction in $T_h - S$ plane holds for the first order phase transition in Fig. 3. The corresponding formula is

$$A_1 = \int_{S_1}^{S_2} T_h(S, q) dS = T^*(S_2 - S_1) = A_2,$$

where $T_h(S, q)$ is defined in Eq. (14). By solving the equation $T_h(S, q) = T^*$, we can get the values of $S_1$ and $S_2$, which corresponds to the smallest and largest roots. Numerically, we have $S_1 = 0.40952$ and $S_2 = 2.57401$. Thus, we find the values at both sides of Eq. (20) are $A_1 = 0.969416$ and $A_2 = 0.969477$, respectively. Consequently, the Maxwell’s equal area law is explicitly valid in the temperature–entropy plane. On the other hand, for the hairy black hole’s second order phase transition, it is also very interesting to find out the critical exponent with respect to the heat capacity as defined in (19). Near the critical point, setting $S = S_C + \sigma$ and expanding the Hawking temperature in small $\sigma$, we have

$$T_h(S, q) = \frac{1}{2^{1/3} \pi^{5/3} (5\pi^2 q + 4S_C)^{4/3}}$$

$$\times \left[ 6\pi^{10/3} q + 4\pi^{4/3} S_C + 5\sqrt[4]{2\pi^2 q} (5\pi^2 q + S_C)^{2/3} 
+ 4\sqrt[5]{2S_C} (5\pi^2 q + 4S_C)^{2/3} \right]$$

$$= \frac{2\sqrt[2]{2}\sigma}{18\pi^{5/3} (5\pi^2 q + 4S_C)^{10/3}}$$

$$\times \left[ 2^{1/3} (5\pi^2 q + 4S_C)^{5/3} - \pi^2 \sqrt[4]{2} (9\pi^2 q + 4S_C) \right]$$

$$= \frac{32\sqrt[3]{2}\sigma}{81\pi^{5/3} (5\pi^2 q + 4S_C)^{13/3}} \left[ 5\sqrt[2]{2}(5\pi^2 q + 4S_C)^{5/3} 
- 14\pi^2 \sqrt[4]{4S_C} (15\pi^2 q + 4S_C)^{5/3} \right].$$

Using Eqs. (14) and (15), we find

$$T_h - T_C = \frac{32\sqrt[2]{2}(S - S_C)^3}{81\pi^{5/3} (5\pi^2 q + 4S_C)^{13/3}} \left[ 5\sqrt[2]{2} \times (5\pi^2 q + 4S_C)^{5/3} 
- 14\pi^2 \sqrt[4]{4S_C} (15\pi^2 q + 4S_C)^{5/3} \right].$$

As a result, from Eqs. (19) and (22), we get $C_q \sim (T_h - T_C)^{-2/3}$, that is, the critical exponent of the second order phase is $-2/3$, which is consistent with the mean field theory. Taking logarithm to the expression (22), we discover a linear relation

$$\log |T_h - T_C| = 3 \log |S - S_C| + \text{constant},$$

where, the value corresponding to the slope is 3.
In the following sections, we would like to investigate the phase structure of the hairy black hole by a geodesic length and minimal surface area in the bulk, which correspond to the two point correlation function and entanglement entropy in the dual conformal field theory. Recently, the two point correlation function and entanglement entropy have been used to probe the non-equilibrium thermalization behaviour, and the results show that all of them have the same effect [37–40]. Now, in the cases of the two point correlation function and entanglement entropy, we will verify whether a similar relation holds.

3. Two point correlation function across the phase transition

Motivated by holography, in this section, we would like to detect whether the two point correlation function possesses the Van der Waals-like phase transition. According to the AdS/CFT correspondence, if $\Delta$ is large enough, the equal time two point correlation function has the following form [41]

$$\langle O(t_0, x_i)O(t_0, x_j) \rangle \approx e^{-\Delta L},$$

(24)

in which $\Delta$ is the conformal dimension of scalar operator $O$, and $L$ is the length of the bulk geodesic between the point $(t_0, x_i)$ and $(t_0, x_j)$ on the AdS boundary. Considering the hairy black hole’s symmetry, we can simply order $(\phi = \pi / 2, \theta = 0, \psi = \psi_0)$ and $(\phi = \pi / 2, \theta = \theta_0, \psi = \psi_0)$ as the two boundary points. Then we employ $\theta$ to parameterize the trajectory, consequently, the proper length is written as

$$L = \int_0^{t_0} \mathcal{L}(r(\theta), \theta) d\theta, \quad \mathcal{L} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + \frac{r^2}{f(r)}}.$$

(25)

where $r' \equiv dr/d\theta$. Here, $\mathcal{L}$ is taken as the Lagrangian and $\theta$ is imagined as time. By solving the Euler–Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{d\theta} \left( \frac{\partial \mathcal{L}}{\partial r'} \right),$$

(26)

the motion equation of $r(\theta)$ can be found to be

$$f'(r)r''(\theta)^2 - 2f(r)r''(\theta) + 2f^2(r)r'(\theta) = 0,$$

(27)

with the boundary conditions

$$r(0) = r_0, \quad r'(0) = 0.$$  

(28)

Thus, $r(\theta)$ can be numerically solved. Also, since the geodesic length is divergent for a fixed $\theta_0$, it has to be regularized by subtracting the geodesic length of the minimal surface $L'$ in pure AdS with same boundary $\theta = \theta_0$ (denoted by $L'$). Namely, the regularized geodesic length becomes $\delta L = L - L'$. In order to attain this objective, we are required to set a UV cutoff $\theta_0$ (which is close to $\theta_0$, that is $\theta_0 \approx \theta_0$). We get $L$ through integrating the length function in Eq. (25) from zero to UV cutoff $\theta_0$. Then, by ordering $m = q = 0$ in the hairy black hole background, we obtain the spacetime of the pure AdS

$$ds^2 = - \left(1 + \frac{r^2}{l^2}\right)dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1}dr^2 + r^2 d\Omega^2.$$

(29)

For this metric, adopting the same procedure as above, we can easily get $L'$ by numerical computation. Thus, we obtain the renormalized geodesic length $\delta L$. For convenience, in this paper, we set the AdS radius $l = 1$ during calculating numerically. We will take $\theta_0 = 0.35, 0.42$ and 0.50 as examples to survey the hairy black hole’s phase structure, and the corresponding UV cutoffs are chosen to be $\theta_C = 0.349, 0.419$ and 0.499, respectively.

![Fig. 6](image-url)

Fig. 6. Scalar isochors in the $T_h - \delta L$ plane for $\theta_0 = 0.35$ (top), $\theta_0 = 0.42$ (intermediate), $\theta_0 = 0.50$ (bottom). In every figure, the curves from top to bottom correspond to $|q| = -0.02 < |q_C|$, $|q| = |q_C|$ and $|q| = -0.05 > |q_C|$, respectively. As can be seen in these plots, the Van der Waals-like phase transition can also be observed in the $T_h - \delta L$ plane. Compared these plots with the previous one in Fig. 3, the phase structure of the two point correlation function in the $T_h - \delta L$ plane is completely similar to that of the black hole entropy in the $T_h - S$ plane. Moreover, this conclusion holds regardless of $\theta_0$ in a reasonable region.

In order to verify the Maxwell’s equal area construction in $T_h - \delta L$ plane, we can choose 3 different values of $\theta_0$. As is done in
a black hole entropy case, in the $T_h - \delta L$ plane, we construct the equal area law as
\[
A_1 \equiv \int_{\delta L_1}^{\delta L_2} T_h(\delta L, q)d\delta L = T^*(\delta L_2 - \delta L_1) \equiv A_2.
\tag{30}
\]
Here, $T^*$ is equal to the equal area temperature $T^*$ found in Section 2, $T_h(\delta L, q)$ is an interpolating function given by the numeric result, and $\delta L_1$ and $\delta L_2$ are the smallest and largest values which can be obtained by solving equation $T_h(\delta L, q) = T^*$. In Fig. 4, we also draw the transition isotherm of a first order phase transition in red line, whose temperature is obtained from the free energy in Fig. 2, and draw the transition isotherm of a second order phase transition in green line, which is just the critical temperature obtained in Eq. (18). For different $\theta_0$, we tabulate the values of $\delta L_1$, $\delta L_2$ and $A_1$, $A_2$ in Table 1. As can be seen from this table, it is obvious that the equal area law holds for the two point correlation function of the hairy black hole.

For the second order phase transition in $T_h - \delta L$ plane, we verify whether the critical exponent is the same with that of the black hole entropy. First, we define a heat capacity which is analogous to black hole entropy, that is
\[
C_q = T_h \frac{\partial \delta L}{\partial T_h} q.
\tag{31}
\]
Then, we calculate numerically the logarithm of the quantities $T_h - T_C$, $\delta L - \delta L_C$, with respect to $C_q$ in Eq. (31). Here, $\delta L_C$ is given numerically by the equation $T_h(\delta L, q) = T_C$. For different $\theta_0$, the relationship between $T_h - T_C$ and $\log |\delta L - \delta L_C|$ is plotted in Fig. 7. The corresponding results can be fitted as follows:
\[
\log |T_h - T_C| = \\
\begin{array}{l}
24.2783 + 3.01206 \log |\delta L - \delta L_C|, \theta_0 = 0.35 \\
21.8051 + 2.98973 \log |\delta L - \delta L_C|, \theta_0 = 0.42 \\
20.0834 + 3.02263 \log |\delta L - \delta L_C|, \theta_0 = 0.50
\end{array}
\tag{32}
\]
As a result, we find the slope is around 3, which agrees with that in Eq. (23). In other words, like black hole entropy, the two point correlation function does exist second order phase transition.

4. Phase transition and equal area law of entanglement entropy

Entanglement entropy is a very useful tool that can be used to detect different phases in various of backgrounds. In this section, in asymptotically AdS spacetime of Einstein – A theory conformally coupled to a scalar field, we attempt to employ entanglement entropy to discuss on the phase transition of a five-dimensional hairy black hole.

Generally speaking, it is relatively complicated to calculate entanglement entropy in field theory. However, according to the AdS/CFT correspondence, S. Ryu et al. [42–44] presented an elegant geometric description for computing holographic entanglement entropy, which can be given by the area $A_{\Sigma}$ of a bulk minimal surface $\Sigma$ anchored on the boundary entangling surface $\partial \Sigma$, which is
\[
S = \frac{A_{\Sigma}(t)}{4G},
\tag{33}
\]
in which $G$ is the Newton's constant. Here, we take $\phi = \phi_0$ as entangling surface and choose the values of $\phi_0$: 0.35, 0.42 and 0.50. The minimal surface can be parameterized by $r(\phi)$, and corresponding function is independent of $\theta$ and $\psi$ by virtue of the symmetry. So in this gravity background, the holographic entanglement entropy can be expressed as
\[
S = \pi \int_0^{\phi_0} \phi^2 \sqrt{f' + r^2 d\phi},
\tag{34}
\]
where \( r' = \frac{dr}{d\phi} \). Proceeding as for in the two point correlation function case, we are able to obtain the motion equation of \( r(\phi) \) by employing the Euler–Lagrange equation, and get the numeric result of \( r(\phi) \) with a boundary condition. In order to regularize entanglement entropy, we again integrate \( S \) in Eq. (34) up to cutoff (which is close to \( \phi_0 \)), and subtract the pure AdS entanglement entropy (denoted by \( S' \)) with a same entangling surface \( \phi_0 \) at the boundary. Here, the corresponding regularized entanglement entropy is denoted by \( \Delta S = S - S' \). In Fig. 8, we plot the scalar isocharges in the \( T_h - \Delta S \) plane for \( \phi_0 = 0.35, 0.42 \) and 0.50. Compared with Fig. 3 and Fig. 6, the entanglement entropy also presents Van der Waals-like phase transition, which is the same as the two point correlation function and the black hole entropy.

Table 2

<table>
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<th>( \phi_0 )</th>
<th>( \Delta S_1 )</th>
<th>( \Delta S_2 )</th>
<th>( A_1 )</th>
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</tr>
</tbody>
</table>

Repeating the procedure as for the two point correlation function case, we proceed to verify that Maxwell equal area law in the \( T_h - \Delta S \) plane works for the first order phase transition, and \( T_C \) is indeed transition temperature of the second order phase transition in a \( T_h - \Delta S \) plane. We construct analogous Maxwell’s equal area law as

\[
A_1 = \int_{\Delta S_1}^{\Delta S_2} T_h(\Delta S, q) d\Delta S = T^*(\Delta S_2 - \Delta S_1) = A_2, \quad \text{(35)}
\]

in which \( T^* \) is the same with the equal area temperature \( T^* \) found in Section 2, \( \Delta S_2 \) and \( \Delta S_1 \) are the largest and smallest roots of the equation \( T_h(\Delta S, q) = T^* \), and \( T_h(\Delta S, q) \) is an Interpolating Function given by the numerical result. The first order phase transition temperature \( T^* \) can be read off from Fig. 4. Again for different \( \phi_0 \), we tabulate the values of \( \Delta S_1, \Delta S_2 \) and \( A_1, A_2 \) in Table 2. Obviously, \( A_1 \) is indeed equal to \( A_2 \) for different \( \phi_0 \) within the numerical accuracy. Consequently the first order phase transition of entanglement entropy also obeys Maxwell equal area law.

For the second order phase transition, defining the heat capacity of the entanglement entropy as

\[
C_q = T_h \frac{\partial \Delta S}{\partial T_h} \bigg|_q, \quad \text{(36)}
\]

We are able to find out the critical exponent from the slope of the relationship between \( \log |T_h - T_C| \) and \( \log |\Delta S - \Delta S_C| \), here \( \Delta S_C \) is the critical entanglement entropy given numerically by an equation \( T_h(\Delta S, q) = T_C \). For different \( \phi_0 \), we plot the relationship between \( \log |T_h - T_C| \) and \( \log |\Delta S - \Delta S_C| \) in Fig. 9. The corresponding results can be fitted as follows:

\[
\log |T_h - T_C| = \begin{cases} 
15.3075 + 3.00481 \log |\Delta S - \Delta S_C| & \phi_0 = 0.35 \\
13.2489 + 3.03497 \log |\Delta S - \Delta S_C| & \phi_0 = 0.42 \\
10.6644 + 2.99220 \log |\Delta S - \Delta S_C| & \phi_0 = 0.50
\end{cases}, \quad \text{(37)}
\]

Again, like the two point correlation function, we discovery the slope is around 3. Therefore, we conclude that entanglement entropy of the hairy black hole exists a second order phase transition at the critical temperature \( T_C \), and the critical exponent coincides with that of the black hole entropy.

5. Conclusion and discussion

The research on holographic phase transition can make us deeply understand a black hole as thermodynamical system, although the nature of the microscopic structure has not been made clear until now. In this paper, in a fixed scalar charge ensemble, employing black hole entropy, entanglement entropy and two point correlation function, we have specifically investigated the holographic phase transition of a five-dimensional AdS black hole including the effect of the scalar field hair. Since the holographic interpretation of the \( T - S \) plane is very well-understood [26,45],
here, in the $\mathbf{T} - \mathbf{S}$ plane, we have presented the Van der Waals-like phase transition by analyzing scalar isocharges. Furthermore, in this paper, the Maxwell's equal area construction of a first order phase transition has been checked, and the critical exponent of a second order phase transition has been obtained. It is emphasized that, by making use of entanglement entropy and two point correlation function, the Van der Waals-like phase transition can also be observed in the $\mathbf{T} - \Delta \mathbf{S}$ plane and $\mathbf{T} - \Delta \mathbf{l}$ plane, which is completely analogous of that in the $\mathbf{T} - \mathbf{S}$ plane with thermal entropy. Here, our study is limited to the uncharged hairless black hole in five-dimensional AdS space. It would be very interesting to generalize the holographic research to other more complicated gravity backgrounds. We hope to report this in a forthcoming paper.

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References