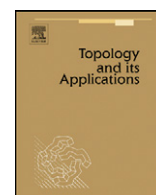




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Group-valued continuous functions with the topology of pointwise convergence

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ABSTRACT

Let G be a topological group with the identity element e . Given a space X , we denote by $C_p(X, G)$ the group of all continuous functions from X to G endowed with the topology of pointwise convergence, and we say that X is: (a) G -regular if, for each closed set $F \subseteq X$ and every point $x \in X \setminus F$, there exist $f \in C_p(X, G)$ and $g \in G \setminus \{e\}$ such that $f(x) = g$ and $f(F) \subseteq \{e\}$; (b) G^* -regular provided that there exists $g \in G \setminus \{e\}$ such that, for each closed set $F \subseteq X$ and every point $x \in X \setminus F$, one can find $f \in C_p(X, G)$ with $f(x) = g$ and $f(F) \subseteq \{e\}$. Spaces X and Y are G -equivalent provided that the topological groups $C_p(X, G)$ and $C_p(Y, G)$ are topologically isomorphic.

We investigate which topological properties are preserved by G -equivalence, with a special emphasis being placed on characterizing topological properties of X in terms of those of $C_p(X, G)$. Since \mathbb{R} -equivalence coincides with l -equivalence, this line of research “includes” major topics of the classical C_p -theory of Arhangel'skiĭ as a particular case (when $G = \mathbb{R}$). We introduce a new class of TAP groups that contains all groups having no small subgroups (NSS groups). We prove that: (i) for a given NSS group G , a G -regular space X is pseudocompact if and only if $C_p(X, G)$ is TAP, and (ii) for a metrizable NSS group G , a G^* -regular space X is compact if and only if $C_p(X, G)$ is a TAP group of countable tightness. In particular, a Tychonoff space X is pseudocompact (compact) if and only if $C_p(X, \mathbb{R})$ is a TAP group (of countable tightness). Demonstrating the limits of the result in (i), we give an example of a precompact TAP group G and a G -regular countably compact space X such that $C_p(X, G)$ is not TAP.

We show that Tychonoff spaces X and Y are \mathbb{T} -equivalent if and only if their free precompact Abelian groups are topologically isomorphic, where \mathbb{T} stays for the quotient group \mathbb{R}/\mathbb{Z} . As a corollary, we obtain that \mathbb{T} -equivalence implies G -equivalence for every Abelian precompact group G . We establish that \mathbb{T} -equivalence preserves the following topological properties: compactness, pseudocompactness, σ -compactness, the property of being a Lindelöf Σ -space, the property of being a compact metrizable space, the (finite) number of connected components, connectedness, total disconnectedness. An example of \mathbb{R} -equivalent (that is, l -equivalent) spaces that are not \mathbb{T} -equivalent is constructed.

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In notation and terminology we follow [6] and [9] if not stated otherwise. All topological spaces are assumed to be Tychonoff (that is, completely regular T_1 spaces) and nonempty, and all topological groups are assumed to be Hausdorff.

By \mathbb{N} we denote the set of all natural numbers, ω stays for the least nonzero limit ordinal, \mathbb{Z} is the discrete additive group of integers, \mathbb{R} is the additive group of reals with its usual topology, \mathbb{T} stays for the quotient group \mathbb{R}/\mathbb{Z} , and $\mathbb{Z}(n)$ denotes the cyclic group of order n (with the discrete topology). The identity element of a group G is denoted by e_G , or simply by e when there is no danger of confusion.

If G is a topological group, then the symbol \widehat{G} stays for the completion of G with respect to the two-sided uniformity. If $G = \widehat{G}$, then G is called complete. It is well known that \widehat{G} always exists, \widehat{G} is a topological group, G is dense in \widehat{G} , and if G is a dense subgroup of a complete group H , then $\widehat{G} = H$. If G is a subgroup of some compact group, then G is called precompact.

1. Introduction

Definition 1.1. Let X be a space and G a topological group.

- (i) We shall use $C(X, G)$ to denote the group of all continuous functions from X to G , equipped with the “pointwise group operations”. That is, the product of $f \in C(X, G)$ and $g \in C(X, G)$ is the function $fg \in C(X, G)$ defined by $fg(x) = f(x)g(x)$ for all $x \in X$, and the inverse element of f is the function $h \in C(X, G)$ defined by $h(x) = (f(x))^{-1}$ for all $x \in X$.
- (ii) The family

$$\{W(x, U): x \in X, U \text{ is an open subset of } G\},$$

where

$$W(x, U) = \{f \in C(X, G): f(x) \in U\},$$

forms a subbase of the topology of pointwise convergence on $C(X, G)$. We use the symbol $C_p(X, G)$ to denote the set $C(X, G)$ endowed with this topology.

One can easily see that $C_p(X, G)$ is a topological group.

Definition 1.2. Let G and H be topological groups.

- (i) Recall that G and H are said to be topologically isomorphic if there exists a bijection $f : G \rightarrow H$ which is both a group homomorphism and a homeomorphism. We write $G \cong H$ whenever G and H are topologically isomorphic.
- (ii) We say that spaces X and Y are G -equivalent, and denote this by $X \overset{G}{\sim} Y$, provided that $C_p(X, G) \cong C_p(Y, G)$.
- (iii) Let \mathcal{C} be a class of spaces. We say that a topological property \mathcal{E} is preserved by G -equivalence within the class \mathcal{C} provided that the following condition holds: If $X \in \mathcal{C}$, $Y \in \mathcal{C}$, $X \overset{G}{\sim} Y$ and X has the property \mathcal{E} , then Y must have the property \mathcal{E} as well. The sentence “ \mathcal{E} is preserved by G -equivalence” is used as an abbreviation for “ \mathcal{E} is preserved by G -equivalence within the class of Tychonoff spaces”.
- (iv) Given a class \mathcal{C} of spaces, we say that G -equivalence implies H -equivalence within the class \mathcal{C} provided that the following statement holds: If $X \in \mathcal{C}$, $Y \in \mathcal{C}$ and $X \overset{G}{\sim} Y$, then $X \overset{H}{\sim} Y$. The sentence “ G -equivalence implies H -equivalence” shall be used as an abbreviation for “ G -equivalence implies H -equivalence within the class of Tychonoff spaces”.

In [15] Markov has introduced the free topological group $F(X)$ of a space X and defined spaces X and Y to be M -equivalent if $F(X) \cong F(Y)$. Thereafter, a significant effort went into an investigation of how topological properties of $F(X)$ depend on those of X , as well as which topological properties are preserved by M -equivalence.

Every continuous function $f : X \rightarrow G$ from a space X to a topological group G can be (uniquely) extended to a continuous group homomorphism $\widehat{f} : F(X) \rightarrow G$. This elementary fact (with \mathbb{T} as G) was applied by Graev to show that the closed unit interval and the circle are not M -equivalent [10]. Tkachuk noticed in [22] that M -equivalence implies G -equivalence for every Abelian topological group G . He then applied this observation to $G = \mathbb{Z}(2)$ to show that connectedness is preserved by M -equivalence [22].

Later on, many properties of M -equivalence were discovered by means of the notion of l -equivalence; see [1]. Recall that spaces X and Y are called l -equivalent provided that $C_p(X, \mathbb{R})$ and $C_p(Y, \mathbb{R})$ are topologically isomorphic as topological vector spaces. A fundamental observation pertinent to the subject of this paper has been made in [22] by Tkachuk: spaces X and Y are l -equivalent if and only if $C_p(X, \mathbb{R})$ and $C_p(Y, \mathbb{R})$ are topologically isomorphic as topological groups. In other words, l -equivalence of spaces coincides with their \mathbb{R} -equivalence (in our notation). A far reaching conclusion that one might get from this fact is that, despite a significant emphasis on the topological vector space structure commonly placed in the C_p -theory [1], this structure is largely irrelevant to the study of the notion of l -equivalence, and in fact may as well be replaced by the topological group structure. It is this conclusion that led us to an idea of introducing the general notion of G -equivalence, for an arbitrary topological group G .

This opens up a topic of studying the properties of the topological group $C_p(X, G)$, for a given space X and a topological group G . Let us outline major problems that appear to be of particular interest in this new area of research.

Problem 1.3. Given a topological group G , characterize topological properties of a space X in terms of algebraic and/or topological properties of $C_p(X, G)$.

Problem 1.4. Given a topological group G , a class \mathcal{C} of spaces and a topological property \mathcal{E} , investigate when the property \mathcal{E} is preserved by G -equivalence within the class \mathcal{C} .

In the particular case when $G = \mathbb{R}$, these two problems are well-known (and major) problems of the C_p -theory. Therefore, one can view Problems 1.3 and 1.4 as a natural generalization of major topics of the C_p -theory to the case of an arbitrary topological group G .

Since the C_p -theory provides a large supply of topological properties that are preserved by \mathbb{R} -equivalence (that is, l -equivalence) within the class of Tychonoff spaces, the following particular version of Problem 1.4 seems to be worth studying:

Problem 1.5. Let G be one of the “important” topological groups such as, for example, the circle group \mathbb{T} , the dual group \mathbb{Q}^* of the discrete group \mathbb{Q} of rational numbers, or the group \mathbb{Z}_p of p -adic integers. Assume also that \mathcal{E} is a topological property preserved by \mathbb{R} -equivalence within (some subclass of) the class of Tychonoff spaces. Is it then true that \mathcal{E} is also preserved by G -equivalence within (an appropriate subclass of) the class of Tychonoff spaces?

One can formulate the most ambitious version of Problem 1.4:

Problem 1.6. Let \mathcal{C} be a class of spaces and \mathcal{E} a topological property. Describe the class $\mathbf{G}(\mathcal{C}, \mathcal{E})$ of topological groups G such that the property \mathcal{E} is preserved by G -equivalence within the class \mathcal{C} .

As may be expected, Problem 1.6 turns out to be difficult even in the case of major topological properties \mathcal{E} such as compactness and pseudocompactness.

Problem 1.7. Given a class \mathcal{C} of spaces and topological groups G, H , when does G -equivalence imply H -equivalence within the class \mathcal{C} ?

In this manuscript we build a foundation for studying these problems.

Section 2 collects necessary preliminaries and basic results, most of which are elementary and have counterparts in the classical C_p -theory. Example 2.1 demonstrates that, in order to obtain nice results, it is important to have enough continuous maps from a space X to a given topological group G . This fact naturally leads to an introduction of three notions of “regularity”, G -regularity, G^* -regularity and G^{**} -regularity, in Definition 2.2. One of the basic results (Proposition 2.7) states that G -equivalence and H -equivalence combined together imply $(G \times H)$ -equivalence. The converse implication fails in general (Example 7.5). The main goal of Section 3 is to show that $(G \times H)$ -equivalence does imply both G -equivalence and H -equivalence for “sufficiently different” topological groups G and H (Theorems 3.2 and 3.3).

In Section 4 we introduce a new class of topological groups (that we call TAP groups) and prove that every group without small subgroups (an NSS group) is TAP; see Theorem 4.9. The class of TAP groups has many common properties with that of NSS groups. For example, this class is closed under taking subgroups and finite products, and a TAP group does not contain any subgroup topologically isomorphic to an infinite product of nontrivial topological groups (Proposition 4.6). Every topological group without nontrivial convergent sequences is TAP (Proposition 4.10), so the class of TAP groups contains many “peculiar” topological groups.

In Section 5 we show that a space X is pseudocompact if and only if $C_p(X, \mathbb{R})$ is a TAP group (Theorem 5.3), thereby providing a short, elementary, “group-theoretic” proof of the result of Arhangel’skiĭ about preservation of pseudocompactness by l -equivalence.

In Section 6 we further generalize Theorem 5.3 by proving that, for every NSS group G , a G -regular space X is pseudocompact if and only if $C_p(X, G)$ is TAP; see Theorem 6.5. Emphasizing the limits of this result, we construct a precompact TAP group G and a countably compact G^* -regular space X such that $C_p(X, G)$ is not TAP (Theorem 6.8).

The main result of Section 7 is Corollary 7.3 saying that, for a metrizable NSS group G , a G^* -regular space X is compact if and only if $C_p(X, G)$ is a TAP group of countable tightness. In particular, G -equivalence preserves compactness within the class of G^* -regular spaces for every NSS metric group G . The classes of topological groups G such that G -equivalence preserves compactness and pseudocompactness, respectively, are closed under taking finite powers (Corollary 2.17) but are not closed under taking finite products (Example 7.5). Moreover, we give an example demonstrating that G -equivalence can preserve both compactness and pseudocompactness without G being NSS, or even TAP (Example 7.6).

Section 8 provides some sufficient conditions on a topological group G that guarantee that G -equivalence preserves total disconnectedness, connectedness and (finite) number of connected components.

In Section 9 we recall a general categorical machinery that leads to a definition of a free object $F_{\mathcal{G}}(X)$ of a space X in a given class \mathcal{G} of topological groups (closed under taking products and subgroups), and we define spaces X and Y to be \mathcal{G} -equivalent provided that $F_{\mathcal{G}}(X) \cong F_{\mathcal{G}}(Y)$. When \mathcal{G} is the class of all topological groups (all topological Abelian groups,

respectively), the free object $F_{\mathcal{G}}(X)$ coincides with the free topological group (respectively, the free Abelian topological group) of a space X in the sense of Markov, and \mathcal{G} -equivalence coincides with the classical M -equivalence (A -equivalence, respectively). When $G \in \mathcal{G}$ is Abelian, then \mathcal{G} -equivalence implies G -equivalence (Corollary 9.11).

Section 10 is devoted to the study of properties of \mathbb{T} -equivalence. A nontrivial connection with the previous section is based on the so-called “precompact duality theorem” (Theorem 10.2) that allows us to prove that \mathbb{T} -equivalence coincides with \mathcal{P} -equivalence, where \mathcal{P} is the class of all precompact Abelian groups (Corollary 10.4). Combining this with Corollary 9.11, we conclude that \mathbb{T} -equivalence implies G -equivalence for every precompact Abelian group G (Corollary 10.5). Theorem 10.7 lists major topological properties that are preserved by \mathbb{T} -equivalence. As a consequence, all these properties are also preserved by \mathcal{G} -equivalence whenever $\mathbb{T} \in \mathcal{G}$ (Corollary 10.8). In particular, it follows that total disconnectedness is preserved by A -equivalence and M -equivalence (Corollary 10.10), which seems to be a new result. Since l -equivalence (aka \mathbb{R} -equivalence) does not preserve connectedness, while \mathbb{T} -equivalence does, it follows that l -equivalence does not imply \mathbb{T} -equivalence (Proposition 10.11).

Section 11 lists some concrete open problems that are related to our results.

2. Basic results

Example 2.1. Let G be a topological group with the trivial connected component (for example, a zero-dimensional group). Then $C_p(X, G) \cong G$ for every connected space X . In particular, any two connected spaces X and Y are G -equivalent, and so most major topological properties are not preserved by G -equivalence.

This example clearly demonstrates that, in order to obtain meaningful theorems about preservation of some topological property by G -equivalence within a given class \mathcal{C} , one has to require each member of \mathcal{C} to have “sufficiently many” continuous functions to the target topological group G . Our next definition exhibits three possible ways of doing so.

Definition 2.2. Given a topological group G , we say that a space X is:

- (i) G -regular if, for each closed set $F \subseteq X$ and every point $x \in X \setminus F$, there exist $f \in C_p(X, G)$ and $g \in G \setminus \{e\}$ such that $f(x) = g$ and $f(F) \subseteq \{e\}$;
- (ii) G^* -regular if there exists $g \in G \setminus \{e\}$ such that, for every closed set $F \subseteq X$ and each point $x \in X \setminus F$, one can find $f \in C_p(X, G)$ such that $f(x) = g$ and $f(F) \subseteq \{e\}$;
- (iii) G^{**} -regular provided that, whenever F is a closed subset of X , $x \in X \setminus F$ and $g \in G$, there exists $f \in C_p(X, G)$ such that $f(x) = g$ and $f(F) \subseteq \{e\}$.

It is clear that

$$X \text{ is } G^{**}\text{-regular} \rightarrow X \text{ is } G^*\text{-regular} \rightarrow X \text{ is } G\text{-regular.} \tag{1}$$

Since the topological group $G = \mathbb{R} \times \mathbb{Z}(2)$ is not connected, and a continuous image of a connected space is connected, one can easily see that the unit interval $[0, 1]$ is G^* -regular but not G^{**} -regular, so the first implication in (1) cannot be reversed. The authors have no example of a group G witnessing that the second implication in (1) cannot be reversed, although they are convinced that such an example must exist; see Question 11.13.

Our next proposition describes three obvious cases when some kind of G -regularity “comes for free”:

Proposition 2.3. Let X be a space and G a topological group.

- (i) If G is pathwise connected, then X is G^{**} -regular.
- (ii) If G contains a homeomorphic copy of the unit interval $[0, 1]$, then X is G^* -regular.
- (iii) If X is zero-dimensional in the sense of ind, then X is G^{**} -regular.

In particular, in all three cases, X is G -regular by (1).

It should be noted that our terminology differs from that of [13], where a pair (X, G) consisting of a space X and a topological group G is called G -regular if it satisfies the condition (iii) of Definition 2.2. The same manuscript [13] states explicitly (but using different terminology) item (i) of Proposition 2.3.

For a cardinal $\tau \geq 1$ we denote by D_τ the discrete space of size τ .

Proposition 2.4. Let G be a topological group and $\tau \geq 1$ be a cardinal.

- (i) Spaces X and Y are G^τ -equivalent if and only if $X \times D_\tau$ and $Y \times D_\tau$ are G -equivalent.
- (ii) If τ is infinite, then every space X is G^τ -equivalent to $X \times D_\tau$.

Proof. (i) follows from $C_p(X \times D_\tau, G) \cong C_p(X, G^{D_\tau}) \cong C_p(X, G^\tau)$ and $C_p(Y \times D_\tau, G) \cong C_p(Y, G^{D_\tau}) \cong C_p(Y, G^\tau)$.
(ii) follows from $C_p(X \times D_\tau, G^\tau) \cong C_p(X, (G^\tau)^{D_\tau}) \cong C_p(X, G^{\tau \times D_\tau}) \cong C_p(X, G^\tau)$ because $|\tau \times D_\tau| = \tau$. \square

Applying Propositions 2.3(iii) and 2.4(ii), we get the following

Corollary 2.5. For every topological group G , a singleton and the countable discrete space D_ω are G^{**} -regular and G^ω -equivalent. In particular, G^ω -equivalence preserves neither pseudocompactness (compactness), nor (pathwise) connectedness within the class of G^{**} -regular spaces.

Corollary 2.6. If G -equivalence preserves the finite number of connected components, then so does G^k -equivalence for all $k \in \mathbb{N} \setminus \{0\}$.

Proof. Fix $k \in \mathbb{N} \setminus \{0\}$, and suppose that X and Y are G^k -equivalent. Then $X \times D_k$ and $Y \times D_k$ are G -equivalent by Proposition 2.4(i). By the assumption of our corollary, $X \times D_k$ and $Y \times D_k$ have the same (finite) number of connected components. Clearly, this implies that X and Y must also have the same number of connected components. \square

Proposition 2.7. Let $\{G_i: i \in I\}$ be a family of topological groups. If spaces X and Y are G_i -equivalent for all $i \in I$, then X and Y are also $(\prod_{i \in I} G_i)$ -equivalent.

Proof. $C_p(X, \prod_{i \in I} G_i) \cong \prod_{i \in I} C_p(X, G_i)$. \square

Corollary 2.8. Let G be a topological group.

- (i) G -equivalence implies G^κ -equivalence for every cardinal $\kappa \geq 1$.
- (ii) Suppose that τ and κ are cardinals such that $1 \leq \tau \leq \kappa$ and $\kappa \geq \omega$. Then G^τ -equivalence implies G^κ -equivalence.

Proposition 2.9. $C_p(X, G)$ contains a closed subgroup topologically isomorphic to G .

Definition 2.10. For a topological group G , a topological property \mathcal{E} and a class \mathcal{C} of spaces define

$$\mathbf{I}(G, \mathcal{E}, \mathcal{C}) = \{ \tau : \tau \geq 1 \text{ is a cardinal such that } G^\tau\text{-equivalence preserves property } \mathcal{E} \text{ within the class } \mathcal{C} \}.$$

Lemma 2.11. Let G be a topological group, \mathcal{E} a topological property and \mathcal{C} a class of spaces. Then $\mathbf{I}(G, \mathcal{E}, \mathcal{C}) \neq \emptyset$ implies $1 \in \mathbf{I}(G, \mathcal{E}, \mathcal{C})$.

Proof. Suppose $\mathbf{I}(G, \mathcal{E}, \mathcal{C}) \neq \emptyset$, and let $\kappa \in \mathbf{I}(G, \mathcal{E}, \mathcal{C})$. Assume that $X \in \mathcal{C}$ and $Y \in \mathcal{C}$ are G -equivalent spaces such that X has property \mathcal{E} . Then X and Y are also G^κ -equivalent by Corollary 2.8(i). Since $\kappa \in \mathbf{I}(G, \mathcal{E}, \mathcal{C})$ and X has property \mathcal{E} , Y must have it as well. Thus, $1 \in \mathbf{I}(G, \mathcal{E}, \mathcal{C})$. \square

Lemma 2.12. Assume that G is a topological group, $\tau \geq 1$ is a cardinal, \mathcal{E} is a topological property and \mathcal{C} is a class of spaces satisfying the following conditions:

- (i) a space X has property \mathcal{E} if and only if the product $X \times D_\tau$ has property \mathcal{E} ;
- (ii) if $X \in \mathcal{C}$, then $X \times D_\tau \in \mathcal{C}$.

Then $1 \in \mathbf{I}(G, \mathcal{E}, \mathcal{C})$ implies $\tau \in \mathbf{I}(G, \mathcal{E}, \mathcal{C})$.

Proof. Assume that $X \in \mathcal{C}$ and $Y \in \mathcal{C}$ are G^τ -equivalent spaces such that X has property \mathcal{E} . Then $X \times D_\tau$ and $Y \times D_\tau$ are G -equivalent by Proposition 2.4(i). Furthermore, $X \times D_\tau \in \mathcal{C}$ and $Y \times D_\tau \in \mathcal{C}$ by (ii). Since $X \times D_\tau$ has property \mathcal{E} by (i), and $1 \in \mathbf{I}(G, \mathcal{E}, \mathcal{C})$, we conclude that $Y \times D_\tau$ must have property \mathcal{E} . Applying (i) once again, we conclude that Y has property \mathcal{E} . This proves that $\tau \in \mathbf{I}(G, \mathcal{E}, \mathcal{C})$. \square

Proposition 2.13. Let G be a topological group, \mathcal{E} a topological property and \mathcal{C} a class of spaces satisfying the following conditions:

- (i) for every space X and each discrete space D , the space X has property \mathcal{E} if and only if $X \times D$ has property \mathcal{E} ;
- (ii) if $X \in \mathcal{C}$ and D is a discrete space, then $X \times D \in \mathcal{C}$.

Assume that G^κ -equivalence preserves property \mathcal{E} within the class \mathcal{C} for some cardinal $\kappa \geq 1$. Then G^τ -equivalence preserves property \mathcal{E} within the class \mathcal{C} for each cardinal $\tau \geq 1$.

Proof. The assumption of our proposition yields $\mathbf{I}(G, \mathcal{E}, \mathcal{C}) \neq \emptyset$, and so $1 \in \mathbf{I}(G, \mathcal{E}, \mathcal{C})$ by Lemma 2.11. Applying the assumption of our proposition and Lemma 2.12 to each cardinal $\tau \geq 1$, we conclude that $\mathbf{I}(G, \mathcal{E}, \mathcal{C}) = \{\tau \geq 1: \tau \text{ is a cardinal}\}$. \square

The above proposition is applicable to many “local” properties.

Corollary 2.14. *Let \mathcal{C} be a class of spaces such that $X \times D \in \mathcal{C}$ whenever $X \in \mathcal{C}$ and D is a discrete space. Let \mathcal{E} be one of the following properties: metrizability, paracompactness, weak paracompactness, local compactness, first countability, countable tightness, Fréchet–Urysohn property, sequentiality, total disconnectedness, (property of having a given) covering dimension \dim , large inductive dimension Ind , small inductive dimension ind . Assume also that G is a topological group such that G^κ -equivalence preserves property \mathcal{E} within the class \mathcal{C} for some cardinal $\kappa \geq 1$. Then for every cardinal $\tau \geq 1$, G^τ -equivalence preserves property \mathcal{E} within the class \mathcal{C} .*

The list of properties \mathcal{E} in the above corollary can be easily extended.

Proposition 2.15. *Let G be a topological group and κ an infinite cardinal. Assume that \mathcal{E} is a topological property and \mathcal{C} is a class of spaces such that:*

- (i) *for every space X , the space $X \times D_\tau$ has property \mathcal{E} if and only if X has property \mathcal{E} and $1 \leq \tau < \kappa$,*
- (ii) *if $X \in \mathcal{C}$ and D is a discrete space, then $X \times D \in \mathcal{C}$, and*
- (iii) *there exists at least one space $X \in \mathcal{C}$ satisfying property \mathcal{E} .*

Then either $\mathbf{I}(G, \mathcal{E}, \mathcal{C}) = \emptyset$, or

$$\mathbf{I}(G, \mathcal{E}, \mathcal{C}) = \{\tau: \tau \text{ is a cardinal satisfying } 1 \leq \tau < \kappa\}. \tag{2}$$

Proof. Let $\tau \geq \kappa$. In particular, τ is infinite. By item (iii), there exists a space $X \in \mathcal{C}$ satisfying property \mathcal{E} . Then $X \times D_\tau \in \mathcal{C}$ by item (ii). Since $\tau \geq \kappa$, from item (i) we conclude that $X \times D_\tau$ does not have property \mathcal{E} . Since X and $X \times D_\tau$ are G^τ -equivalent by Proposition 2.4(ii), it follows that $\tau \notin \mathbf{I}(G, \mathcal{E}, \mathcal{C})$. We have proved that

$$\mathbf{I}(G, \mathcal{E}, \mathcal{C}) \subseteq \{\tau: \tau \text{ is a cardinal satisfying } 1 \leq \tau < \kappa\}. \tag{3}$$

Assume now that $\mathbf{I}(G, \mathcal{E}, \mathcal{C}) \neq \emptyset$. Then $1 \in \mathbf{I}(G, \mathcal{E}, \mathcal{C})$ by Lemma 2.11. If τ is a cardinal satisfying $1 \leq \tau < \kappa$, then from (i), (ii) and Lemma 2.12 we get $\tau \in \mathbf{I}(G, \mathcal{E}, \mathcal{C})$. Together with (3) this proves (2). \square

Corollary 2.16. *Let G, κ, \mathcal{E} and \mathcal{C} be as in the assumption of Proposition 2.15. Assume also that G^σ -equivalence preserves property \mathcal{E} within the class \mathcal{C} for some cardinal $\sigma \geq 1$. Then the following conditions are equivalent:*

- (i) *G^τ -equivalence preserves property \mathcal{E} within the class \mathcal{C} ;*
- (ii) *$1 \leq \tau < \kappa$.*

Proof. The assumption of our corollary yields $\sigma \in \mathbf{I}(G, \mathcal{E}, \mathcal{C}) \neq \emptyset$, so (2) holds by Proposition 2.15. It remains only to show that G^0 -equivalence does not preserve property \mathcal{E} within the class \mathcal{C} . Since $G^0 = \{e\}$ is the trivial group, so is $C_p(Y, G^0)$ for every space Y . Thus, any two spaces are G^0 -equivalent. In particular, $X \overset{G^0}{\sim} X \times D_\kappa$, where $X \in \mathcal{C}$ is the space from item (iii) of Proposition 2.15. Then $X \times D_\kappa \in \mathcal{C}$ by Proposition 2.15(ii), while $X \times D_\kappa$ does not have property \mathcal{E} by Proposition 2.15(i). Since X has property \mathcal{E} , we conclude that G^0 -equivalence does not preserve property \mathcal{E} within the class \mathcal{C} . \square

Corollary 2.17. *Let G be a topological group, $m \geq 1$ an integer number and \mathcal{C} a class of spaces satisfying the condition from item (ii) of Proposition 2.15. If G^m -equivalence preserves compactness (countable compactness, pseudocompactness, σ -compactness) within the class \mathcal{C} , then G^k -equivalence preserves the corresponding property within the class \mathcal{C} for every $k \in \mathbb{N} \setminus \{0\}$.*

Proof. It suffices to note that $\kappa = \omega$, \mathcal{E} and \mathcal{C} satisfy the assumptions of Proposition 2.15, where \mathcal{E} is one of the four properties listed in the statement of our corollary. Indeed, item (i) of Proposition 2.15 obviously holds. If no space $X \in \mathcal{C}$ satisfies property \mathcal{E} , then G -equivalence trivially preserves property \mathcal{E} within the class \mathcal{C} . Otherwise, item (iii) of Proposition 2.15 holds. Since $m \in \mathbf{I}(G, \mathcal{E}, \mathcal{C}) \neq \emptyset$, the conclusion of our corollary follows now from that of Proposition 2.15. \square

3. When does $(G \times H)$ -equivalence imply both G -equivalence and H -equivalence?

Let G and H be topological groups. If two spaces are both G -equivalent and H -equivalent, then they are also $(G \times H)$ -equivalent (Proposition 2.7). In Example 7.5 we exhibit topological groups G and H such that $(G \times H)$ -equivalence implies

neither G -equivalence, nor H -equivalence, thereby demonstrating that the converse implication fails in general. In this section we will prove that this implication holds for sufficiently different topological groups G and H ; see Theorems 3.2 and 3.3. These results turn out to be useful in constructing numerous examples in Sections 7 and 8.

Lemma 3.1. *Suppose that X and Y are spaces, G and H are topological groups and*

$$\varphi : C_p(X, G) \times C_p(X, H) \rightarrow C_p(Y, G) \times C_p(Y, H) \text{ is a topological isomorphism} \quad (4)$$

satisfying

$$\varphi(C_p(X, G) \times \{1_X\}) = C_p(Y, G) \times \{1_Y\}, \quad (5)$$

where 1_X and 1_Y denote the identity elements of $C_p(X, H)$ and $C_p(Y, H)$, respectively. Then X and Y are both G -equivalent and H -equivalent.

Proof. From (4) and (5) we get $C_p(X, G) \cong C_p(Y, G)$. This proves that X and Y are G -equivalent. Applying (4) and (5) once again, one easily obtains that

$$C_p(X, H) \cong (C_p(X, G) \times C_p(X, H)) / (C_p(X, G) \times \{1_X\}) \cong (C_p(Y, G) \times C_p(Y, H)) / (C_p(Y, G) \times \{1_Y\}) \cong C_p(Y, H),$$

which proves that X and Y are H -equivalent. \square

Theorem 3.2. *Suppose that G is a precompact group and H is a topological group without nontrivial precompact subgroups. If two spaces are $(G \times H)$ -equivalent, then they are both G -equivalent and H -equivalent.*

Proof. Assume that X and Y are $(G \times H)$ -equivalent spaces. Then

$$C_p(X, G) \times C_p(X, H) \cong C_p(X, G \times H) \cong C_p(Y, G \times H) \cong C_p(Y, G) \times C_p(Y, H),$$

so we can fix φ satisfying (4). We continue using notations from Lemma 3.1. For $y \in Y$ let $\varpi_y : C_p(Y, G) \times C_p(Y, H) \rightarrow H$ be the continuous homomorphism defined by $\varpi_y(g, h) = h(y)$ for $(g, h) \in C_p(Y, G) \times C_p(Y, H)$.

Since G is precompact, so is G^X . Being a subgroup of the precompact group G^X , the group $C_p(X, G)$ is precompact as well. Being an image of the precompact group under a continuous group homomorphism, $H_y = \varpi_y(\varphi(C_p(X, G) \times \{1_X\}))$ is a precompact subgroup of H for every $y \in Y$. By our assumption, each H_y must be the trivial subgroup of H , which yields the inclusion $\varphi(C_p(X, G) \times \{1_X\}) \subseteq C_p(Y, G) \times \{1_Y\}$. By the symmetry, the inclusion $\varphi^{-1}(C_p(Y, G) \times \{1_Y\}) \subseteq C_p(X, G) \times \{1_X\}$ holds as well. This proves (5). Now the conclusion follows from Lemma 3.1. \square

Let G be a group. Recall that $g \in G$ is called a *torsion element* of G if there exists some $n \in \mathbb{N} \setminus \{0\}$ such that $g^n = e$. The subset of all torsion elements of G is called the *torsion part* of G and denoted by $\text{tor}(G)$. If $\text{tor}(G) = \{e\}$, then G is called *torsion-free*. For a given $n \in \mathbb{N}$, let $G^{(n)} = \{g^n : g \in G\}$.

Theorem 3.3. *Let G and H be topological groups satisfying one of the following two conditions:*

- (i) $\text{tor}(G)$ is dense in G and \widehat{H} is torsion-free;
- (ii) there exists $n \in \mathbb{N}$ such that $\widehat{G}^{(n)} = \widehat{G}$ and $H^{(n)} = \{e\}$.

*If G^{**} -regular spaces are $(G \times H)$ -equivalent, then they are both G -equivalent and H -equivalent.*

Proof. Assume that G^{**} -regular spaces X and Y are $(G \times H)$ -equivalent. Arguing as in the beginning of the proof of Theorem 3.2, we can fix φ satisfying (4). For typographical reasons, define

$$G(X) = \widehat{C_p(X, G)}, \quad G(Y) = \widehat{C_p(Y, G)}, \quad H(X) = \widehat{C_p(X, H)} \quad \text{and} \quad H(Y) = \widehat{C_p(Y, H)}.$$

Let $\Phi : G(X) \times H(X) \rightarrow G(Y) \times H(Y)$ be the (unique) topological isomorphism extending φ . We claim that

$$\Phi(G(X) \times \{e_{H(X)}\}) = G(Y) \times \{e_{H(Y)}\}. \quad (6)$$

Since X is G^{**} -regular, one can easily see that $C_p(X, G)$ is dense in G^X . Therefore,

$$G(X) = \widehat{C_p(X, G)} = \widehat{G^X} = \widehat{G^X}. \quad (7)$$

We need to consider two cases.

Case 1. Item (i) holds. Since $\text{tor}(G)$ is dense in G , and the latter group is dense in \widehat{G} , we conclude that $\text{tor}(\widehat{G^X})$ is dense in $\widehat{G^X}$. Combining this with (7), we conclude that $\text{tor}(G(X))$ is dense in $G(X)$. Since Φ is a topological isomorphism, it follows that

$$\Phi(\text{tor}(G(X)) \times \{e_{H(X)}\}) \text{ is dense in } \Phi(G(X) \times \{e_{H(X)}\}) \quad (8)$$

and

$$\Phi(\text{tor}(G(X)) \times \{e_{H(X)}\}) \subseteq \text{tor}(G(Y) \times H(Y)). \tag{9}$$

Since $H(Y) = C_p(\widehat{Y}, H) \subseteq \widehat{H}^Y$ and \widehat{H} is torsion-free, $H(Y)$ must be torsion-free as well. In particular,

$$\text{tor}(G(Y) \times H(Y)) \subseteq G(Y) \times \{e_{H(Y)}\}. \tag{10}$$

Since the latter set is closed in $G(Y) \times H(Y)$, from (8)–(10) one concludes that

$$\Phi(G(X) \times \{e_{H(X)}\}) \subseteq G(Y) \times \{e_{H(Y)}\}. \tag{11}$$

Applying the same arguments to the inverse map Φ^{-1} of Φ , we get

$$\Phi^{-1}(G(Y) \times \{e_{H(Y)}\}) \subseteq G(X) \times \{e_{H(X)}\}. \tag{12}$$

Case 2. Item (ii) holds. Fix $n \in \mathbb{N}$ as in item (ii). Choose $g \in G(X)$ arbitrarily. From $\widehat{G}^{(n)} = \widehat{G}$ one gets $(\widehat{G}^X)^{(n)} = \widehat{G}^X$. Combining this with (7), we obtain that $G(X)^{(n)} = G(X)$. Hence there exists $g_0 \in G(X)$ such that $g_0^n = g$. Let $\Phi(g_0, e_{H(X)}) = (f, h)$.

Since $H^{(n)} = \{e_H\}$, by the “principle of extending of equations”, $\widehat{H}^{(n)} = \{e_{\widehat{H}}\}$ holds as well. Since $h \in H(Y) \subseteq \widehat{H}^Y$, we conclude that $h^n = e_{H(Y)}$. Thus,

$$\Phi(g, e_{H(X)}) = \Phi((g_0, e_{H(X)})^n) = \Phi(g_0, e_{H(X)})^n = (f, h)^n = (f^n, h^n) = (f^n, e_{H(Y)}).$$

This proves the inclusion (11). Applying the same arguments to the inverse map Φ^{-1} of Φ , we get the inclusion (12).

Going back to the common proof, note that (11) and (12) yield (6). Furthermore, since Φ extends the isomorphism φ , from (6) one gets (5), and now the application of Lemma 3.1 finishes the proof. \square

Recall that, for a prime number p , a group G has exponent p if $G^{(p)} = \{e_G\}$ and $G \neq \{e_G\}$.

Corollary 3.4. *Let p and q be distinct prime numbers. Suppose that a topological group G has exponent p and a topological group H has exponent q . Suppose also that spaces X and Y are G^{**} -regular and $(G \times H)$ -equivalent. Then X and Y are both G -equivalent and H -equivalent.*

Proof. Since G has exponent p , the “principle of extending of equations” implies $\widehat{G}^{(p)} = \{e_{\widehat{G}}\}$. That is, \widehat{G} has exponent p as well. One can easily see that this yields $\widehat{G}^{(q)} = \widehat{G}$. Since $H^{(q)} = \{e_H\}$, the conclusion of our corollary follows from Theorem 3.3(ii). \square

Example 3.5. Let $(\mathbb{C} \setminus \{0\}, \cdot)$ denote the multiplicative group of all nonzero complex numbers with its usual topology. Then spaces X and Y are $(\mathbb{C} \setminus \{0\}, \cdot)$ -equivalent if and only if they are both \mathbb{R} -equivalent and \mathbb{T} -equivalent. Indeed, it suffices to realize that $\mathbb{C} \setminus \{0\} \cong \mathbb{T} \times \mathbb{R}$. The rest follows from Proposition 2.7, combined with either Theorem 3.2 or Theorem 3.3(i).

4. TAP and NSS groups

Definition 4.1. We say that a subset A of a topological group G is *absolutely productive in G* provided that, for every injection $a: \mathbb{N} \rightarrow A$ and each mapping $z: \mathbb{N} \rightarrow \mathbb{Z}$, the sequence

$$\left\{ \prod_{n=0}^k a(n)^{z(n)} : k \in \mathbb{N} \right\} \tag{13}$$

of elements of G converges to some $g \in G$. In such a case we will also say that the (infinite) product $\prod_{n=0}^{\infty} a(n)^{z(n)}$ converges to g and write

$$g = \prod_{n=0}^{\infty} a(n)^{z(n)}. \tag{14}$$

The proofs of the next three lemmas are straightforward.

Lemma 4.2.

- (i) *A subset of an absolutely productive set is absolutely productive.*
- (ii) *Let $\phi: G \rightarrow H$ be a continuous homomorphism between topological groups G and H . If a set $A \subseteq G$ is absolutely productive in G , then $\phi(A)$ is absolutely productive in H .*

Lemma 4.3. Let H be a subgroup of a topological group G and $A \subseteq H$.

- (i) If A is absolutely productive in H , then it is absolutely productive in G as well.
- (ii) If H is sequentially closed in G , then A is absolutely productive in H if and only if A is absolutely productive in G .

Item (i) of the next lemma gives a typical example of an infinite absolutely productive set, while item (ii) shows that “sequentially closed” cannot be omitted in Lemma 4.3(ii).

Lemma 4.4. Let $\{G_i \in I\}$ be an infinite family consisting of nontrivial topological groups G_i . Let $G = \prod_{i \in I} G_i$ and

$$H = \{g \in G : \text{the set } \{i \in I : g(i) \neq e_{G_i}\} \text{ is finite}\}.$$

For each $i \in I$ choose $g_i \in G \setminus \{e\}$ such that $g_i(j) = e_{G_j}$ for every $j \in I \setminus \{i\}$, and consider the infinite set $A = \{g_i : i \in I\} \subseteq H$. Then:

- (i) A is absolutely productive in G , but
- (ii) A is not absolutely productive in H .

Definition 4.5. We say that a topological group G is TAP (an abbreviation for “trivially absolutely productive”) if every absolutely productive set in G is finite.

Proposition 4.6.

- (i) The class of all TAP groups is closed under taking finite products and subgroups.
- (ii) Let G_i be a nontrivial topological group for every $i \in \mathbb{N}$. Then $G = \prod_{i=0}^{\infty} G_i$ is not TAP.
- (iii) A TAP group does not contain any subgroup topologically isomorphic to a Cartesian product $\prod_{i=0}^{\infty} G_i$ of nontrivial topological groups G_i .

Proof. Item (i) follows from Lemmas 4.2(ii) and 4.3(i), item (ii) follows from Lemma 4.4(i), and item (iii) follows from items (i) and (ii). \square

Remark 4.7. The converse of Proposition 4.6(iii) does not hold in general. Indeed, the group \mathbb{Z}_p of p -adic integers does not contain any subgroup topologically isomorphic to an infinite product of nontrivial groups, yet \mathbb{Z}_p is not TAP [8].

Recall that a topological group G is an NSS group, or has an NSS property (an abbreviation for “no small subgroups”) if G has an open neighborhood of the identity containing no nontrivial subgroups of G . The following lemma provides a simple reformulation of the NSS property.

Lemma 4.8. Let G be a topological group. Then the following conditions are equivalent:

- (i) G is an NSS group;
- (ii) there exists an open neighborhood U of the identity e of G such that for every $g \in G \setminus \{e\}$ and each $f, h \in G$ one can find $z \in \mathbb{Z}$ with $hg^z \notin fU$.

Proof. (i) \Rightarrow (ii) Let V be a neighborhood of the identity e witnessing that G is NSS. Choose a neighborhood U of e with $U^{-1}U \subseteq V$. Let $g \in G \setminus \{e\}$ and $f, h \in G$ be arbitrary. If $f^{-1}h \notin U$, then $f^{-1}hg^0 = f^{-1}h \notin U$, and consequently $hg^0 \notin fU$, so $z = 0$ works. Suppose now that $f^{-1}h \in U$. Then $h^{-1}fU \subseteq U^{-1}U \subseteq V$. By the choice of V , we can find $z \in \mathbb{Z}$ such that $g^z \notin V$ (otherwise V would contain the nontrivial cyclic subgroup generated by g). In particular, $g^z \notin h^{-1}fU$, and so $hg^z \notin fU$.

(ii) \Rightarrow (i) Applying (ii) with $f = h = e$, we conclude that for every $g \in G \setminus \{e\}$ there exists $z \in \mathbb{Z}$ such that $g^z \notin U$. This means that U is an open neighborhood of e which contains no nontrivial subgroup. Thus G is NSS. \square

Theorem 4.9. An NSS group is TAP.

Proof. Assume that A is an infinite subset of an NSS group G . We must show that A is not absolutely productive in G . Fix an injection $a: \mathbb{N} \rightarrow A \setminus \{e\}$. The subgroup H of G generated by $a(\mathbb{N})$ is countable, so we can choose a map $f: \mathbb{N} \rightarrow H$ such that $f^{-1}(h)$ is infinite for every $h \in H$. Since G is NSS, we can fix U satisfying item (ii) of Lemma 4.8. We are going to define a map $z: \mathbb{N} \rightarrow \mathbb{Z}$ such that

$$\prod_{i=0}^k a(i)^{z(i)} \notin f(k)U \tag{*}_k$$

holds for every $k \in \mathbb{N}$. Since $a(0) \neq e$, applying item (ii) of Lemma 4.8 to $g = a(0)$, $f = f(0)$ and $h = e$, we can choose $z(0) \in \mathbb{Z}$ satisfying $(*_0)$. For $n \in \mathbb{N} \setminus \{0\}$, assume that $z(j) \in \mathbb{Z}$ satisfying $(*_j)$ has already been defined for all $j < n$. Since $a(n) \neq e$, applying item (ii) of Lemma 4.8 to $f = f(n)$, $g = a(n)$ and $h = \prod_{i=0}^{n-1} a(i)^{z(i)}$ we can choose $z(n) \in \mathbb{Z}$ satisfying $(*_n)$. This finishes the inductive construction.

Suppose now that (14) holds for some $g \in G$. Since $\prod_{i=0}^k a(i)^{z(i)} \in H$ for every $k \in \mathbb{N}$, g must belong to the sequential closure of H . In particular, $g \in \overline{H} \subseteq HU$, and so $g \in hU$ for some $h \in H$. On the other hand, $(*_k)$ holds for every $k \in \mathbb{N}$, which gives

$$f^{-1}(h) \subseteq \left\{ k \in \mathbb{N} : \prod_{i=0}^k a(i)^{z(i)} \notin hU \right\}.$$

Since the set $f^{-1}(h)$ is infinite and $g \in hU$, we conclude that the sequence (13) cannot converge to g , in contradiction with (14). This proves that A is not absolutely productive in G . \square

Proposition 4.10.

- (i) Every topological group without nontrivial convergent sequences is TAP.
- (ii) An infinite pseudocompact group without nontrivial convergent sequences is TAP but not NSS.

Proof. (i) Let A be an infinite subset of a topological group G without nontrivial convergent sequences. Fix an injection $a: \mathbb{N} \rightarrow A$. By induction on $m \in \mathbb{N}$ one can easily choose an increasing sequence $\{n_m: m \in \mathbb{N}\} \subseteq \mathbb{N}$ such that

$$\prod_{i=0}^m a(n_i) \notin \left\{ \prod_{i=0}^k a(n_i) : k \in \mathbb{N}, k < m \right\}. \tag{15}$$

Define $z: \mathbb{N} \rightarrow \mathbb{Z}$ by $z(n) = 1$ if $n \in \{n_m: m \in \mathbb{N}\}$ and $z_n = 0$ otherwise. It follows from (15) that the sequence (13) is nontrivial, and so it cannot converge by our assumption on G . Thus, A is not absolutely productive in G .

(ii) Assume, in addition, that G is pseudocompact and infinite. Suppose that G is NSS, and choose an open neighborhood U of the identity e that contains no nontrivial subgroups of G . Starting with $U_0 = U$, choose a sequence $\{U_n: n \in \mathbb{N}\}$ of open neighborhoods of e such that $U_{n+1}^{-1}U_{n+1} \subseteq U_n$ for every $n \in \mathbb{N}$. Then $H = \bigcap_{n=0}^{\infty} U_n \subseteq U_0 = U$ is a subgroup of G , which gives $H = \{e\}$. Since G is pseudocompact, we conclude that G must be metrizable and hence compact. Being infinite, G must contain a nontrivial convergent sequence, a contradiction. \square

Remark 4.11.

- (i) There is an infinite pseudocompact Abelian group without nontrivial convergent sequences [21]. Therefore, from Proposition 4.10 we conclude that *there exists a pseudocompact Abelian TAP group that is not NSS*.
- (ii) There are consistent examples of infinite countably compact Abelian groups without nontrivial convergent sequences; see [7] for references. Applying Proposition 4.10, we conclude that *the existence of a countably compact Abelian TAP group that is not NSS is consistent with ZFC*.
- (iii) Theorem 4.9 can sometimes be reversed. Indeed, it has been proved recently in [8] that *a locally compact TAP group is NSS*. Moreover, *a totally disconnected compact TAP group is finite* [8].
- (iv) It is proved in [8] that *a σ -compact complete Abelian TAP group need not be NSS*.

We refer the reader to Proposition 6.9(ii) for other examples of TAP groups that are not NSS.

5. A group-theoretic proof that l -equivalence preserves pseudocompactness

Lemma 5.1. *Assume that X is a space, G is a TAP group and A is a subset of $C_p(X, G)$. If A is absolutely productive in $C_p(X, G)$, then the set $\{f \in A: f(x) \neq e\}$ is finite for every $x \in X$.*

Proof. Suppose, by the way of contradiction, that there exists $x \in X$ such that $f(x) \neq e$ for infinitely many $f \in A$. Define $\pi_x: C_p(X, G) \rightarrow G$ by $\pi_x(f) = f(x)$ for all $f \in C_p(X, G)$. Since π_x is a continuous homomorphism, $\pi_x(A)$ is absolutely productive in G by Lemma 4.2(ii). Since G is TAP, $\pi_x(A)$ must be finite, so by the pigeon hole principle, there exists $g \in G \setminus \{e\}$ and an infinite set $\{f_i: i \in \mathbb{N}\} \subseteq A$ such that $f_i(x) = g$ for every $i \in \mathbb{N}$. Since the sequence $\{\prod_{i=0}^k f_i^{(-1)^i}(x): k \in \mathbb{N}\}$ alternates between e and $g \neq e$, the product $\prod_{i=0}^{\infty} f_i^{(-1)^i}$ does not exist. This contradicts the fact that A is absolutely productive. \square

Lemma 5.2. Let X be a space, G a TAP group and A an infinite absolutely productive subset of $C_p(X, G)$. Then there exist two (necessarily faithfully indexed) sequences $\{x_i: i \in \mathbb{N}\} \subseteq X$ and $\{f_i: i \in \mathbb{N}\} \subseteq A$ such that

$$f_i(x_i) \neq e \text{ and } f_i(x_j) = e \text{ whenever } i, j \in \mathbb{N} \text{ and } j < i. \tag{16}$$

Proof. We use induction on $i \in \mathbb{N}$. First, choose $f_0 \in A$ and $x_0 \in X$ such that $f_0(x_0) \neq e$. Let $n \in \mathbb{N} \setminus \{0\}$, and suppose that $\{x_0, x_1, \dots, x_{n-1}\} \subseteq X$ and $\{f_0, f_1, \dots, f_{n-1}\} \subseteq A$ have already been selected so that $f_i(x_i) \neq e$ and $f_i(x_j) = e$ whenever $i, j \in \mathbb{N}$ and $j < i \leq n-1$. The set $B_n = \bigcup_{i=0}^{n-1} \{f \in A: f(x_i) \neq e\}$ is finite by Lemma 5.1, and hence there exists $f_n \in A \setminus B_n \neq \emptyset$. Without loss of generality, f_n is not the identity element of $C_p(X, G)$, and so $f_n(x_n) \neq e$ for some $x_n \in X$. \square

Theorem 5.3. A space X is pseudocompact if and only if $C_p(X, \mathbb{R})$ has the TAP property.

Proof. Since the group $C_p(X, \mathbb{R})$ is Abelian, in this proof we shall use the additive notation.

To prove the “if” part, suppose that X is not pseudocompact. Fix an infinite discrete family $\mathcal{U} = \{U_i: i \in \mathbb{N}\}$ of non-empty open subsets of X . For each $i \in \mathbb{N}$ choose $x_i \in U_i$ and $f_i \in C_p(X, \mathbb{R})$ such that $f(x_i) \neq 0$ and $f(X \setminus U_i) \subseteq \{0\}$. Clearly, $A = \{f_i: i \in \mathbb{N}\}$ is faithfully indexed (and thus infinite). Let $s: \mathbb{N} \rightarrow \mathbb{N}$ be an injection and $z: \mathbb{N} \rightarrow \mathbb{Z}$ a map. Since \mathcal{U} is discrete, $f = \sum_{i=0}^{\infty} z(i) f_{s(i)} \in C_p(X, \mathbb{R})$ and $f = \lim_{k \rightarrow \infty} \sum_{i=0}^k z(i) f_{s(i)}$. This shows that A is absolutely productive in $C_p(X, \mathbb{R})$, and so $C_p(X, \mathbb{R})$ is not TAP.

Being an NSS group, \mathbb{R} has the TAP property by Theorem 4.9. A simpler direct proof can be obtained as follows. Let A be an infinite subset of \mathbb{R} . Fix an injection $a: \mathbb{N} \rightarrow A \setminus \{0\}$. By induction on $k \in \mathbb{N}$ choose $z(n) \in \mathbb{Z}$ such that $\sum_{n=0}^k z(n)a(n) > k$. Then the series $\sum_{n=0}^{\infty} z(n)a(n)$ diverges, thereby proving that A is not absolutely productive in \mathbb{R} .

To prove the “only if” part, assume that $C_p(X, \mathbb{R})$ is not TAP and choose an infinite absolutely productive set $A \subseteq C_p(X, \mathbb{R})$. Let $\{x_i: i \in \mathbb{N}\} \subseteq X$ and $\{f_i: i \in \mathbb{N}\} \subseteq A$ be as in the conclusion of Lemma 5.2 (with 0 instead of e due to the additive notation). By induction on $n \in \mathbb{N}$ select $z(n) \in \mathbb{Z}$ so that

$$\sum_{i=0}^n z(i) f_i(x_n) > n. \tag{17}$$

Since A is absolutely productive, there exists $f \in C_p(X, \mathbb{R})$ such that $f = \lim_{k \rightarrow \infty} \sum_{i=0}^k z(i) f_i$. From (16) and (17) we get $f(x_n) = \sum_{i=0}^n z(i) f_i(x_n) > n$ for every $n \in \mathbb{N}$. Thus, the function f is unbounded on X , and so X is not pseudocompact. \square

Since \mathbb{R} -equivalence coincides with l -equivalence, from Theorem 5.3 we obtain the following well-known result of Arhangel'skiĭ.

Corollary 5.4. ([2]) l -equivalence preserves pseudocompactness.

6. Pseudocompactness of X and TAP property of $C_p(X, G)$

The main goal of this section is to generalize Theorem 5.3 by replacing the real line \mathbb{R} in it with an arbitrary NSS group G (provided that X is G -regular), see Theorem 6.5.

Lemma 6.1. If X is a countably compact space and G is an NSS group, then $C_p(X, G)$ has the TAP property.

Proof. Theorem 4.9 yields that G is TAP. Suppose that A is an infinite absolutely productive set in $C_p(X, G)$. Let $\{x_i: i \in \mathbb{N}\} \subseteq X$ and $\{f_i: i \in \mathbb{N}\} \subseteq A$ be as in the conclusion of Lemma 5.2. Since X is countably compact, the set $\{x_i: i \in \mathbb{N}\}$ has a cluster point $x \in X$. By Lemma 5.1 the set $J = \{j \in \mathbb{N}: f_j(x) \neq e\}$ is finite. Let $j = \max J$. After deleting the first $(j + 1)$ -many f_i 's and renumbering, we can assume, without loss of the generality, that

$$f_i(x) = e \text{ for every } i \in \mathbb{N}. \tag{18}$$

Since G is NSS, we can fix U as in item (ii) of Lemma 4.8. By recursion on $n \in \mathbb{N}$ we will choose $z_n \in \mathbb{Z}$ such that

$$\prod_{i=0}^n f_i(x_n)^{z_i} \notin U. \tag{**_n}$$

Indeed, applying item (ii) of Lemma 4.8 to $g = f_0(x_0) \neq e$ and $f = h = e$, we can select $z_0 \in \mathbb{Z}$ satisfying (**₀). Let $n \in \mathbb{N} \setminus \{0\}$, and suppose that $z_i \in \mathbb{Z}$ satisfying (**_i) have already been selected for each $i \in \mathbb{N}$ with $i < n$. Applying item (ii) of Lemma 4.8 to $f = e$, $g = f_n(x_n)$ and $h = \prod_{i=0}^{n-1} f_i(x_n)^{z_i}$, we can find $z_n \in \mathbb{Z}$ satisfying (**_n).

Since A is an absolutely productive subset of $C_p(X, G)$, there exists $f \in C_p(X, G)$ such that

$$f = \prod_{i=0}^{\infty} f_i^{z_i}. \tag{19}$$

From this and (16) we conclude that

$$f(x_n) = \lim_{k \rightarrow \infty} \prod_{i=0}^k f_i(x_n)^{z_i} = \prod_{i=0}^n f_i(x_n)^{z_i} \quad \text{for every } n \in \mathbb{N}.$$

Combining this with $(**_n)$, we get $f(x_n) \notin U$ for every $n \in \mathbb{N}$. Since x is a cluster point of the set $\{x_n : n \in \mathbb{N}\}$ and f is continuous, $f(x)$ must be a cluster point of the set $\{f(x_n) : n \in \mathbb{N}\}$, which yields $f(x) \notin U$. On the other hand, from (18) and (19) we should have

$$f(x) = \lim_{k \rightarrow \infty} \prod_{i=0}^k f_i(x)^{z_i} = e \in U,$$

a contradiction. This proves that all absolutely productive subsets of $C_p(X, G)$ are finite. \square

Lemma 6.2. *If X is a pseudocompact space and G is a metrizable NSS group, then $C_p(X, G)$ is TAP.*

Proof. Assume that $C_p(X, G)$ is not TAP, and let \mathcal{F} be an infinite absolutely productive subset of $C_p(X, G)$. Lemma 4.2(i) allows us to assume, without loss of generality, that \mathcal{F} is countable, so we can fix a faithful enumeration $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ of \mathcal{F} . Let $h = \Delta_{n \in \mathbb{N}} f_n : X \rightarrow G^{\mathbb{N}}$ be the diagonal product. Since each f_n is continuous, so is h . Since X is pseudocompact, $Y = h(X)$ is pseudocompact as well. Being a subspace of the metrizable space $G^{\mathbb{N}}$, Y is metrizable. It follows that Y is compact.

For $n \in \mathbb{N}$ let $p_n : G^{\mathbb{N}} \rightarrow G$ be the projection on n th coordinate (defined by $p_n(\phi) = \phi(n)$ for $\phi \in G^{\mathbb{N}}$). Since p_n is continuous, $g_n = p_n \upharpoonright Y \in C_p(Y, G)$. Clearly, $f_n = g_n \circ h$. If $m, n \in \mathbb{N}$ and $m \neq n$, then $f_m \neq f_n$, and so $g_m(h(x)) = g_m \circ h(x) = f_m(x) \neq f_n(x) = g_n \circ h(x) = g_n(h(x))$ for some $x \in X$, which yields $g_m \neq g_n$. Therefore, the family $\mathcal{G} = \{g_n : n \in \mathbb{N}\} \subseteq C_p(Y, G)$ is faithfully indexed (in particular, infinite).

Let us show that \mathcal{G} is absolutely productive in $C_p(Y, G)$, in contradiction with Lemma 6.1. Let $s : \mathbb{N} \rightarrow \mathbb{N}$ be an injection and $z : \mathbb{N} \rightarrow \mathbb{Z}$ a map. Since \mathcal{F} is absolutely productive, there exists $f \in C_p(X, G)$ such that $f = \prod_{n=0}^{\infty} f_{s(n)}^{z(n)}$.

Assume that $x, x' \in X$ and $h(x) = h(x')$. For each $n \in \mathbb{N}$ we have $f_n = g_n \circ h$, which yields $f_n(x) = f_n(x')$. Therefore,

$$f(x) = \lim_{k \rightarrow \infty} \prod_{n=0}^k f_{s(n)}^{z(n)}(x) = \lim_{k \rightarrow \infty} \prod_{n=0}^k f_{s(n)}^{z(n)}(x') = f(x').$$

It follows that there exists a unique function $g : Y \rightarrow G$ such that $f = g \circ h$. Since X is pseudocompact, G is metrizable and $f \in C_p(X, G)$, from [4, Theorem 7] we conclude that $g \in C_p(Y, G)$.

Let $y \in Y$ be arbitrary. Choose $x \in X$ such that $y = h(x)$. Then

$$\begin{aligned} g(y) &= g(h(x)) = f(x) = \lim_{k \rightarrow \infty} \prod_{n=0}^k f_{s(n)}^{z(n)}(x) = \lim_{k \rightarrow \infty} \prod_{n=0}^k (g_{s(n)} \circ h)^{z(n)}(x) \\ &= \lim_{k \rightarrow \infty} \prod_{n=0}^k (g_{s(n)}(h(x)))^{z(n)} = \lim_{k \rightarrow \infty} \prod_{n=0}^k g_{s(n)}^{z(n)}(y). \end{aligned}$$

Therefore, $g = \prod_{n=0}^{\infty} g_{s(n)}^{z(n)}$. Thus, \mathcal{G} is absolutely productive in $C_p(Y, G)$. \square

Theorem 6.3. *If X is a pseudocompact space and G is an NSS group, then $C_p(X, G)$ is TAP.*

Proof. Assume that $C_p(X, G)$ is not TAP, and let A be an infinite absolutely productive subset of $C_p(X, G)$. Lemma 4.2(i) allows us to assume, without loss of generality, that A is countable, so we can fix a faithful enumeration $A = \{f_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, the subset $f_n(X)$ of G is pseudocompact, being a continuous image of the pseudocompact space X . The smallest closed subgroup K of G containing $\bigcup \{f_n(X) : n \in \mathbb{N}\}$ must be ω -bounded. (Recall that, according to Guran [11], a topological group G is called ω -bounded if, for any open set $U \subseteq G$, there exists a countable set $S \subseteq G$ such that $SU = \{su : s \in S, u \in U\} = G$.) Note that $A \subseteq C_p(X, K)$ and $C_p(X, K)$ is a closed subgroup of $C_p(X, G)$, so A is absolutely productive in $C_p(X, K)$ by Lemma 4.3(ii). Being a subgroup of an NSS group G , K itself is an NSS group. Therefore, without loss of generality, we may (and will) assume that $K = G$, i.e., G itself is ω -bounded. Thus, there exists a family $\{G_\beta : \beta \in B\}$ consisting of separable metric groups G_β such that G is a subgroup of $\prod_{\beta \in B} G_\beta$; see [11].

Let $n, m \in \mathbb{N}$ and $n < m$. Since $f_n \neq f_m$, there is $x_{n,m} \in X$ such that $g_{n,m} = f_n(x_{n,m}) \neq f_m(x_{n,m}) = g_{m,n}$. So we can pick some $\beta_{n,m} \in B$ such that $g_{n,m}(\beta_{n,m}) \neq g_{m,n}(\beta_{n,m})$. Since G is an NSS group, we can find $k \in \mathbb{N}$, $\beta_1, \dots, \beta_k \in B$ and an open neighborhood U_i of the identity in G_{β_i} for $i \leq k$, such that the open neighborhood

$$U = G \cap \left\{ g \in \prod_{\beta \in B} G_\beta : g(\beta_i) \in U_i \text{ for } i = 1, \dots, k \right\}$$

of the identity of G contains no nontrivial subgroup of G . Define

$$C = \{\beta_{n,m} : n, m \in \mathbb{N}, n < m\} \cup \{\beta_1, \dots, \beta_k\}$$

and consider the projection $q : \prod_{\beta \in B} G_\beta \rightarrow \prod_{\beta \in C} G_\beta$. Let $H = q(G)$ and $\phi = q|_G$. As a subspace of a countable product of metrizable spaces, H is metrizable. Moreover, $\phi(U)$ is an open neighborhood of the identity of H which contains no nontrivial subgroup of H . Hence, H is NSS.

Let $\Phi : C_p(X, G) \rightarrow C_p(X, H)$ be the continuous homomorphism defined by $\Phi(f) = \phi \circ f$ for $f \in C_p(X, G)$. Then $\Phi(A)$ is an absolutely productive subset of $C_p(X, H)$ by Lemma 4.2(ii).

Let $m, n \in \mathbb{N}$ and $n < m$. Since $\beta_{n,m} \in C$ and $g_{n,m}(\beta_{n,m}) \neq g_{m,n}(\beta_{n,m})$, we have $q(g_{n,m}) \neq q(g_{m,n})$, which yields

$$\Phi(f_n)(x_{n,m}) = \phi(f_n(x_{n,m})) = \phi(g_{n,m}) = q(g_{n,m}) \neq q(g_{m,n}) = \phi(g_{m,n}) = \phi(f_m(x_{n,m})) = \Phi(f_m)(x_{n,m}).$$

Hence $\Phi(f_n) \neq \Phi(f_m)$. This shows that $\Phi(A) = \{\Phi(f_n) : n \in \mathbb{N}\}$ is a faithfully indexed (and thus infinite) set. It follows that $C_p(X, H)$ is not TAP, in contradiction with Lemma 6.2. \square

Lemma 6.4. *Let G be a topological group and X a G -regular space which is not pseudocompact. Then $C_p(X, G)$ contains a subgroup topologically isomorphic to a product $H = \prod_{i \in \mathbb{N}} H_i$ of nontrivial topological groups H_i .*

Proof. Since X is not pseudocompact, there exists a discrete family $\mathcal{U} = \{U_i : i \in \mathbb{N}\}$ consisting of non-empty open subsets of X . For each $i \in \mathbb{N}$ choose $x_i \in U_i$ and use G -regularity of X to fix $f_i \in C_p(X, G)$ such that $f_i(X \setminus U_i) \subseteq \{e\}$ and $f_i(x_i) \neq e$. Let H_i be the cyclic subgroup of $C_p(X, G)$ generated by f_i (equipped with the subspace topology inherited from $C_p(X, G)$). Clearly, H_i is nontrivial. Let $H = \prod_{i \in \mathbb{N}} H_i$. Since the family \mathcal{U} is discrete, for each $h \in H$ the infinite product $\theta(h) = \prod_{i=0}^{\infty} h(i) = \lim_{k \rightarrow \infty} \prod_{i=0}^k h(i)$ is well defined and $\theta(h) \in C_p(X, G)$. A straightforward verification of the fact that $\theta : H \rightarrow C_p(X, G)$ is a topological isomorphism between H and $\theta(H)$ is left to the reader. \square

Theorem 6.5. *For an NSS group G and a G -regular space X , the following conditions are equivalent:*

- (i) X is pseudocompact;
- (ii) $C_p(X, G)$ is TAP;
- (iii) $C_p(X, G)$ does not contain a subgroup which is topologically isomorphic to an infinite product of nontrivial topological groups.

Proof. (i) \Rightarrow (ii) follows from Theorem 6.3.

(ii) \Rightarrow (iii) follows from Proposition 4.6(iii).

(iii) \Rightarrow (i) follows from Lemma 6.4. \square

Corollary 6.6. *Let G be an NSS group. Then G -equivalence preserves pseudocompactness within the class of G -regular spaces.*

The proof of the following lemma is straightforward.

Lemma 6.7. *Let X and Y be spaces and H a topological group. For $f \in C_p(X \times Y, H)$ and $x \in X$ define $f_x \in C_p(Y, H)$ by $f_x(y) = f(x, y)$ for every $y \in Y$. Consider the map $\theta : C_p(X \times Y, H) \rightarrow C_p(X, C_p(Y, H))$ which assigns to every $f \in C_p(X \times Y, H)$ the function $\theta(f) \in C_p(X, C_p(Y, H))$ defined by $\theta(f)(x) = f_x$ for each $x \in X$. Then θ is a topological isomorphism between $C_p(X \times Y, H)$ and $\theta(C_p(X \times Y, H))$.*

It follows from Propositions 2.9 and 4.6(i) that G must be TAP whenever $C_p(X, G)$ is TAP. Our next theorem shows that the TAP property of G is not sufficient to ensure that $C_p(X, G)$ is TAP, even for a countably compact space X .

Theorem 6.8. *There exist a precompact TAP group G and a countably compact G^* -regular space X such that $C_p(X, G)$ is not TAP.*

Proof. Let X be a countably compact Tychonoff space and Y a pseudocompact Tychonoff space such that $X \times Y$ is not pseudocompact (see, for example, [9, Example 3.10.19]).

By Proposition 2.3, Y is \mathbb{T} -regular (in fact, even \mathbb{T}^{**} -regular). Since Y is pseudocompact and \mathbb{T} is NSS, Theorem 6.3 yields that $G = C_p(Y, \mathbb{T})$ is a TAP group. Since G is a subgroup of the compact group \mathbb{T}^Y , G is precompact. By Proposition 2.9, G contains a subgroup topologically isomorphic to \mathbb{T} , so X is G^* -regular by Proposition 2.3(ii).

By Proposition 2.3, the space $X \times Y$ is \mathbb{T} -regular (in fact, even \mathbb{T}^{**} -regular). Since $X \times Y$ is not pseudocompact, it follows from Theorem 6.5 that $C_p(X \times Y, \mathbb{T})$ contains an infinite set A that is absolutely productive in $C_p(X \times Y, \mathbb{T})$. According to Lemma 6.7, $C_p(X \times Y, \mathbb{T})$ is topologically isomorphic to a subgroup of $C_p(X, C_p(Y, \mathbb{T})) = C_p(X, G)$. Applying Lemma 4.3(i), we conclude that A is absolutely productive in $C_p(X, G)$. Since A is infinite, it follows that $C_p(X, G)$ is not TAP. \square

Theorem 6.8 demonstrates that the conclusion of Theorem 6.5 is no longer valid if we replace the NSS property of G in its assumption by the weaker TAP property.

Proposition 6.9. Let G be a topological group and X an infinite G -regular space.

- (i) $C_p(X, G)$ is not NSS.
- (ii) If, in addition, X is pseudocompact and G is NSS, then $C_p(X, G)$ is TAP but not NSS.

Proof. (i) Let U be any neighborhood of the identity in $C_p(X, G)$. Then there exist an integer $n \in \mathbb{N} \setminus \{0\}$, points $x_1, \dots, x_n \in X$ and an open set $V \subseteq G$ with $e \in V$, such that $H = \{f \in C_p(X, G) : f(x_i) = e \text{ for } i = 1, \dots, n\} \subseteq U$. Since X is infinite and G -regular, H is a nontrivial subgroup of $C_p(X, G)$. It follows that $C_p(X, G)$ is not NSS.

(ii) follows from (i) and Theorem 6.3. \square

7. Compactness-like properties and G -equivalence

For a space X , $t(X)$ stays for the tightness of X , $l(X)$ denotes the Lindelöf number of X , and $l^*(X) = \sup\{l(X^n) : n \in \mathbb{N}\}$.

Our first lemma is a “ G -analogue” of the well-known theorem of Pytkeev from the C_p -theory; see [18, Theorem 1]. Its proof essentially follows the original proof, with necessary adaptations to take into account the G^* -regularity condition.

Lemma 7.1. If G is a topological group and X is a G^* -regular space, then $l^*(X) \leq t(C_p(X, G))$.

Proof. To start with, we claim that one can find $g \in G \setminus \{e\}$ such that, for every open subset U of X and each non-empty finite set $K \subseteq U$, there exists $f_{K,U} \in C_p(X, G)$ satisfying $f_{K,U}(K) \subseteq \{e\}$ and $f_{K,U}(X \setminus U) \subseteq \{g^{-1}\}$. Indeed, let $g \in G \setminus \{e\}$ be the element witnessing G^* -regularity of X . Let U be an open subset of X and $K \neq \emptyset$ a finite subset of U . Let $K = \{x_0, \dots, x_k\}$ be a faithful enumeration of K , and let U_0, \dots, U_k be pairwise disjoint open subsets of U with $x_i \in U_i$ for $i \leq k$. For every $i \leq k$ we can choose $\varphi_i \in C_p(X, G)$ such that $\varphi_i(x_i) = g$ and $\varphi_i(X \setminus U_i) \subseteq \{e\}$. Then the function $f_{K,U} \in C_p(X, G)$ defined by $f_{K,U}(x) = g^{-1} \cdot \prod_{i=0}^k \varphi_i(x)$ for $x \in X$, is as required.

Given two families \mathcal{A} and \mathcal{B} , we write $\mathcal{A} < \mathcal{B}$ provided that, for every $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $A \subseteq B$.

Fix $n \in \mathbb{N} \setminus \{0\}$ and an open cover \mathcal{V} of X^n . Let \mathbf{U} denote the set of all finite families \mathcal{U} of open subsets of X satisfying $\Pi(\mathcal{U}) < \mathcal{V}$, where $\Pi(\mathcal{U}) = \{U_1 \times \dots \times U_n : (U_1, \dots, U_n) \in \mathcal{U}^n\}$. For every $\mathcal{U} \in \mathbf{U}$ choose a finite subfamily $\mathcal{V}_{\mathcal{U}}$ of \mathcal{V} such that $\Pi(\mathcal{U}) < \mathcal{V}_{\mathcal{U}}$. Let

$$F = \left\{ f \in C_p(X, G) : f\left(X \setminus \bigcup \mathcal{U}\right) \subseteq \{g^{-1}\} \text{ for some } \mathcal{U} \in \mathbf{U} \right\}. \tag{20}$$

We claim that $\mathbf{1} \in \bar{F}$, where $\mathbf{1} \in C_p(X, G)$ is defined by $\mathbf{1}(x) = e$ for all $x \in X$. Indeed, let K be a non-empty finite subset of X and O an arbitrary open neighborhood of e in G . Since \mathcal{V} is an open cover of X^n , there exists some $\mathcal{U} \in \mathbf{U}$ with $K \subseteq U$, where $U = \bigcup \mathcal{U}$. Now $f_{K,U} \in F \cap \bigcap_{x \in K} W(x, O) \neq \emptyset$.

Since $\mathbf{1} \in \bar{F}$, we can choose $F^* \subseteq F$ such that $|F^*| \leq t(C_p(X, G))$ and $\mathbf{1} \in \bar{F}^*$. For every $f \in F^*$ use (20) to select $\mathcal{U}_f \in \mathbf{U}$ such that

$$f\left(X \setminus \bigcup \mathcal{U}_f\right) \subseteq \{g^{-1}\}. \tag{21}$$

Define $\mathcal{V}^* = \bigcup_{f \in F^*} \mathcal{V}_{\mathcal{U}_f} \subseteq \mathcal{V}$. Clearly, $|\mathcal{V}^*| \leq |F^*| \leq t(C_p(X, G))$. It remains only to prove that \mathcal{V}^* covers X^n . Indeed, let $(x_1, \dots, x_n) \in X^n$ be arbitrary. Define $K = \{x_1, \dots, x_n\}$ and $O^* = G \setminus \{g^{-1}\}$. Then $\bigcap_{x \in K} W(x, O^*)$ is an open neighborhood of $\mathbf{1}$, so we can pick

$$f \in F^* \cap \bigcap_{x \in K} W(x, O^*). \tag{22}$$

From (22) and (21) we conclude that $K \subseteq \bigcup \mathcal{U}_f$. For $i = 1, \dots, n$ choose $U_i \in \mathcal{U}_f$ with $x_i \in U_i$. Then $(x_1, \dots, x_n) \in U_1 \times \dots \times U_n \in \Pi(\mathcal{U}_f)$. Since $\Pi(\mathcal{U}_f) < \mathcal{V}_{\mathcal{U}_f}$, we have $U_1 \times \dots \times U_n \subseteq V$ for some $V \in \mathcal{V}_{\mathcal{U}_f} \subseteq \mathcal{V}^*$. Hence $(x_1, \dots, x_n) \in \bigcup \mathcal{V}^*$. \square

Our next proposition is a “ G -analogue” of the classical theorem of Arhangel’skiĭ–Pytkeev from the C_p -theory.

Proposition 7.2. If G is a metric group and X is a G^* -regular space, then $l^*(X) = t(C_p(X, G))$.

Proof. The inequality $l^*(X) \leq t(C_p(X, G))$ was proved in Lemma 7.1. The converse inequality can be proved by a straightforward modification of the proof of [1, Theorem II.1.1]. \square

Theorem 6.5 and Proposition 7.2 give the following

Corollary 7.3. Let G be an NSS metric group. Then a G^* -regular space X is compact if and only if $C_p(X, G)$ is a TAP group of countable tightness. In particular, G -equivalence preserves compactness within the class of G^* -regular spaces.

For a separable metric group G and a G -regular space X , one can prove via standard arguments that $nw(X) = nw(C_p(X, G))$, where $nw(X)$ is the network weight of X . Combining this with Theorem 6.5, we obtain the following

Proposition 7.4. *Let G be an NSS separable metric group. Then a G -regular space X is compact and metrizable if and only if $C_p(X, G)$ is a TAP group with a countable network. In particular, G -equivalence preserves the property “to be compact metrizable” within the class of G -regular spaces.*

Our next example shows that $(G \times H)$ -equivalence need not imply either G -equivalence or H -equivalence.

Example 7.5. Let $G = \mathbb{T}^\omega \times \mathbb{R}$ and $H = \mathbb{T} \times \mathbb{R}^\omega$. Then both G -equivalence and H -equivalence preserve pseudocompactness and compactness, but $(G \times H)$ -equivalence does not preserve either of these properties. By Proposition 2.3(i), every space is both G^{**} -regular and H^{**} -regular. Applying either Theorem 3.2 or Theorem 3.3(i), we conclude that G -equivalence implies \mathbb{R} -equivalence, and H -equivalence implies \mathbb{T} -equivalence. Both \mathbb{R} -equivalence and \mathbb{T} -equivalence preserve pseudocompactness (Corollary 6.6) and compactness (Corollary 7.3). Thus, both G -equivalence and H -equivalence preserve pseudocompactness and compactness as well. On the other hand, since $G \times H \cong (\mathbb{R} \times \mathbb{T})^\omega$, Corollary 2.5 yields that $(G \times H)$ -equivalence preserves neither pseudocompactness, nor compactness.

In connection with Corollary 6.6, our next example shows that neither NSS nor TAP property of G is necessary for G -equivalence to preserve pseudocompactness (and compactness as well).

Example 7.6.

- (i) For every infinite zero-dimensional pseudocompact space X , the group $H = \mathbb{R} \times C_p(X, \mathbb{Z}(2))$ is TAP but not NSS, and H -equivalence preserves both pseudocompactness and compactness. Indeed, according to Theorem 6.3, $C_p(X, \mathbb{Z}(2))$ is TAP. Since \mathbb{R} is TAP by Theorem 4.9, it follows from Proposition 4.6(i) that H is TAP as well. From Proposition 6.9(i) we get that $C_p(X, \mathbb{Z}(2))$, and consequently H , is not NSS. Furthermore, applying Theorem 3.3(ii), we obtain that H -equivalence implies \mathbb{R} -equivalence. Since \mathbb{R} -equivalence (that is l -equivalence) preserves pseudocompactness (Corollary 5.4) and compactness (Corollary 7.3), so does H -equivalence.
- (ii) The group $G = \mathbb{R} \times \mathbb{Z}(2)^\omega$ is not TAP (and consequently not NSS by Theorem 4.9), but G -equivalence preserves both pseudocompactness and compactness. Proposition 4.6(iii) guarantees that G is not TAP. Applying Theorem 3.3(ii), we conclude that G -equivalence implies \mathbb{R} -equivalence. Now we finish the argument as in item (i).

If one replaces \mathbb{R} by \mathbb{T} in Example 7.6, then the resulting groups G and H would become even precompact, without losing any of the properties mentioned in Example 7.6.

Remark 7.7. Examples 7.5 and 7.6 show that the class of all (Abelian) topological groups G for which G -equivalence preserves compactness is not finitely productive and is larger than that of NSS, metrizable groups.

Now we turn to a particular version of Problem 1.6 by considering the class **PSC** of all topological groups G for which G -equivalence preserves pseudocompactness.

By Corollary 6.6, $\mathbb{Z}(2)$ -equivalence preserves pseudocompactness within the class of $\mathbb{Z}(2)$ -regular spaces, and yet $\mathbb{Z}(2) \notin \mathbf{PSC}$, by Example 2.1. Therefore, it is reasonable to investigate whether G -equivalence preserves pseudocompactness only within the class of G -regular spaces. On the other hand, if a space X is G -regular for every topological group G , then X must be zero-dimensional. One possible way to avoid such a restriction on X is to require our groups G to be the elements of the class **I** of all topological groups that contain a homeomorphic copy of the closed unit interval $[0, 1]$ as a subspace. Indeed, by Proposition 2.3(ii), this would make the condition of G -regularity automatically satisfied for every space. Therefore, one may expect that the subclass $\mathbf{PSC} \cap \mathbf{I}$ of the class **PSC** should have especially nice properties. Let us summarize what we know about the properties of this class.

Proposition 7.8. *Denote by **NSS** the class of all NSS groups and by **TAP** the class of all TAP groups. Then:*

- (i) $\mathbf{NSS} \cap \mathbf{I} \subseteq \mathbf{PSC} \cap \mathbf{I}$ (Corollary 6.6);
- (ii) $\mathbf{PSC} \cap \mathbf{I} \neq \mathbf{I}$ (Corollary 2.5);
- (iii) $(\mathbf{PSC} \cap \mathbf{I}) \setminus \mathbf{NSS} \neq \emptyset$ (Example 7.6(i));
- (iv) $(\mathbf{PSC} \cap \mathbf{I}) \setminus \mathbf{TAP} \neq \emptyset$ (Example 7.6(ii));
- (v) both **PSC** and $\mathbf{PSC} \cap \mathbf{I}$ are closed under taking finite powers (Corollary 2.17);
- (vi) both **PSC** and $\mathbf{PSC} \cap \mathbf{I}$ are not closed under taking finite products (Example 7.5).

8. (Dis)connectedness and G-equivalence

Recall that a topological space X is *totally disconnected* if every quasi-component of X is a singleton, or equivalently, if for every pair x, y of distinct points of X there exists a clopen set $F \subseteq X$ such that $x \in F \not\ni y$.

Theorem 8.1. *Let G be a topological group with the dense and totally disconnected torsion part $\text{tor}(G)$. Then a G -regular space X is totally disconnected if and only if $\text{tor}(C_p(X, G))$ is dense in $C_p(X, G)$.*

Proof. Suppose that X is totally disconnected. Let O be a non-empty open subset of $C_p(X, G)$. Choose $f \in O$. Then there exist $n \in \mathbb{N} \setminus \{0\}$, pairwise distinct elements x_1, \dots, x_n of X and non-empty open subsets U_1, \dots, U_n of G such that $f \in \bigcap_{i=1}^n W(x_i, U_i) \subseteq O$. As $\text{tor}(G)$ is dense in G , for every $i = 1, \dots, n$ we can choose $t_i \in \text{tor}(G) \cap U_i$. Since X is totally disconnected, there exists a disjoint partition $X = \bigcup_{i=1}^n F_i$ of X into clopen subsets F_i such that $x_i \in F_i$ for $i = 1, \dots, n$. Define $h \in C_p(X, G)$ by letting $h(x) = t_i$ whenever $x \in F_i$. Clearly, $h \in \text{tor}(C_p(X, G))$ and $h \in \bigcap_{i=1}^n W(x_i, U_i) \subseteq O$. Therefore, $O \cap \text{tor}(C_p(X, G)) \neq \emptyset$. This proves that $\text{tor}(C_p(X, G))$ is dense in $C_p(X, G)$.

To prove the reverse implication, assume that $\text{tor}(C_p(X, G))$ is dense in $C_p(X, G)$. Suppose that $x, y \in X$ and $x \neq y$. Since X is G -regular, there exists $f \in C_p(X, G)$ with $f(x) \neq f(y)$. Since $\text{tor}(C_p(X, G))$ is dense in $C_p(X, G)$, we may assume that $f \in \text{tor}(C_p(X, G))$, and thus $f \in C_p(X, \text{tor}(G))$. Since $\text{tor}(G)$ is totally disconnected, there exists a clopen set $W \subseteq \text{tor}(G)$ with $f(x) \in W \not\ni f(y)$. Consequently, $f^{-1}(W)$ is a clopen subset of X satisfying $x \in f^{-1}(W) \not\ni y$. This shows that X is totally disconnected. \square

Corollary 8.2. *Let G be a topological group with the dense and totally disconnected torsion part $\text{tor}(G)$. Then G -equivalence preserves total disconnectedness within the class of G -regular spaces.*

Recall that the *order* of an element g of a group G is defined to be the smallest $n \in \mathbb{N} \setminus \{0\}$ satisfying $g^n = e$ (if such n exists).

Proposition 8.3. *Suppose that a topological group G has exactly m elements of order p , for a suitable integer $m \in \mathbb{N} \setminus \{0\}$ and some prime number p . Let $k \in \mathbb{N} \setminus \{0\}$. Then a space X has precisely k connected components if and only if $C_p(X, G)$ has exactly $(m + 1)^k - 1$ elements of order p .*

Proof. To prove the “only if” part, it suffices to realize that an element $f \in C_p(X, G)$ of order p must be constant on every connected component of X . The rest is a simple computation.

To prove the “if” part, it suffices to show that X has finitely many connected components, since the (finite) number m of connected components of X is uniquely determined by $(m + 1)^k - 1$ (this follows from the “only if” part of our proof). Assume that X has infinitely many connected components. Then for every natural number $n \geq 2$ one can find a decomposition $X = \bigcup_{i=1}^n X_i$ of X into pairwise disjoint non-empty clopen subsets X_i of X , so $C_p(X, G) \cong \prod_{i=1}^n C_p(X_i, G)$. Since each $C_p(X_i, G)$ has at least one element of order p by Proposition 2.9, it follows that $C_p(X, G)$ must have at least n elements of order p . Since n was chosen arbitrarily, this contradicts our assumption that $C_p(X, G)$ has finitely many elements of order p . \square

Corollary 8.4. *Let G be a topological group that contains at least one, but finitely many, elements of prime order p (for a suitable p). Then G -equivalence preserves the finite number of connected components.*

Corollary 8.4 is a slight generalization of the result of Tkachuk, who proved in [22] that, for every $n \in \mathbb{N} \setminus \{0\}$, the property “to consist of n -many connected components” is preserved by $\mathbb{Z}(2)$ -equivalence (and thus, by M -equivalence as well).

The following example shows that there are groups G which do not satisfy the assumption of Proposition 8.3, but for which G -equivalence nevertheless preserves connectedness.

Example 8.5. *The group $G = \mathbb{T}^\omega \times \mathbb{Z}(2)$ contains infinitely many elements of each order $n \geq 2$, yet G -equivalence preserves the finite number of connected components. Indeed, since every space is $(\mathbb{T}^\omega)^{**}$ -regular by Proposition 2.3(i), it follows from Theorem 3.3(ii) that G -equivalence implies $\mathbb{Z}(2)$ -equivalence, and the latter equivalence preserves the finite number of connected components by Corollary 8.4.*

The last example in this section shows that the class of all groups G for which G -equivalence preserves the finite number of connected components is not closed under taking finite products, even though this class is closed under taking finite powers by Corollary 2.6.

Example 8.6. *If $G = \mathbb{T}^\omega \times \mathbb{Z}(2)$ and $H = \mathbb{T} \times \mathbb{Z}(2)^\omega$, then both G -equivalence and H -equivalence preserve the finite number of connected components, but $(G \times H)$ -equivalence does not. The fact that G -equivalence preserves the finite number of connected components was proved in Example 8.5. Since every space is \mathbb{T}^{**} -regular by Proposition 2.3(i), it follows from*

Theorem 3.3(ii) that H -equivalence implies \mathbb{T} -equivalence. Since the latter equivalence preserves the finite number of connected components by Corollary 8.4, we conclude that so does H -equivalence. On the other hand, since $G \times H \cong (\mathbb{T} \times \mathbb{Z}(2))^\omega$, it follows from Corollary 2.5 that $(G \times H)$ -equivalence does not preserve the finite number of connected components.

9. \mathcal{G} -equivalence and its relation with G -equivalence

Definition 9.1, as well as Propositions 9.2 and 9.3, are well known in category theory. We include the proofs of these propositions only for the reader's convenience.

Definition 9.1.

- (i) For a class \mathcal{G} of topological groups we denote by $\overline{\mathcal{G}}$ the smallest (with respect to inclusion) class of topological groups containing \mathcal{G} which is closed under taking arbitrary products and subgroups.
- (ii) Given a topological group H and a class \mathcal{G} of topological groups, we will say that $r_{H,\mathcal{G}}(H) \in \overline{\mathcal{G}}$ is a reflection of H in \mathcal{G} provided that there exists a continuous homomorphism $r_{H,\mathcal{G}} : H \rightarrow r_{H,\mathcal{G}}(H)$ (called the reflection homomorphism) satisfying the following condition: For every $G \in \overline{\mathcal{G}}$ and each continuous homomorphism $h : H \rightarrow G$ one can find a continuous homomorphism $g : r_{H,\mathcal{G}}(H) \rightarrow G$ such that $h = g \circ r_{H,\mathcal{G}}$.

Proposition 9.2. For every topological group H and each class \mathcal{G} of topological groups, the reflection $r_{H,\mathcal{G}}(H)$ of H in \mathcal{G} exists and is unique up to a topological isomorphism.

Proof. There exists an indexed set $\{(G_s, h_s) : s \in S\}$ such that:

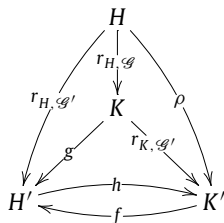
- (a) for each $s \in S$, $G_s \in \overline{\mathcal{G}}$ and $h_s : H \rightarrow G_s$ is a continuous surjective homomorphism;
- (b) if $G \in \overline{\mathcal{G}}$ and $h : H \rightarrow G$ is a continuous homomorphism, then there exist $t \in S$, a subgroup G'_t of G and a topological isomorphism $i_t : G_t \rightarrow G'_t$ such that $h = i_t \circ h_t$.

The diagonal product $r_{H,\mathcal{G}} = \Delta_{s \in S} h_s : H \rightarrow \prod_{s \in S} G_s$ of the family $\{h_s : s \in S\}$ is a continuous group homomorphism. Clearly, $r_{H,\mathcal{G}}(H) \in \overline{\mathcal{G}}$. Let $G \in \overline{\mathcal{G}}$, and let $h : H \rightarrow G$ be a continuous homomorphism. Let $t \in S$ and i_t be as in the conclusion of item (b), and let $\pi_t : \prod_{s \in S} G_s \rightarrow G_t$ be the projection on t 's coordinate. Then $g = i_t \circ \pi_t \upharpoonright_{r_{H,\mathcal{G}}(H)} : r_{H,\mathcal{G}}(H) \rightarrow G'_t$ is a continuous group homomorphism such that $g \circ r_{H,\mathcal{G}} = i_t \circ \pi_t \upharpoonright_{r_{H,\mathcal{G}}(H)} \circ r_{H,\mathcal{G}} = i_t \circ h_t = h$. This proves the existence of $r_{H,\mathcal{G}}(H)$.

To show its uniqueness, assume that $r_0 : H \rightarrow r_0(H) \in \overline{\mathcal{G}}$ and $r_1 : H \rightarrow r_1(H) \in \overline{\mathcal{G}}$ are continuous homomorphisms such that, for every $i \in \{0, 1\}$, each $G \in \overline{\mathcal{G}}$ and every continuous homomorphism $h : H \rightarrow G$, one can find a continuous homomorphism $g_{i,h} : r_i(H) \rightarrow G$ such that $h = g_{i,h} \circ r_i$. In particular, $r_{1-i} = g_{i,r_{1-i}} \circ r_i$ for $i \in \{0, 1\}$. Fix $i \in \{0, 1\}$. Then $r_i = g_{1-i,r_i} \circ r_{1-i} = g_{1-i,r_i} \circ g_{i,r_{1-i}} \circ r_i$, which yields that $g_{1-i,r_i} \circ g_{i,r_{1-i}}$ is the identity map on $r_i(H)$. Therefore, $g_{1,r_0} : r_1(H) \rightarrow r_0(H)$ is the inverse map of the map $g_{0,r_1} : r_0(H) \rightarrow r_1(H)$. Hence, $r_0(H) \cong r_1(H)$. \square

Proposition 9.3. Suppose that H is a topological group and $\mathcal{G}, \mathcal{G}'$ are classes of topological groups. If $\mathcal{G}' \subseteq \mathcal{G}$, then $r_{H,\mathcal{G}'}(H) \cong r_{\mathcal{G}',r_{H,\mathcal{G}}(H)}(r_{H,\mathcal{G}}(H))$.

Proof. Let $K = r_{H,\mathcal{G}}(H)$, $K' = r_{K,\mathcal{G}'}(K)$ and $H' = r_{H,\mathcal{G}'}(H)$. We need to prove that $H' \cong K'$.



Since $H' \in \overline{\mathcal{G}'} \subseteq \overline{\mathcal{G}}$ and $K = r_{H,\mathcal{G}}(H)$, there exists a continuous homomorphism $g : K \rightarrow H'$ such that $r_{H,\mathcal{G}'} = g \circ r_{H,\mathcal{G}}$. Since $H' \in \overline{\mathcal{G}'}$ and $K' = r_{K,\mathcal{G}'}(K)$, there exists a continuous homomorphism $f : K \rightarrow H'$ such that $g = f \circ r_{K,\mathcal{G}'}$. Define $\rho = r_{K,\mathcal{G}'} \circ r_{H,\mathcal{G}} : H \rightarrow K'$. Then $r_{H,\mathcal{G}'} = g \circ r_{H,\mathcal{G}} = f \circ r_{K,\mathcal{G}'} \circ r_{H,\mathcal{G}} = f \circ \rho$. Since $K' \in \overline{\mathcal{G}'}$ and $H' = r_{H,\mathcal{G}'}(H)$, there exists a continuous homomorphism $h : H' \rightarrow K'$ such that $\rho = h \circ r_{H,\mathcal{G}'} = h \circ f \circ \rho$. Thus, $h \circ f$ is the identity map on $\rho(H) = K'$. Finally, $r_{H,\mathcal{G}'} = f \circ \rho = f \circ h \circ r_{H,\mathcal{G}'}$, which yields that $f \circ h$ is the identity map on $r_{H,\mathcal{G}'} = H'$. Therefore, $h = f^{-1}$, and so $H' \cong K'$. \square

Definition 9.4. For a space X and a class \mathcal{G} of topological groups, we define $F_{\mathcal{G}}(X) = r_{F(X),\mathcal{G}}(F(X))$ and call $F_{\mathcal{G}}(X)$ the free object over X in \mathcal{G} .

Since the free topological group $F(X)$ of X is unique up to a topological isomorphism, from Definition 9.4 and Proposition 9.2 we obtain the following

Proposition 9.5. *For every space X and each class \mathcal{G} of topological groups, the free object $F_{\mathcal{G}}(X)$ over X in \mathcal{G} exists and is unique up to a topological isomorphism.*

One also has the “usual properties” of the free object:

Proposition 9.6. *For a space X and a class \mathcal{G} of topological groups, there exists a continuous mapping $\varphi_{X,\mathcal{G}} : X \rightarrow F_{\mathcal{G}}(X)$ satisfying the following conditions:*

- (i) $\varphi_{X,\mathcal{G}}(X)$ algebraically generates $F_{\mathcal{G}}(X)$;
- (ii) for every $G \in \mathcal{G}$ and each continuous map $f : X \rightarrow G$ there exists a (unique) continuous homomorphism $g : F_{\mathcal{G}}(X) \rightarrow G$ such that $f = g \circ \varphi_{X,\mathcal{G}}$.

Proof. Define $\varphi_{X,\mathcal{G}} = r_{F(X),\mathcal{G}} \upharpoonright_X$. Since X algebraically generates $F(X)$ and $r_{F(X),\mathcal{G}}$ is a homomorphism, we get (i). To prove (ii), assume that $G \in \mathcal{G}$ and $f : X \rightarrow G$ is a continuous map. Let $h : F(X) \rightarrow G$ be the unique continuous homomorphism extending f . Since $G \in \mathcal{G}$ and $F_{\mathcal{G}}(X) = r_{F(X),\mathcal{G}}(F(X))$, there exists a continuous homomorphism $g : F_{\mathcal{G}}(X) \rightarrow G$ such that $h = g \circ r_{F(X),\mathcal{G}}$. Now $f = h \upharpoonright_X = g \circ r_{F(X),\mathcal{G}} \upharpoonright_X = g \circ \varphi_{X,\mathcal{G}}$. The uniqueness of g follows from (i). \square

When \mathcal{G} forms a (wide) variety of topological groups, the free object over X in \mathcal{G} was defined and investigated by Morris in [17]. Comfort and van Mill have generalized this concept in [5]. In fact, when a space X admits a homeomorphic embedding into a Cartesian product of a family of members of \mathcal{G} , $F_{\mathcal{G}}(X)$ coincides with the free topological group in the class \mathcal{G} defined in [5].

From Definition 9.4 and Proposition 9.3 one immediately gets the following

Proposition 9.7. *Let X be a topological space and $\mathcal{G}, \mathcal{G}'$ two classes of topological groups. If $\mathcal{G}' \subseteq \mathcal{G}$, then $F_{\mathcal{G}'}(X) \cong r_{F_{\mathcal{G}}(X),\mathcal{G}'}(F_{\mathcal{G}}(X))$.*

Definition 9.8. Let \mathcal{G} be a class of topological groups. We say that spaces X and Y are \mathcal{G} -equivalent (and we write $X \overset{\mathcal{G}}{\sim} Y$) provided that $F_{\mathcal{G}}(X) \cong F_{\mathcal{G}}(Y)$.

When \mathcal{G} is the class of all topological groups, one has $F_{\mathcal{G}}(X) \cong F(X)$, and so \mathcal{G} -equivalence in this case coincides with the classical M -equivalence of Markov. Similarly, when \mathcal{A} is the class of all Abelian topological groups, then $F_{\mathcal{A}}(X)$ coincides with the free Abelian group in the sense of Markov, and so \mathcal{A} -equivalence in this case coincides with the classical A -equivalence of Markov. Since $F_{\overline{\mathcal{G}}}(X) \cong F_{\mathcal{G}}(X)$, $\overline{\mathcal{G}}$ -equivalence is the same as \mathcal{G} -equivalence.

Theorem 9.9. *If $\mathcal{G}' \subseteq \mathcal{G}$ are two classes of topological groups, then \mathcal{G} -equivalence implies \mathcal{G}' -equivalence.*

Proof. Immediately follows from Definition 9.8 and Proposition 9.7. \square

Given topological groups G and H , we denote by $\text{Chom}_p(G, H)$ the subspace of $C_p(G, H)$ consisting of homomorphisms from G to H . If H is Abelian, then $\text{Chom}_p(G, H)$ is a topological group.

Theorem 9.10. *Assume that \mathcal{G} is a class of topological groups and $G \in \mathcal{G}$ is Abelian. Then $C_p(X, G) \cong \text{Chom}_p(F_{\mathcal{G}}(X), G)$ for every space X .*

Proof. Let $\varphi_{X,\mathcal{G}}$ be as in Proposition 9.6. For every $f \in C_p(X, G)$ put $\phi(f) = \widehat{f}$, where $\widehat{f} : F_{\mathcal{G}}(X) \rightarrow G$ is the unique continuous homomorphism such that $f = \widehat{f} \circ \varphi_{X,\mathcal{G}}$; see Proposition 9.6(ii). Since G is Abelian, $\phi : C_p(X, G) \rightarrow \text{Chom}_p(F_{\mathcal{G}}(X), G)$ is an isomorphism. We need to show that ϕ is a homeomorphism as well.

If $x \in X$ and V is an open neighborhood of the identity in G , then

$$\begin{aligned} \phi(W(x, V)) &= \{ \widehat{f} : f \in C_p(X, G), f(x) \in V \} = \{ \widehat{f} : f \in C_p(X, G), \widehat{f}(\varphi_{X,\mathcal{G}}(x)) \in V \} \\ &= \{ \pi \in \text{Chom}_p(F_{\mathcal{G}}(X), G) : \pi(\varphi_{X,\mathcal{G}}(x)) \in V \} \end{aligned}$$

is an open set in $\text{Chom}_p(F_{\mathcal{G}}(X), G)$. Since the family $\{W(x, V) : x \in X, V \text{ is an open neighborhood of } e \text{ in } G\}$ forms a subbase of open neighborhoods of the identity map of $C_p(X, G)$, this proves that ϕ is an open map.

For $a \in F_{\mathcal{G}}(X)$ and an open neighborhood O of the identity in G , define $O_a = \{\pi \in \text{Chom}_p(F_{\mathcal{G}}(X), G) : \pi(a) \in O\}$. Let us show that the set $\phi^{-1}(O_a)$ is open in $C_p(X, G)$. Since $\varphi_{X, \mathcal{G}}(X)$ generates $F_{\mathcal{G}}(X)$, there exist $n \in \mathbb{N} \setminus \{0\}$, $x_1, \dots, x_n \in X$ and $z_1, \dots, z_n \in \mathbb{Z}$ such that $a = \prod_{i=1}^n \varphi_{X, \mathcal{G}}(x_i)^{z_i}$. Choose $f_0 \in \phi^{-1}(O_a)$ arbitrarily. Then $\phi(f_0) \in O_a$, and therefore

$$\phi(f_0)(a) = \widehat{f_0}(a) = \widehat{f_0} \left(\prod_{i=1}^n \varphi_{X, \mathcal{G}}(x_i)^{z_i} \right) = \prod_{i=1}^n (\widehat{f_0}(\varphi_{X, \mathcal{G}}(x_i)))^{z_i} = \prod_{i=1}^n f_0(x_i)^{z_i} \in O.$$

Using the continuity of group operations in G , for each $i = 1, \dots, n$ we can choose an open neighborhood U_i of $f_0(x_i)$ in G such that $\prod_{i=1}^n U_i^{z_i} \subseteq O$. Now $W = \bigcap_{i=1}^n W(x_i, U_i)$ is an open subset of $C_p(X, G)$ satisfying $f_0 \in W \subseteq \phi^{-1}(O_a)$. This proves that $\phi^{-1}(O_a)$ is open in $C_p(X, G)$. Since the family $\{O_a : a \in F_{\mathcal{G}}(X), O \text{ is an open neighborhood of } e \text{ in } G\}$ is a subbase of open neighborhoods of the identity in $\text{Chom}_p(F_{\mathcal{G}}(X), G)$, we conclude that ϕ is continuous. \square

Let us note that the condition “ G is Abelian” in the above theorem cannot be omitted. Indeed, $\text{Chom}_p(F_{\mathcal{G}}(X), G)$ need not be a group when G is not Abelian, while $C_p(X, G)$ is always a group.

Corollary 9.11. *If \mathcal{G} is a class of topological groups and $G \in \mathcal{G}$ is Abelian, then \mathcal{G} -equivalence implies G -equivalence.*

Proof. If X and Y are \mathcal{G} -equivalent, then $F_{\mathcal{G}}(X) \cong F_{\mathcal{G}}(Y)$, and so

$$C_p(X, G) \cong \text{Chom}_p(F_{\mathcal{G}}(X), G) \cong \text{Chom}_p(F_{\mathcal{G}}(Y), G) \cong C_p(Y, G)$$

by Theorem 9.10. Thus, X and Y are G -equivalent. \square

This corollary allows us to distinguish between $\{G\}$ -equivalence and G -equivalence for many Abelian topological groups G . Indeed, if H is an Abelian topological group such that H -equivalence preserves pseudocompactness within the class of H -regular spaces and $G = H^\omega$, then $\{G\}$ -equivalence implies G -equivalence, while G -equivalence does not imply $\{G\}$ -equivalence. The first statement follows directly from Corollary 9.11. To check the second statement, note that $\overline{\{G\}} = \overline{\{H\}}$, so $\{G\}$ -equivalence coincides with $\{H\}$ -equivalence. Hence, $\{G\}$ -equivalence implies H -equivalence by Corollary 9.11. Since H -equivalence preserves pseudocompactness within the class of H -regular spaces, so does $\{G\}$ -equivalence. On the other hand, it follows from Corollary 2.5 that G -equivalence does not preserve pseudocompactness within the class of H -regular spaces. Thus, G -equivalence does not imply $\{G\}$ -equivalence.

Lemma 9.12. *Let \mathcal{G} be a class of topological groups and X a topological space such that X is G^* -regular for some $G \in \mathcal{G}$. Then $\varphi_{X, \mathcal{G}} : X \rightarrow F_{\mathcal{G}}(X)$ is a homeomorphic embedding such that $\varphi_{X, \mathcal{G}}(X)$ is closed in $F_{\mathcal{G}}(X)$.*

Proof. Since X is G^* -regular for some $G \in \mathcal{G}$, $\varphi_{X, \mathcal{G}}$ is a homeomorphic embedding. We identify X with $\varphi_{X, \mathcal{G}}(X)$. Fix $a \in F_{\mathcal{G}}(X) \setminus X$. We have to prove that a is not in the closure of X . Since X algebraically generates $F_{\mathcal{G}}(X)$, there exist $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $z_1, \dots, z_n \in \mathbb{Z}$ such that $a = \prod_{i=1}^n x_i^{z_i}$. Define $s = \sum_{i=1}^n z_i$ if $\{x_1, \dots, x_n\} \neq \emptyset$ and let $s = 0$ otherwise. We consider two cases.

Case 1. There exist $G \in \mathcal{G}$ and $g \in G$ such that $g^s \neq g$. In this case take the constant map $f \in C_p(X, G)$ defined by $\widehat{f}(x) = g$ for all $x \in X$. This map extends to a continuous homomorphism $\widehat{f} : F_{\mathcal{G}}(X) \rightarrow G$. Clearly, $\widehat{f}(a) = g^s \neq g$ and $\widehat{f}(X) = f(X) \subseteq \{g\}$, so $V = \widehat{f}^{-1}(G \setminus \{g\})$ is an open neighborhood of a disjoint from X .

Case 2. $g^s = g$ for each $G \in \mathcal{G}$ and every $g \in G$. Fix pairwise disjoint open subsets U, U_1, \dots, U_n of $F_{\mathcal{G}}(X)$ such that $a \in U$, and $x_i \in U_i$ for $i = 1, \dots, n$. Take $G \in \mathcal{G}$ such that X is G^* -regular. Then there exist $g \in G \setminus \{e\}$ and $f_i \in C_p(X, G)$ such that $f_i(x_i) = g$ and $f_i(X \setminus U_i) \subseteq \{e\}$ for every $i = 1, \dots, n$. The function $f = \prod_{i=1}^n f_i$ extends to a continuous homomorphism $\widehat{f} : F_{\mathcal{G}}(X) \rightarrow G$. Since $\widehat{f}(a) = g^s = g \neq e$, $V = U \cap \widehat{f}^{-1}(G \setminus \{e\})$ is an open neighborhood of a in $F_{\mathcal{G}}(X)$. If $x \in X \setminus \bigcup_{i=1}^n U_i$, then $\widehat{f}(x) = e$, so $x \notin V$. If $x \in \bigcup_{i=1}^n U_i$, then $x \notin U$, so again $x \notin V$. This proves that $V \cap X = \emptyset$.

In both cases we have found an open neighborhood V of a which does not intersect X . Since a was chosen arbitrarily, we conclude that X is closed in $F_{\mathcal{G}}(X)$. \square

Proposition 9.13. *Let X be a space and \mathcal{G} a class of topological groups such that X is G^* -regular for some $G \in \mathcal{G}$. Then X is σ -compact (Lindelöf Σ -space) if and only if $F_{\mathcal{G}}(X)$ is σ -compact (Lindelöf Σ -space, respectively).*

Proof. We start with the “if” part. By Lemma 9.12, X is a closed subspace of a σ -compact space (Lindelöf Σ -space) $F_{\mathcal{G}}(X)$. Hence X is σ -compact (Lindelöf Σ -space, respectively). Let us prove the “only if” part. Since X algebraically generates $F_{\mathcal{G}}(X)$, there exists a representation $F_{\mathcal{G}}(X) = \bigcup_{i=0}^{\infty} F_i$ where each F_i is a continuous image of some finite power X^{n_i} of X . Now it remains only to note that the class of σ -compact spaces (Lindelöf Σ -spaces) is closed under taking countable unions, finite powers and continuous images. \square

10. \mathbb{T} -equivalence

In the theory of topological groups the group \mathbb{T} can be viewed as a counterpart to \mathbb{R} in the theory of topological vector spaces. In this section we derive some corollaries about \mathbb{T} -equivalence from theorems that we have established so far, and we compare them with the similar statements about l -equivalence (\mathbb{R} -equivalence), in order to emphasize that the properties of \mathbb{T} -equivalence are at least as good as (and often even better than) those of l -equivalence.

Definition 10.1.

- (i) We denote by \mathcal{P} the class of all precompact Abelian groups.
- (ii) For every topological group G , let $G^\dagger = \text{Chom}_p(G, \mathbb{T})$.

The following “precompact duality theorem” is the particular case of a much more general result of Menini and Orsatti [16] that has been later re-discovered in [19] and [12].

Theorem 10.2. $G \cong (G^\dagger)^\dagger$ for each $G \in \mathcal{P}$.

Theorem 10.3. Let X be a space. Then:

- (i) $C_p(X, \mathbb{T}) \cong \text{Chom}_p(F_{\mathcal{P}}(X), \mathbb{T}) = F_{\mathcal{P}}(X)^\dagger$, and
- (ii) $F_{\mathcal{P}}(X) \cong \text{Chom}_p(C_p(X, \mathbb{T}), \mathbb{T})$.

Proof. (i) Since $\overline{\mathcal{P}} = \mathcal{P}$, from Definitions 9.1 and 9.4 it follows that $F_{\mathcal{P}}(X) \in \mathcal{P}$. Since $\mathbb{T} \in \mathcal{P}$, $C_p(X, \mathbb{T}) \cong \text{Chom}_p(F_{\mathcal{P}}(X), \mathbb{T}) = F_{\mathcal{P}}(X)^\dagger$ by Theorem 9.10 and Definition 10.1(ii).

(ii) Applying Theorem 10.2, item (i) and Definition 10.1(ii), we obtain $F_{\mathcal{P}}(X) \cong (F_{\mathcal{P}}(X)^\dagger)^\dagger \cong C_p(X, \mathbb{T})^\dagger = \text{Chom}_p(C_p(X, \mathbb{T}), \mathbb{T})$. \square

It is well known that $C_p(X, \mathbb{R}) \cong \text{Chom}_p(L_p(X), \mathbb{R})$, where $L_p(X) \cong \text{Chom}_p(C_p(X, \mathbb{R}), \mathbb{R})$; see [1]. Consequently, $X \overset{\mathbb{R}}{\sim} Y$ if and only if $L_p(X) \cong L_p(Y)$. Our next corollary establishes a counterpart to this theorem for \mathbb{T} .

Corollary 10.4. \mathbb{T} -equivalence coincides with \mathcal{P} -equivalence.

Proof. Let X and Y be spaces.

Assume that $X \overset{\mathbb{T}}{\sim} Y$. Then $C_p(X, \mathbb{T}) \cong C_p(Y, \mathbb{T})$, and so $F_{\mathcal{P}}(X)^\dagger \cong F_{\mathcal{P}}(Y)^\dagger$ by Theorem 10.3(i). Applying Theorem 10.2, we obtain $F_{\mathcal{P}}(X) \cong (F_{\mathcal{P}}(X)^\dagger)^\dagger \cong (F_{\mathcal{P}}(Y)^\dagger)^\dagger \cong F_{\mathcal{P}}(Y)$. Therefore, $X \overset{\mathcal{P}}{\sim} Y$.

Assume now that $X \overset{\mathcal{P}}{\sim} Y$. Then $F_{\mathcal{P}}(X) \cong F_{\mathcal{P}}(Y)$, and so $F_{\mathcal{P}}(X)^\dagger \cong F_{\mathcal{P}}(Y)^\dagger$. Applying Theorem 10.3(i) once again, we conclude that $C_p(X, \mathbb{T}) \cong C_p(Y, \mathbb{T})$. This proves $X \overset{\mathbb{T}}{\sim} Y$. \square

In general, G -equivalence is weaker than \mathcal{G} -equivalence for an Abelian group $G \in \mathcal{G}$; see the discussion following Corollary 9.11. However, Corollary 10.4 shows that this is not the case for the class \mathcal{P} of all precompact Abelian groups and $\mathbb{T} \in \mathcal{P}$.

Recall that the compact group $\widehat{F_{\mathcal{P}}(X)}$ is called the *free compact Abelian group* of a space X ; see [14]. Our Corollary 10.4 should be compared with the following result from [14]: two spaces X, Y generate the same free compact Abelian group if and only if $C(X, \mathbb{T})$ and $C(Y, \mathbb{T})$ are algebraically isomorphic.

Corollary 10.5. \mathbb{T} -equivalence implies G -equivalence for every precompact Abelian group G .

Proof. Combine Corollaries 10.4 and 9.11. \square

Theorem 10.6. Let X be a space.

- (i) X is pseudocompact if and only if $C_p(X, \mathbb{T})$ is TAP.
- (ii) X is compact if and only if $C_p(X, \mathbb{T})$ is a TAP group of countable tightness.
- (iii) X is compact metrizable if and only if $C_p(X, \mathbb{T})$ is a TAP group with a countable network.
- (iv) $l^*(X) = t(C_p(X, \mathbb{T}))$.

- (v) X is totally disconnected if and only if $\text{tor}(C_p(X, \mathbb{T}))$ is dense in $C_p(X, \mathbb{T})$.
- (vi) For a given integer $n \in \mathbb{N} \setminus \{0\}$, the space X has precisely n connected components if and only if for every (equivalently, for some) prime number p the group $C_p(X, \mathbb{T})$ has exactly $p^n - 1$ elements of order p .

Proof. Since \mathbb{T} is pathwise connected, it follows from Proposition 2.3(i) that every space X is \mathbb{T}^{**} -regular (and thus, both \mathbb{T}^* -regular and \mathbb{T} -regular). Since \mathbb{T} is a separable metric NSS group, item (i) follows from Theorem 6.5, item (ii) follows from Corollary 7.3, item (iii) follows from Proposition 7.4, and item (iv) follows from Proposition 7.2. Item (v) follows from Theorem 8.1, and item (vi) follows from Proposition 8.3. \square

Theorem 10.7. \mathbb{T} -equivalence preserves the following properties:

- (i) pseudocompactness;
- (ii) the cardinal invariant l^* (defined at the beginning of Section 7);
- (iii) property of being a Lindelöf Σ -space;
- (iv) σ -compactness;
- (v) compactness;
- (vi) the property of being compact metrizable;
- (vii) the (finite) number of connected components;
- (viii) connectedness;
- (ix) total disconnectedness.

Proof. (i) and (ii) follow from items (i) and (iv) of Theorem 10.6, respectively. (iii) and (iv) follow from Proposition 9.13 and Corollary 10.4, as every space is \mathbb{T}^* -regular by Proposition 2.3(ii). (v) follows from items (i) and (iv). (vi) follows from Theorem 10.6(iii). (vii) follows from Theorem 10.6(vi). (viii) follows from (vii). (ix) follows from Theorem 10.6(v). \square

From Corollary 9.11 and Theorem 10.7 we immediately get:

Corollary 10.8. Let \mathcal{G} be a class of groups such that $\mathbb{T} \in \mathcal{G}$. Then \mathcal{G} -equivalence preserves properties (i)–(ix) listed in Theorem 10.7.

For an infinite cardinal τ , let \mathcal{B}_τ be the class of topological groups G such that for every open neighborhood U of e there exists a set $F \subseteq G$ with $G = FU$ and $|F| < \tau$. Let \mathcal{A}_τ be the class consisting of Abelian members of \mathcal{B}_τ . Since $\mathbb{T} \in \mathcal{A}_\tau \subseteq \mathcal{B}_\tau$ for every infinite cardinal τ , from Corollary 10.8 we get

Corollary 10.9. For every infinite cardinal τ , both \mathcal{A}_τ -equivalence and \mathcal{B}_τ -equivalence preserve each of the properties (i)–(ix) listed in Theorem 10.7.

To the best of our knowledge, the above corollary is new even for $\tau = \omega$. Similarly, the following particular case of Corollary 10.8 appears to be new as well.

Corollary 10.10. Total disconnectedness is preserved by \mathcal{A} -equivalence and \mathcal{M} -equivalence.

Note that l -equivalence preserves properties from items (i)–(vi) of Theorem 10.7. To the best of the author's knowledge, it is not known whether l -equivalence preserves total disconnectedness. On the other hand, it is known that l -equivalence does not preserve properties from items (vii) and (viii) of Theorem 10.7. This allows us to distinguish between l -equivalence and \mathbb{T} -equivalence.

Proposition 10.11. \mathbb{R} -equivalence (that is, l -equivalence) does not imply \mathbb{T} -equivalence.

Proof. It is well known that every connected metrizable space X is l -equivalent to the topological sum $X \oplus \{x\}$ of X with a singleton; see [3]. Hence, l -equivalence does not preserve connectedness. Combining this with Theorem 10.7(viii), we conclude that l -equivalence does not imply \mathbb{T} -equivalence. \square

Remark 10.12. Our proof that \mathbb{T} -equivalence preserves pseudocompactness necessarily differs from the existing proofs that \mathbb{R} -equivalence preserves pseudocompactness. Indeed, the original proof in [2] uses tools from functional analysis (namely,

barreled spaces) which are not available in the case of \mathbb{T} . An alternative proof in [23] is based on the following characterization: a space X is pseudocompact if and only if $C_p(X, \mathbb{R})$ is σ -precompact. This approach is not applicable in the case of \mathbb{T} since $C_p(X, \mathbb{T})$ is precompact for every space X .

Remark 10.13. We prove in our forthcoming paper [20] that the covering dimension \dim is preserved by \mathbb{T} -equivalence, so one can add this property to the list of properties in Theorem 10.7, as well as in Corollaries 10.8 and 10.9.

11. Open questions

Besides general problems listed in Section 1, there are many concrete questions that can be asked. In this section we list only a small sample of those.

In connection with Theorem 6.8 it seems natural to ask the following

Question 11.1. If G is a TAP group and X is a G -regular (G^* -regular, G^{**} -regular) compact space, must $C_p(X, G)$ have the TAP property?

Since the product $X \times Y$ of a compact space X and a pseudocompact space Y is pseudocompact, the trick from the proof of Theorem 6.8 cannot be used to solve this question.

Question 11.2. Can one replace “ G^* -regularity” by “ G -regularity” in Corollary 7.3?

Question 11.3. Is there an NSS group G such that G -equivalence does not preserve compactness within the class of G -regular (or G^* -regular) spaces?

Question 11.4. Is there a TAP group G such that G -equivalence does not preserve compactness (or pseudocompactness) within the class of G -regular spaces?

The following question arises in connection with Proposition 10.11:

Question 11.5. Does \mathbb{T} -equivalence imply \mathbb{R} -equivalence (that is, l -equivalence)?

Question 11.6. Let $\mathcal{R} = \{\mathbb{R}\}$ be the class consisting of a single group \mathbb{R} of reals. Do \mathcal{R} -equivalence and \mathbb{R} -equivalence (that is, l -equivalence) coincide?

Note that a similar question for the torus \mathbb{T} has a positive answer. Indeed, $\overline{\{\mathbb{T}\}} = \mathcal{P}$, and since \mathbb{T} -equivalence coincides with \mathcal{P} -equivalence by Corollary 10.4, it follows that \mathbb{T} -equivalence coincides with \mathcal{T} -equivalence, where $\mathcal{T} = \{\mathbb{T}\}$ is the class consisting of a single group \mathbb{T} .

In connection with Corollary 10.5, the following question seems to be interesting:

Question 11.7.

- (i) Does \mathbb{Q}^* -equivalence imply \mathbb{T} -equivalence? (Here \mathbb{Q}^* denotes the Pontryagin dual of the discrete group \mathbb{Q} of rational numbers.)
- (ii) For a prime number p , does \mathbb{Z}_p -equivalence imply \mathbb{T} -equivalence within the class of ind-zero-dimensional Tychonoff spaces?

Question 11.8. Let G be a topological group and $n \geq 1$ an integer number. Does G^n -equivalence imply G -equivalence within the class of G -regular (G^* -regular or G^{**} -regular) spaces?

The answer to this question seems to be unknown even for $G = \mathbb{R}$ and $n = 2$.

Let us recall another notion from the C_p -theory. Spaces X and Y are called u -equivalent provided that the topological groups $C_p(X, \mathbb{R})$ and $C_p(Y, \mathbb{R})$ considered with their (two-sided) uniformities, are uniformly homeomorphic (that is, there exists a bijection between $C_p(X, \mathbb{R})$ and $C_p(Y, \mathbb{R})$ that preserves the uniform structures). This motivates the following definition.

Definition 11.9. For a given topological group G , let us say that spaces X and Y are G_u -equivalent (G'_u -equivalent) provided that topological groups $C_p(X, G)$ and $C_p(Y, G)$ considered with their left uniformities (two-sided uniformities, respectively), are uniformly homeomorphic.

This definition leads to natural “uniform analogues” of our Problems 1.3 and 1.4.

Problem 11.10. Given a topological group G , characterize topological properties of X in terms of the uniform properties of $C_p(X, G)$.

Problem 11.11. Given a topological group G , a class \mathcal{C} of spaces and a topological property \mathcal{E} , investigate when the property \mathcal{E} is preserved by G_u -equivalence (G'_u -equivalence) within the class \mathcal{C} .

Recall that spaces X and Y are called t -equivalent provided that spaces $C_p(X, \mathbb{R})$ and $C_p(Y, \mathbb{R})$ are homeomorphic [1]. Given a topological group G , let us say that spaces X and Y are G_t -equivalent provided that spaces $C_p(X, G)$ and $C_p(Y, G)$ are homeomorphic. Clearly, one can also investigate the version of Problem 11.11 obtained by replacing G_u -equivalence in it with G_t -equivalence.

Remark 4.11 motivates the following question:

Question 11.12. ([8]) Is there a ZFC example of a countably compact Abelian TAP group that is not NSS?

Question 11.13. Is there a space X and a topological group G such that X is G -regular but not G^* -regular?

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