Qualitative analysis of a reaction–diffusion system

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Received 1 October 2003; accepted 1 May 2004

Abstract

In this paper, we study the qualitative analysis of a reaction–diffusion system. The local stability of the trivial steady state, as well as the blow-up behavior of the solution are obtained.

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MSC: 35K20; 35B40

Keywords: Reaction–diffusion system; Stability; Blow-up in finite time; Upper and lower solutions

1. Introduction

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). Consider the following initial boundary-value problem

\[
\begin{aligned}
    u_t + Lu &= c_1(e^{u_1} - 1), & x \in \Omega, & t > 0, \\
    v_t + Lv &= c_2(e^{v_2} - 1), & x \in \Omega, & t > 0, \\
    Bu &= Bv = 0, & x \in \partial \Omega, & t > 0, \\
    u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x \in \Omega, \\
\end{aligned}
\]

(1)

\textsuperscript{*}This work was supported by the Ministry of Education of China Science and Technology Major Projects Grant 104090.
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doi:10.1016/j.aml.2004.05.004
where $L = -(a_{ij}(x)\partial_{x_i})_{ij} + b_i \partial_{x_i}$ is a uniformly elliptic operator in $\Omega$, $a_{ij}(x)$ are continuous functions and $b_i$ are constants. $Bu = \gamma \frac{\partial u}{\partial t} + u$ with constant $\gamma \geq 0$. Parameters $c_1$, $c_2$, $a_1$ and $a_2$ are positive constants. The initial data $u_0(x)$ and $v_0(x)$ are non-negative continuous functions and satisfy the compatibility condition.

When $c_1 = c_2$, $a_1 = a_2$ and $u_0(x) = v_0(x)$, system (1) becomes a single equation which arises from the thermal ignition of mixture of gases and some nonlinear diffusion problems, and has been studied by many authors, see [1–3, 5–8].

Let $\lambda_1$ be the first eigenvalue of the eigenvalue problem

$$L \varphi = \lambda_1 \varphi \text{ in } \Omega, \quad B \varphi = 0 \text{ on } \partial \Omega,$$

and $\varphi(x)$ be the corresponding eigenfunction, then $\lambda_1 > 0$, $\varphi(x) > 0$ in $\Omega$, and $\frac{\partial \varphi}{\partial \nu} < 0$ on $\partial \Omega$, normalized by $\int_{\Omega} \varphi \, dx = 1$.

The main results of the present paper read as follows.

**Theorem 1.** If $\lambda_1^2 < a_1 a_2 c_1 c_2$, then the solution $(u, v)$ of (1) blows up in finite time provided that $(u_0, v_0) \neq (0, 0)$.

**Theorem 2.** If $\lambda_1^2 > a_1 a_2 c_1 c_2$, then the trivial steady-state solution $(0, 0)$ is local stable. In addition, the solution $(u, v)$ of (1) blows up in finite time for the suitable large initial data $(u_0(x), v_0(x))$.

2. Proof of theorems

**Proof of Theorem 1.** As the initial data $(u_0, v_0)$ is non-negative, so is the solution $(u, v)$. Let $\alpha$ and $\beta$ be positive constants which will be determined later. Multiplying the first and second equation of (1) by $\alpha \varphi(x)$ and $\beta \varphi(x)$ respectively, and integrating the results over $\Omega$ by parts and using $L \varphi = \lambda_1 \varphi$, we have

$$\alpha \int_{\Omega} \varphi u \, dx + \lambda_1 \alpha \int_{\Omega} u \varphi \, dx = \alpha c_1 \int_{\Omega} \varphi (e^{a_1 u} - 1) \, dx \geq c_1 \alpha \int_{\Omega} \left( a_1 u + \frac{a_1^2 u^2}{2} \right) \varphi \, dx,$$

$$\beta \int_{\Omega} \varphi v \, dx + \lambda_1 \beta \int_{\Omega} v \varphi \, dx = \beta c_2 \int_{\Omega} \varphi (e^{a_2 v} - 1) \, dx \geq c_2 \beta \int_{\Omega} \left( a_2 v + \frac{a_2^2 v^2}{2} \right) \varphi \, dx.$$

Adding these two inequalities, we have

$$\int_{\Omega} \varphi (\alpha u + \beta v) \, dx + \int_{\Omega} \lambda_1 \varphi (\alpha u + \beta v) \, dx$$

$$\geq \int_{\Omega} (c_1 a_1 \alpha u + c_2 a_2 \beta u) \varphi \, dx + \int_{\Omega} \left( \frac{c_1^2 a_1^2 u^2}{2} + \frac{c_2^2 a_2^2 u^2}{2} \right) \varphi \, dx,$$

i.e.,

$$\int_{\Omega} \varphi (\alpha u + \beta v) \, dx \geq \int_{\Omega} [(c_1 a_1 \alpha - \lambda_1 \beta) u \varphi + (c_2 a_2 \beta - \lambda_1 \alpha) v \varphi] \, dx$$

$$+ \int_{\Omega} \left( \frac{c_1^2 a_1^2 u^2}{2} + \frac{c_2^2 a_2^2 u^2}{2} \right) \varphi \, dx.$$

(2)

If $\lambda_1^2 < a_1 a_2 c_1 c_2$, we choose $\alpha > 0$ satisfying $\lambda_1/(c_1 a_1) < \alpha^2 < c_2 a_2 / \lambda_1$ and $\beta = 1/\alpha$. If $\lambda_1^2 = a_1 a_2 c_1 c_2$, we choose $\alpha, \beta > 0$ satisfying $\alpha/\beta = \lambda_1/(c_1 a_1)$. Then $\alpha c_1 a_1 \geq \beta \lambda_1$ and $\beta a_2 c_2 \geq \alpha \lambda_1$. Taking

$$\ell = \min \{ c_1 a_1^2 \alpha / \beta^2, c_2 a_2^2 \beta / \alpha^2 \},$$
it follows from (2) that
\[
\int_\Omega \varphi(\alpha u + \beta v)\,dx \geq \ell \left( \int_\Omega (\beta v)^2\varphi\,dx + \int_\Omega (\alpha u)^2\varphi\,dx \right)
\geq \ell \left\{ \left( \int_\Omega \beta v\varphi\,dx \right)^2 + \left( \int_\Omega \alpha u\varphi\,dx \right)^2 \right\}
\geq \frac{\ell}{2} \left( \int_\Omega (\alpha u + \beta v)\varphi\,dx \right)^2.
\]

Let \( f(t) = \int_\Omega (\alpha u + \beta v)\varphi\,dx \), then we have that
\[
\frac{df}{dt} \geq \frac{\ell}{2} f^2(t), \quad f(0) = \int_\Omega (\alpha u_0 + \beta v_0)\varphi\,dx > 0.
\]

Integrating this inequality from 0 to \( t \), we get
\[
f(t) \geq \frac{2}{2f(0) - \epsilon t},
\]
which implies that \( f(t) \to \infty \) as \( t \to 2f(0)/\ell \), i.e., \( \int_\Omega \varphi(\alpha u + \beta v)\,dx \to \infty \) as \( t \to 2f(0)/\ell \). This shows that the solution blows up in finite time. This completes the proof of Theorem 1. \( \square \)

**Proof of Theorem 2.** We first prove the local stability of the trivial steady-state solution \((0, 0)\). To this aim, we will seek an upper solution of (1) with the form
\[
\begin{align*}
\hat{u}(x, t) &= \rho_1 e^{-\alpha t} \varphi(x), \\
\hat{v}(x, t) &= \rho_2 e^{-\alpha t} \varphi(x),
\end{align*}
\]
where \( \rho_1, \rho_2 \) and \( \alpha \) are all positive constants which will be determined later. By the definition of the upper solution of the system (1), it suffices to have \((\hat{u}, \hat{v})\) satisfy
\[
\begin{align*}
\hat{u}_t + \hat{L}\hat{u} &\geq c_1 (e^{\alpha \hat{v}} - 1), & x \in \Omega, \ t > 0, \\
\hat{v}_t + \hat{L}\hat{v} &\geq c_2 (e^{\alpha \hat{u}} - 1), & x \in \Omega, \ t > 0, \\
B\hat{u} &\geq 0, \ B\hat{v} \geq 0, & x \in \partial\Omega, \ t > 0, \\
\hat{u}(x, 0) &\geq u_0(x), \ \hat{v}(x, 0) \geq v_0(x), & x \in \hat{\Omega}.
\end{align*}
\]
(3)

By the direct computation we see that (3) holds if
\[
\begin{align*}
-\alpha \rho_1 e^{-\alpha t} \varphi(x) + \lambda_1 \rho_1 e^{-\alpha t} \varphi(x) &\geq c_1 [\exp\{a_1 \rho_2 e^{-\alpha t} \varphi(x)\} - 1], & x \in \Omega, \ t > 0, \\
-\alpha \rho_2 e^{-\alpha t} \varphi(x) + \lambda_1 \rho_2 e^{-\alpha t} \varphi(x) &\geq c_2 [\exp\{a_2 \rho_1 e^{-\alpha t} \varphi(x)\} - 1], & x \in \Omega, \ t > 0, \\
\rho_1 \varphi(x) &\geq u_0(x), \ \rho_2 \varphi(x) \geq v_0(x), & x \in \hat{\Omega}.
\end{align*}
\]
(4)

A simple analysis shows that, for small \( \epsilon > 0 \), there exists \( \rho_0 > 0 \) such that when \( 0 < \rho_1, \rho_2 \leq \rho_0 \) the following hold:
\[
\frac{c_1 [\exp\{a_1 \rho_2 e^{-\alpha t} \varphi(x)\} - 1]}{a_1 \rho_2 e^{-\alpha t} \varphi(x)} < (1 + \epsilon)c_1, \quad \frac{c_2 [\exp\{a_2 \rho_1 e^{-\alpha t} \varphi(x)\} - 1]}{a_2 \rho_1 e^{-\alpha t} \varphi(x)} < (1 + \epsilon)c_2.
\]

Therefore, the first and second inequalities of (4) hold provided that
\[
\begin{align*}
-\alpha \rho_1 e^{-\alpha t} \varphi(x) + \lambda_1 \rho_1 e^{-\alpha t} \varphi(x) &\geq (1 + \epsilon)c_1 a_1 \rho_2 e^{-\alpha t} \varphi(x), \\
-\alpha \rho_2 e^{-\alpha t} \varphi(x) + \lambda_1 \rho_2 e^{-\alpha t} \varphi(x) &\geq (1 + \epsilon)c_2 a_2 \rho_1 e^{-\alpha t} \varphi(x),
\end{align*}
\]
(5)
which is equivalent to
\[(\lambda_1 - \alpha) \rho_1 \geq (1 + \varepsilon)c_1 a_1 \rho_2, \quad (\lambda_1 - \alpha) \rho_2 \geq (1 + \varepsilon)c_2 a_2 \rho_1. \tag{6}\]

As \(\lambda_1^2 > a_1 a_2 c_1 c_2\), we take the above \(\varepsilon\) smaller if necessary, then there exists \(\alpha > 0\) such that
\[(\lambda_1 - \alpha)^2 \geq (1 + \varepsilon)^2 a_1 a_2 c_1 c_2. \tag{7}\]

Applying the inequality (7), we will prove that the inequality (6) holds for the suitable choices of \(\rho_1\) and \(\rho_2\). We divide the discussion into three cases.

(i) If \(\lambda_1 - \alpha \geq (1 + \varepsilon)a_1 c_1\) and \(\lambda_1 - \alpha \geq (1 + \varepsilon)a_2 c_2\), we take \(\rho_1 = \rho_2\). Then (6) obviously holds.

(ii) If \(\lambda_1 - \alpha < (1 + \varepsilon)a_1 c_1\) and \(\lambda_1 - \alpha > (1 + \varepsilon)a_2 c_2\), we take \(0 < \rho_1, \rho_2 \leq \rho_0\) such that \(\rho = \rho_2/\rho_1\) satisfies \(\lambda_1 - \alpha = (1 + \varepsilon)a_1 c_1 \rho\). In view of (7) we obtain
\[(\lambda_1 - \alpha)(1 + \varepsilon)a_1 c_1 \rho \geq (1 + \varepsilon)^2 a_1 a_2 c_1 c_2,\]
which implies
\[(\lambda_1 - \alpha) \geq a_2 c_2 (1 + \varepsilon)/\rho. \]

Therefore, (6) holds.

(iii) If \(\lambda_1 - \alpha < (1 + \varepsilon)a_2 c_2\) and \(\lambda_1 - \alpha > (1 + \varepsilon)a_1 c_1\). The discussion is similar to case (ii).

So far, we know that the first and second inequalities of (4) hold. If the initial data \((u_0, v_0)\) satisfies \(0 \leq (u_0, v_0) \leq (\rho_1 \varphi(x), \rho_2 \varphi(x))\), the comparison principle shows that the unique solution \((u, v)\) of (1) satisfies \(0 \leq (u(x, t), v(x, t)) \leq (\rho_1 e^{-\alpha t} \varphi(x), \rho_2 e^{-\alpha t} \varphi(x))\). Thus the local stability of the trivial steady-state solution \((0, 0)\) of (1) has been proved.

Now, we show that the solution of (1) will blow up in finite time for the large initial data. Multiplying the first and second equations of (1) by \(\varphi\) and integrating the results over \(\Omega\),
\[
\int_{\Omega} u \varphi dx + \lambda_1 \int_{\Omega} u \varphi dx = c_1 \int_{\Omega} \varphi (\varphi(a_1 u - 1) dx \geq c_1 \int_{\Omega} \left( a_1 v + \frac{a_1^2 v^2}{2} \right) \varphi dx, \]
\[
\int_{\Omega} v \varphi dx + \lambda_1 \int_{\Omega} v \varphi dx = c_2 \int_{\Omega} \varphi (\varphi(a_2 u - 1) dx \geq c_2 \int_{\Omega} \left( a_2 u + \frac{a_2^2 u^2}{2} \right) \varphi dx. \]

Set \(F(t) = \int_{\Omega} u \varphi dx\) and \(G(t) = \int_{\Omega} v \varphi dx\), in view of the Schwarz inequality, we get
\[
\begin{align*}
F'(t) + \lambda_1 F(t) &\geq c_1 a_1 G(t) + \frac{c_1 a_1^2}{2} G^2(t), \quad F(0) = \int_{\Omega} u_0 \varphi dx > 0, \\
G'(t) + \lambda_1 G(t) &\geq c_1 a_1 F(t) + \frac{c_2 a_2^2}{2} F^2(t), \quad G(0) = \int_{\Omega} v_0 \varphi dx > 0.
\end{align*} \tag{8}\]

Define
\[
\mathcal{R} = \left\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0, \frac{c_2 a_2^2}{2 \lambda_1} x^2 + y > \left( \frac{2 \lambda_1}{c_1 a_1^2} x \right)^{1/2} \right\},
\]
and suppose that \((F(0), G(0)) \in \mathcal{R}\). We construct the auxiliary function \((f(t), g(t))\) satisfying
\[
\begin{align*}
f'(t) + \lambda_1 f(t) &= \frac{c_1 a_1^2}{2} g^2(t), \quad f(0) = \varepsilon F(0), \\
g'(t) + \lambda_1 g(t) &= \frac{c_2 a_2^2}{2} f^2(t), \quad g(0) = \varepsilon G(0), \tag{9}\end{align*}
\]
where $0 < \varepsilon < 1$ will be determined later. As $(F(0), G(0)) \in \mathcal{R}$, in view of (8) and (9) we have that $f'(0) < F'(0)$, $g'(0) < G'(0)$. According to [4, Lemma 2.1], we can obtain
\[ f(t) \leq F(t), \quad g(t) \leq G(t). \]
Using $(F(0), G(0)) \in \mathcal{R}$ once again, we know that if $\varepsilon$ is close to 1 then $(f(0), g(0)) \in \mathcal{R}$. Applying Lemma 2.4 and theorem 2.5 of [4] we have that $(f(t), g(t))$ and $(F(t), G(t))$ blow up in finite time. This completes the proof of Theorem 2. □

References