ELSEVIER
Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE @ DIRECT®
Discrete Mathematics 279 (2004) 203–213
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

# One frame and several new infinite families of $Z$-cyclic whist designs

Norman J. Finizio

*Department of Mathematics, University of Rhode Island, Kingston, RI 02881, USA*

Received 9 September 2002; received in revised form 21 November 2002; accepted 9 June 2003

**Abstract**

In 2001, Ge and Zhu published a frame construction which they utilized to construct a large class of $Z$-cyclic triplewhist designs. In this study the power and elegance of their methodology is illustrated in a rather dramatic fashion. Primarily due to the discovery of a single new frame it is possible to combine their techniques with the product theorems of Anderson, Finizio and Leonard along with a few new specific designs to obtain several new infinite classes of $Z$-cyclic whist designs. A sampling of the new results contained herein is as follows: (1) $Z$-cyclic $\mathrm{Wh}(3^3p+1)$, $p$ a prime of the form $4t+1$; (2) $Z$-cyclic $\mathrm{Wh}(3^{2n+1}s+1)$, for all $n \geqslant 1$, $s=5, 13, 17$; (3) $Z$-cyclic $\mathrm{Wh}(3^{2n}s+1)$, for all $n \geqslant 1$, $s=35, 55, 91$; (4) $Z$-cyclic $\mathrm{Wh}(3^{2n+1}s)$, for all $n \geqslant 1$, and for all $s$ for which there exist a $Z$-cyclic $\mathrm{Wh}(3s)$ and a homogeneous $(s, 4, 1)$-DM; and (5) $Z$-cyclic $\mathrm{Wh}(3^{2n}s)$ for all $n \geqslant 1$, $s = 5, 13$. Many other results are also obtained. In particular, there exist $Z$-cyclic $\mathrm{Wh}(3^3v + 1)$ where $v$ is any number for which Ge and Zhu obtained $Z$-cyclic $\mathrm{TWh}(3v + 1)$.


*Keywords:* $Z$-cyclic whist designs; Triplewhist designs; Frames; Resolvable BIBDs; Near resolvable BIBDs

## 1. Introduction

A whist tournament on $v$ players, $\mathrm{Wh}(v)$, is a (near) resolvable $(v, 4, 3)$-BIBD wherein each block $(a, b, c, d)$ is a game in which the partnership $\{a, c\}$ opposes that of $\{b, d\}$. The design is subject to the conditions that for every pair $x$, $y$ of players $x$ partners $y$ exactly once and $x$ opposes $y$ exactly twice. The (near) resolution classes of the design are called rounds. Whist tournaments are known to exist for all $v \equiv 0, 1 \pmod 4$ [3]. A whist tournament is said to be $Z$-cyclic if the players are elements in $Z_N \cup A$, where $N = v$, $A = \emptyset$ if $v \equiv 1 \pmod 4$, $N = v - 1$, $A = \{\infty\}$ if

*E-mail address:* finizio@uriacc.uri.edu (N.J. Finizio).

0012-365X/$ - see front matter © 2003 Published by Elsevier B.V.
doi:10.1016/S0012-365X(03)00270-X

$v \equiv 0 \pmod 4$ and round $j+1$ is obtained by adding $+1 \pmod N$ to every element in round $j$. When $\infty$ is present then $\infty + 1 = \infty$. An attractive feature of $Z$-cyclic whist tournaments is the fact that the entire tournament is obtainable from any one of its rounds. The round chosen to generate the tournament is called an initial round of the tournament. When $v \equiv 0 \pmod 4$ it is conventional to choose the round in which $\infty$ partners 0 as the initial round. When $v \equiv 1 \pmod 4$ then the round that omits 0 is conventionally chosen as the initial round. The opponent pairs $\{a,b\}$ and $\{c,d\}$ are called opponents of the first kind and the opponent pairs $\{a,d\}$ and $\{c,b\}$ are called opponents of the second kind. If a Wh($v$) is such that every player opposes every other player exactly once as an opponent of the first (alt. second) kind then the Wh($v$) is called a triplewhist design, TWh($v$). In the game $(a,b,c,d)$, $b$ is called $a$'s left hand opponent and $d$ is called $a$'s right hand opponent. Similar descriptions apply to the other players. If a Wh($v$) is such that every player has every other player exactly once as a left (alt. right) hand opponent then the Wh($v$) is called a directedwhist design, DWh($v$).

**Example 1.** The initial round of a $Z$-cyclic TWh(16) is $(\infty,5,0,10)$, $(1,4,2,8)$, $(6,13,9,7)$, $(11,12,3,14)$.

**Example 2.** The initial round of a $Z$-cyclic TWh(21) is $(1,15,2,18)$, $(3,4,13,19)$, $(17,8,5,16)$, $(6,10,20,12)$, $(7,9,11,14)$.

One of the results of this study is the existence of $Z$-cyclic Wh($3^3p+1$) for all primes $p=4t+1$, thus providing a complete solution to the problem discussed in [6]. Here the results are brought about by combining the Ge–Zhu frame construction with a $Z$-cyclic WhFrame($3^9$). By incorporating some additional new designs we are able to extend the class of solutions obtained by Ge and Zhu. If $v$ denotes an arbitrary product of elements contained in this extended class then it is established that there exists a $Z$-cyclic Wh($3^3v+1$). Additionally it is proven that there exist $Z$-cyclic Wh($v$) for $v=3^{2n+1}s+1$, for all $n \geqslant 1$, $s \in S_{02}$, $v=3^{2n}s+1$, $s \in S_{13} \cap T_d$, $v=3^{2n}s$, for all $n \geqslant 1$, $s \in S_{02}$ and $v=3^{2n+1}s$, for all $n \geqslant 1$, and for all $s$ for which both a $Z$-cyclic Wh($3s$) and a homogeneous $(s,4,1)$-DM exist. The sets $S_{02}$, $S_{13}$, $T_d$ are defined in the sequel and each is non-vacuous. We also establish the existence of many additional $Z$-cyclic Wh($v$). The simplicity and ease with which these results are obtained offers dramatic testimony to the power and elegance of the frame construction of Ge and Zhu. To be sure we also benefit from the product theorems of Anderson et al. [5] and the $Z$-cyclic WhFrame($3^9$). Indeed these latter two tools are indispensable and it is the combination of all three that allows for such large classes of new solutions.

For convenience we introduce the following sets.

$$P_1 = \{p\colon\ p \text{ is a prime},\ p=4t+1\},$$

$$Q_1 = \{q^2\colon\ q \text{ is a prime},\ q=4t+3,\ 1 \leqslant t < 124\},$$

$$E = \{21,33,77,133,161,781\},$$

$$R = P_1 \cup Q_1 \cup E,$$

$$R^* = R \setminus \{33\},$$

$$T = \{x\text{: there exists a homogeneous } (x, 4, 1)\text{-DM}\},$$

$$T_1 = \{x\text{: there exists a homogeneous } (3x, 4, 1)\text{-DM}\},$$

$$T_2 = \{x\text{: there exists a homogeneous } (3^2x, 4, 1)\text{-DM}\},$$

$$T_a = \{5, 7, 11, 13, 15, 17, 19, 23, 27, 31, 35, 39, 47, 67, 147, 167, 187, 227, 267, 287, 327\},$$

$$T_b = \{5, 13, 29, 49, 89, 109\},$$

$$T_c = \{x\text{: there exists a } Z\text{-cyclic Wh}(3x + 1)\},$$

$$T_d = \{x\text{: there exists a } Z\text{-cyclic Wh}(3^2x + 1)\},$$

$$T_e = \{x\text{: there exists a } Z\text{-cyclic Wh}(3x)\},$$

It is to be noted that the only difference between the set $R$ and the solution set obtained by Ge and Zhu is the presence of $33, 77, 161, 781$ in the set $E$. There is a $Z$-cyclic OWh for each of these new entries [1]. Thus if we momentarily remove 33 from consideration and let $v$ denote an arbitrary product of elements in $R$ the theorems of Ge and Zhu establish the existence of a $Z$-cyclic TWh$(3v + 1)$. Thus extending, slightly, the solution set of Ge and Zhu. When $v$ is divisible by 33 the result is contained in Theorem 26 below. Some of the results contained in this paper depend on new $Z$-cyclic whist designs that, as yet, do not appear in print. We cite only the designs that are utilized in this study. In particular, a $Z$-cyclic TWh(45) is found in [8], $Z$-cyclic TWh$(v)$, $v = 57, 65, 69, 77, 81, 85, 93, 100, 117, 129$ are found in [2] and $Z$-cyclic DWh$(v)$, $v = 45, 57, 69, 77, 81$ are found in [1].

## 2. Preliminaries

In this section we indicate the materials that provide the background and basis for the methods employed.

**Definition 3.** A homogeneous $(v, 4, 1)$-DM (i.e. difference matrix) is a $4 \times v$ array such that each row is a copy of $Z_v$ and the set of differences of any two rows equals $Z_v$.

**Theorem 4.** *If* $v = 2n + 1$, $n \geqslant 1$, *and* $gcd(v, 3) = 1$ *then there exists a homogeneous* $(v, 4, 1)$-*DM*.

**Proof.** Denote the rows of the array as $R_1$, $R_2$, $R_3$, and $R_4$. Set $R_1 = Z_v$, $R_2 = 2Z_v$, $R_3 = -Z_v$ and $R_4 = -2Z_v$. □

Obviously for $v$ an odd integer the only cases for which the existence of a homogeneous $(v,4,1)$-DM is in question are those for which $3|v$. The following theorem and its proof is found in [5].

**Theorem 5.** *If* $v=4n+1$ *and if there exists a Z-cyclic* TWh$(v)$ *then there exists a homogeneous* $(v,4,1)$-DM.

An obvious analog of this theorem is to replace the triplewhist requirement by directedwhist.

**Corollary 6.** *If* $v=4n+1$ *and there exists a Z-cyclic* DWh$(v)$ *then there exists a homogeneous* $(v,4,1)$-*DM*.

**Proof.** In the proof of Theorem 5 (see [5]) replace opponent first kind with left hand opponent and replace opponent second kind with right hand opponent. □

These latter two results are helpful for some cases in which 3 divides $v$. Thus, for example, $T_a \subset T_1$ and $T_b \subset T_2$. The following two theorems are the product theorems of Anderson et al. [5].

**Theorem 7.** *If there exist Z-cyclic* Wh$(P_i)$, $i=1,2$ *where* $P_i \equiv 1 \pmod 4$, *and if there exists a homogeneous* $(P_1,4,1)$-*DM*, *then there exists a Z-cyclic* Wh$(P_1P_2)$. *This* Wh$(P_1P_2)$ *is directed* (*triplewhist*, *ZCPS*) *if both* Wh$(P_i)$ *are*, *provided*, *in the ZCPS case*, *that* $3 \nmid P_1$.

**Theorem 8.** *Let* $Q>3$, $Q \equiv 3 \pmod 4$, $P \equiv 1 \pmod 4$, *where Z-cyclic* Wh$(Q+1)$ *and* Wh$(P)$ *and a homogeneous* $(Q,4,1)$-*DM exist. Then a Z-cyclic* Wh$(QP+1)$ *exists. Further*, *if the* Wh$(Q+1)$ *and the* Wh$(P)$ *are both triplewhist then so is the* Wh$(QP+1)$.

**Theorem 9.** *There exists a homogeneous* $(3^{4n},4,1)$-*DM for all* $n \geqslant 1$.

**Proof.** A $Z$-cyclic TWh(81) has been constructed in [2]. Combining Theorem 5 with Theorem 7 in a recursive fashion yields a $Z$-cyclic TWh$((81)^n)$ and hence the desired result. An alternative proof follows from the $Z$-cyclic DWh(81) found in [1] together with Corollary 6. □

**Theorem 10.** *Let* $s=4t+1$ *be such that there exists* (a) *either a Z-cyclic* TWh$(s)$ *or a Z-cyclic* DWh$(s)$ *and* (b) *either a Z-cyclic* TWh$(3^2s)$ *or a Z-cyclic* DWh$(3^2s)$ *then there exists a homogeneous* $(3^{2n}s,4,1)$-*DM for all* $n \geqslant 0$.

**Proof.** The proof relies heavily upon the fact that there exist both a $Z$-cyclic TWh $((81)^m)$ and a $Z$-cyclic DWh$((81)^m)$ for all $m \geqslant 1$. If $n=0$ the homogeneous $(s,4,1)$-DM follows from Hypothesis (a) and Theorem 5 or Corollary 6 whichever is appropriate. Let $n \geqslant 1$ be fixed. If $n$ is even, say $n=2m$, apply Theorem 7 with $P_1=(81)^m$

and $P_2 = s$ to obtain a $Z$-cyclic TWh$((81)^m s)$ or a $Z$-cyclic DWh$((81)^m s)$. The desired DM follows from either Theorem 5 or Corollary 6. If $n$ is odd, say $n = 2m+1$, apply Theorem 7 with $P_1 = (81)^m$ and $P_2 = 3^2 s$ to obtain a $Z$-cyclic TWh$((81)^m 3^2 s)$ or a $Z$-cyclic DWh$((81)^m 3^2 s)$. The difference matrix follows from either Theorem 5 or Corollary 6. □

**Theorem 11.** *Suppose* $s = 4t+3$ *is such that there exists* (a) *either a $Z$-cyclic* TWh$(3s)$ *or a $Z$-cyclic* DWh$(3s)$ *and* (b) *either a $Z$-cyclic* TWh$(3^3 s)$ *or a $Z$-cyclic* DWh$(3^3 s)$. *Then there exists a homogeneous* $(3^{2n+1}s, 4, 1)$*-DM, for all* $n \geqslant 0$.

**Proof.** The proof is virtually identical to that of Theorem 10 except that when $n$ is even set $P_2 = 3s$ and when $n$ is odd set $P_2 = 3^3 s$. □

Let $S_{02}$ denote the set of all $s$ that satisfy the hypotheses of Theorem 10, let $S_{13}$ denote the set of all $s$ that satisfy the hypotheses of Theorem 11 and let $S_a$ denote the set of all $s$ that satisfy Hypothesis (a) of Theorem 11. The next three theorems are not intended to be exhaustive but rather to exhibit that the sets $S_{02}$, $S_a$ and $S_{13}$ are nonempty.

**Theorem 12.** $\{5, 13, 25, 29, 45, 49, 65, 77, 85, 109\} \subset S_{02}$.

**Proof.** With the exception of 5, 13 there exist both $Z$-cyclic TWh and $Z$-cyclic DWh for each of the remaining entries [1,2,4]. If $p = 4t+1$ is a prime then there is a $Z$-cyclic DWh$(p)$ [1]. If $p \geqslant 29$ then there is a $Z$-cyclic TWh$(p)$ [4]. $Z$-cyclic TWh$(3^2 s)$ for $s = 5, 13$ are found in [2]. $Z$-cyclic DWh$(3^2 s)$ for $s = 29, 109$ are found in [1]. For the remaining cases use Theorem 7 noting that $3^2 \cdot 25 = 5 \cdot 45$, $3^2 \cdot 45 = 5 \cdot 81$, $3^2 \cdot 49 = 21 \cdot 21$, $3^2 \cdot 65 = 5 \cdot 117$, $3^2 \cdot 77 = 21 \cdot 33$ and $3^2 \cdot 85 = 17 \cdot 45$. It is well known that $Z$-cyclic TWh$(v)$ and $Z$-cyclic DWh$(v)$ exist for $v = 21, 33$. □

**Theorem 13.** $\{7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47\} \subset S_a$.

**Proof.** There exist both $Z$-cyclic TWh$(3s)$ and $Z$-cyclic DWh$(3s)$ for each of $s = 7, 11, 15, 19, 23, 27$ [1,2,4]. $Z$-cyclic TWh$(3s)$, $s = 31, 39, 43$ are given in [2]. A $Z$-cyclic DWh$(141)$ is given in [1]. A $Z$-cyclic TWh$(105)$ has been known for some time and a $Z$-cyclic DWh$(105)$ follows from Theorem 7 with $P_1 = 5$ and $P_2 = 21$. □

**Theorem 14.** *If* $v_1 \in S_{02}$ *and* $v_2 \in S_a$ *then* $v_1 v_2 \in S_{13}$ *provided that there is consistency in design type. That is to say both are TWh or both are DWh.*

**Proof.** To verify that $v_1 v_2$ satisfies Hypothesis (a) of Theorem 11 would require, for example, that if $v_1$ is such that there is a $Z$-cyclic TWh$(v_1)$, there must be a $Z$-cyclic TWh$(3v_2)$. If so then an application of Theorem 7 with $P_1 = v_1$ and $P_2 = 3v_2$ demonstrates that Hypothesis (a) is satisfied. A similar consistency is required for Hypothesis (b), in which case apply Theorem 7 with $P_1 = 3v_2$, $P_2 = 3^2 v_1$. □

**Example 15.** Note that $13 \cdot 7 \in S_{13}$. For Hypothesis (a) there is a $Z$-cyclic DWh(13) and a $Z$-cyclic DWh$(3 \cdot 7)$. For Hypothesis (b) there is a $Z$-cyclic TWh$(3 \cdot 7)$ and a $Z$-cyclic TWh$(3^2.13)$.

**Definition 16.** A frame is a group divisible design, $GDD_\lambda(X, \mathscr{G}, \mathscr{B})$ such that (1) the size of each block is the same, say $k$, (2) the block set can be partitioned into a family $\mathscr{F}$ of partial resolution classes and (3) each $F_i \in \mathscr{F}$ can be associated with a group $G_j \in \mathscr{G}$ so that $F_i$ contains every point in $X \setminus G_j$ exactly once.

An excellent source of information on frames is the text by Furino et al. [7]. For our purposes the group sizes of our frames will be uniform, say $g$, and we use the exponential notation, $g^n$, as the group type of the frame to indicate that there are $n$ groups each of size $g$. If the block size $k$ of a frame is 4 then we interpret each block as a whist game. Furthermore, if the blocks have the property that any two players $x, y$ from distinct groups appear together in exactly three blocks and exactly once as partners then the frame is called a whist frame and the partial resolution classes are called the rounds of the frame. For notation such a frame will be denoted by WhFrame$(g^n)$.

**Definition 17.** Suppose $S = Z_v, v = hw$ and $Z_v$ has a subgroup $H$ of order $h$. Suppose a WhFrame$(h^w)$ has a special round $R_1$, called the initial round, whose elements form a partition of $S \setminus H$ and is such that it, together with all the other rounds can be arranged in a cyclic order, say $R_1, R_2, \dots$ so that $R_{j+1}$ can be obtained by adding $+1$ modulo $v$ to every element in $R_j$ then the frame is said to be $Z$-cyclic.

**Theorem 18.** *Suppose $v - 1 = hw$, $h = 4m + 3$ and there exists a $Z$-cyclic* Wh$(v)$ *whose initial round contains $m + 1$ games $(a_i, b_i, c_i, d_i)$, $i = 1, 2, \dots, m + 1$ such that $\{a_i, b_i, c_i, d_i\colon\ i = 1, \dots, m + 1\} = \{0, w, 2w, \dots, (h - 1)w\} \cup \{\infty\}$ then there exists a $Z$-cyclic* WhFrame$(h^w)$.

**Proof.** In the initial round of the $Z$-cyclic Wh$(v)$ remove the $m+1$ games $(a_i, b_i, c_i, d_i)$. The remaining games form the initial round for a $Z$-cyclic WhFrame$(h^w)$ having groups $\{0, w, 2w, \dots, (h - 1)w\} + 0, 1, 2, \dots, w - 1$. □

**Example 19.** Using the construction in the proof of Theorem 18 a $Z$-cyclic WhFrame $(3^5)$ can be constructed from the $Z$-cyclic TWh(16) of Example 1.

**Example 20.** A $Z$-cyclic WhFrame$(3^9)$. Groups are $\{0, 9, 18\} + 0, 1, 2, \dots, 8$. The initial round is given by the six games: $(1, 12, 2, 24)$, $(8, 21, 19, 4)$, $(13, 23, 16, 15)$, $(3, 6, 5, 10)$, $(25, 17, 11, 22)$, $(20, 7, 26, 14)$.

Surprisingly, it is this $Z$-cyclic WhFrame$(3^9)$ that enables us to obtain most of the results of this study. Of course we need the machinery of the frame construction of Ge and Zhu. We list now their pertinent theorems (Theorems 21–25). The proofs can be found in their paper [9]. It is to be noted that the Ge–Zhu theorems relate to

TWhFrames. Since our $3^9$ Frame is not a TWhFrame, we replace TWhFrame with WhFrame.

**Theorem 21.** *Suppose there exists a $Z$-cyclic* WhFrame$(h^{v/h})$ *and a $Z$-cyclic* WhFrame $(u^{h/u})$ *then there exists a $Z$-cyclic* WhFrame$(u^{v/u})$.

**Theorem 22.** *If there exists a $Z$-cyclic* WhFrame$(h^w)$ *and if there exists a homogeneous $(g,4,1)$-DM then there exists a $Z$-cyclic* WhFrame$((hg)^w)$.

In Theorem 22 the process is known as an inflation by $g$.

**Theorem 23.** *Suppose there exists a $Z$-cyclic* WhFrame$(h^w)$ *and a $Z$-cyclic* Wh$(h)$, $h \equiv 1 \pmod 4$. *Then there exists a $Z$-cyclic* Wh$(hw)$.

**Theorem 24.** *Suppose there exists a $Z$-cyclic* WhFrame$(h^w)$ *and a $Z$-cyclic* Wh$(h+1)$, $h \equiv 3 \pmod 4$. *Then there exists a $Z$-cyclic* Wh$(hw+1)$.

**Theorem 25.** *Let $v$ be an arbitrary product of elements in $R \setminus \{33, 77, 161, 781\}$. Then there exists a $Z$-cyclic* TWh$(3v+1)$.

Theorem 25 also holds if $v$ contains any of $77, 161, 781$ as factors since there exist ordered whist tournaments of these orders [1]. The case of 33 being a factor of $v$ requires special consideration.

**Theorem 26.** *Let $v$ denote an arbitrary product of elements of $R$ such that $v = 33w$ where $33 \nmid w$. Then there exists a $Z$-cyclic* TWh$(99w+1)$.

**Proof.** If $w=1$ there is the TWh$(100)$ found in [2]. Otherwise begin with the $Z$-cyclic TWhFrame$(3^w)$ [9], inflate by 33 and use Theorem 24. □

An illustration of Theorem 24 is afforded by Example 20. Set $h=3$, $w=9$. There exists a $Z$-cyclic Wh(4), namely the initial round game $(\infty,1,0,2)$. Constructing this Wh(4) on the multiples of 9 (that is to say $(\infty,9,0,18)$) and adjoining this table to the initial round of the $Z$-cyclic WhFrame$(3^9)$ one obtains the initial round of a $Z$-cyclic Wh$(28)$.

**Theorem 27.** *If $x \in R^*$ and $y \in T_2 \cap R^*$ then there exists a $Z$-cyclic* WhFrame$(3^{9xy})$.

**Proof.** Inflate a $Z$-cyclic WhFrame$(3^x)$ [9] by $3^2 y$ to obtain a $Z$-cyclic WhFrame $((3^3 y)^x)$. Call this frame $F_1$. Take the $Z$-cyclic WhFrame$(3^9)$ and inflate by $y$ to obtain a $Z$-cyclic WhFrame$((3y)^9)$. Call this frame $F_2$. Apply Theorem 21 with $v = 27yx$, $h = 27y$ and $u = 3y$ to obtain from $F_1$ and $F_2$ a $Z$-cyclic WhFrame$((3y)^{9x})$. Call this latter frame $F_3$. Since $y \in R^*$ there exists a $Z$-cyclic WhFrame$(3^y)$ [9]. Applying Theorem 21 to this latter frame and $F_3$ yields the desired frame. □

**Theorem 28.** *Let $v$ be an arbitrary product of factors from $R^*$ such that $v$ contains at least two factors, one of which, say $y$, is an element of $R^* \cap T_2$ then there exists a Z-cyclic* WhFrame($3^{9v}$).

**Proof.** Define $x = v/y$ and apply Theorem 27. □

## 3. $Z$-cyclic Wh($3^3v$ + 1)

**Theorem 29.** *There exists a Z-cyclic* Wh($3^3P + 1$) *for all* $P \in R$.

**Proof.** Choose $P \in R$. Begin with the $Z$-cyclic WhFrame($3^9$) of Example 20 and since $P \in T$ we can inflate (i.e. apply Theorem 22) this frame by $P$ to obtain a $Z$-cyclic WhFrame($(3P)^9$). By the results of Ge–Zhu (or the TWh(100) if $P = 33$) there exists a $Z$-cyclic Wh($3P + 1$) thus an application of Theorem 24 yields a $Z$-cyclic Wh($3^3P + 1$). □

**Corollary 30.** *Let $p$ be a prime of the form $p = 4t + 1$ then there exists a Z-cyclic* Wh($3^3p + 1$).

**Example 31.** Set $p = 5$. Using the Wh(16) of Example 1 and a homogeneous (5, 4, 1)-DM whose $i$th row is $iZ_5$ we obtain the initial round of a $Z$-cyclic Wh($3^3 \cdot 5 + 1 = 136$).
$(\infty, 45, 0, 90)$, $(9, 36, 18, 72)$, $(54, 117, 81, 63)$, $(99, 108, 27, 126)$,
$(1, 12, 2, 24)$, $(28, 66, 83, 132)$, $(55, 120, 29, 105)$, $(82, 39, 110, 78)$,
$(109, 93, 56, 51)$, $(8, 21, 19, 4)$, $(35, 75, 100, 112)$, $(62, 129, 46, 85)$,
$(89, 48, 127, 58)$, $(116, 102, 73, 31)$, $(13, 23, 16, 15)$, $(40, 77, 97, 123)$,
$(67, 131, 43, 96)$, $(94, 50, 124, 69)$, $(121, 104, 70, 42)$, $(3, 6, 5, 10)$,
$(30, 60, 86, 118)$, $(57, 114, 32, 91)$, $(84, 33, 113, 64)$, $(111, 87, 59, 37)$,
$(25, 17, 11, 22)$, $(52, 71, 92, 130)$, $(79, 125, 38, 103)$, $(106, 44, 119, 76)$,
$(133, 98, 65, 49)$, $(20, 7, 26, 14)$, $(47, 61, 107, 122)$, $(74, 115, 53, 95)$,
$(101, 34, 134, 68)$, $(128, 88, 80, 41)$.

**Corollary 32.** *Let $v$ be an arbitrary product of elements from $R^*$ then there exists a Z-cyclic* Wh($3^3v + 1$).

**Theorem 33.** *Let $s \in S_{02}$. Then there exists a Z-cyclic* Wh($3^{2n+1}s + 1$) *for all* $n \geqslant 0$.

**Proof.** The proof is by induction on $n$. If $n = 0$ apply Theorem 25. Assume the theorem true for $n = k$ and consider the case $n = k + 1$, $k \geqslant 0$. Begin with the $Z$-cyclic WhFrame($3^9$) and inflate by $3^{2k}s$ to obtain a $Z$-cyclic WhFrame($(3^{2k+1}s)^9$). Via the induction hypothesis there is a $Z$-cyclic Wh($3^{2k+1}s + 1$). Hence an application of Theorem 24 yields a $Z$-cyclic Wh($3^{2k+3}s + 1$). □

**Corollary 34.** *Theorem* 33 *is non-vacuous.*

**Proof.** See Theorem 12. □

**Corollary 35.** *Let $v$ be an arbitrary product of elements from $R^*$ subject to the condition that $v$ contains at least two factors $y \in R^* \cap T_2$ and $s \in S_{02}$ with $y \neq s$. Then there exists a $Z$-cyclic* $\mathrm{Wh}(3^{2n+1}v+1)$ *for all $n \geqslant 1$.*

**Proof.** Let $n \geqslant 1$ be fixed. Set $v_1 = v/s$. Take the $Z$-cyclic WhFrame$(3^{9v_1})$ of Theorem 28, inflate by $3^{2n-2}s$ and invoke Theorem 24. □

**Corollary 36.** *Theorem* 33 *is also true for $s \in T \cap T_1 \cap T_c$, $3 \nmid s$. In particular, Theorem* 33 *is valid for $s = 17$.*

**Proof.** Let $s \in T \cap T_1 \cap T_c$ be such that $3 \nmid s$. Apply Theorem 8 with $Q = 3s$ and $P = 81$ to obtain a $Z$-cyclic Wh$(3^5 s+1)$. Next, by the product theorem for difference matrices there is a homogeneous $(81s, 4, 1)$-DM. Thus we can inflate the $Z$-cyclic WhFrame$(3^9)$ by $81s$ and apply Theorem 24 to obtain a $Z$-cyclic Wh$(3^7 s+1)$. An application of Theorem 8 with $Q = 3s$ and $P = (81)^2$ produces a $Z$-cyclic Wh$(3^9 s+1)$. Using Theorem 9 and recursively bouncing back and forth between Theorems 24 and 8 yields the $Z$-cyclic Wh$(3^{2n+1}s+1)$. □

**Theorem 37.** *Let $s \in S_{13} \cap T_d$. Then there exists a $Z$-cyclic* $\mathrm{Wh}(3^{2n}s+1)$ *for all $n \geqslant 1$.*

**Proof.** The proof is by induction on $n$. For $n = 1$ there is the $Z$-cyclic Wh$(3^2 s+1)$ of the hypothesis. Assume the theorem true for $n = k$ and consider the case $n = k+1$, $k \geqslant 1$. Begin with the $Z$-cyclic WhFrame$(3^9)$ and inflate by $3^{2k-1}s$ to obtain a $Z$-cyclic WhFrame$((3^{2k}s)^9)$. The induction hypothesis combined with Theorem 24 completes the proof. □

**Corollary 38.** *Theorem* 37 *is not vacuous.*

**Proof.** If $r$ denotes any of the values listed in Theorem 13 and $t$ denotes any of the values listed in Theorem 12 then $s = rt$ satisfies the hypotheses of Theorem 37. To see this consider Theorems 14 and 8 with $Q = r$ and $P = 3^2 t$. □

**Corollary 39.** *Let $v$ be an arbitrary product of elements in $R^*$ subject to the condition that $v$ contains at least two factors $y \in R^* \cap T_2$ and $s$ that satisfies the hypotheses of Theorem* 37, *$y \neq s$. Then there exists a $Z$-cyclic* $\mathrm{Wh}(3^{2n}v+1)$ *for all $n \geqslant 2$.*

**Proof.** Let $n \geqslant 2$ be fixed. Set $v_1 = v/s$. Inflate the $Z$-cyclic WhFrame$(3^{9v_1})$ of Theorem 28 by $3^{2n-3}s$. Apply Theorem 24. □

## 4. $Z$-cyclic Wh$(3^3 s)$, $s \equiv 3 \pmod 4$

In this section we establish the existence of some new $Z$-cyclic whist designs when the number of players is of the form $4n+1$.

**Theorem 40.** *Let $s \in T \cap T_e$ such that $3 \nmid s$. Then there exists a Z-cyclic* $\mathrm{Wh}(3^{2n+1}s)$, *for all $n \geqslant 1$.*

**Proof.** Begin with the $Z$-cyclic WhFrame($3^9$). Inflate by $s$ and apply Theorem 23 to obtain a $Z$-cyclic Wh($3^3s$). To obtain a $Z$-cyclic Wh($3^5s$) apply Theorem 7 with $P_1=81$ and $P_2=3s$. Next begin with the $Z$-cyclic WhFrame($3^9$), inflate by $3^4s$ and apply Theorem 23 to obtain a $Z$-cyclic Wh($3^7s$). Use Theorem 7 with $P_1=(81)^2$ and $P_2=3s$ to obtain a $Z$-cyclic Wh($3^9s$). Inflating the $Z$-cyclic WhFrame($3^9$) by $3^8s$ and applying Theorem 23 produces a $Z$-cyclic Wh($3^{11}s$). Clearly one can repeat these steps recursively to obtain the desired result. □

**Corollary 41.** *Theorem* 40 *is not vacuous.*

**Proof.** $s=7,11,19,23,31,35,43,47,67,167,187,227,287$ satisfy the hypotheses of Theorem 40. The fact that there exist $Z$-cyclic Wh($3s$) for $s=7,11$ has been known for some time. For the remaining cases see [1,2]. Infinite families of solutions can be obtained by setting $s=rv$ where $r$ is any number from the previous list and $v$ is an arbitrary product of elements in $R^*$. One would apply Theorem 7 with $P_1=v$, $P_2=3r$ for the existence of the $Z$-cyclic Wh($3s$). □

**Example 42.** Using the Wh(21) of Example 2, a homogeneous (7,4,1)-DM whose $i$th row is $iZ_7$ and the $Z$-cyclic WhFrame($3^9$) we obtain the following initial round of a $Z$-cyclic Wh($3^3\cdot 7=189$).

(9, 135, 18, 162), (27, 36, 117, 171), (153, 72, 45, 144), (63, 81, 99, 126),
(1, 12, 2, 24), (28, 66, 83, 132), (55, 120, 164, 51), (82, 174, 56, 159),
(109, 39, 137, 78), (136, 93, 29, 186), (163, 147, 110, 105), (8, 21, 19, 4),
(35, 75, 100, 112), (62, 129, 181, 31), (89, 183, 73, 139), (116, 48, 154, 58),
(143, 102, 46, 166), (170, 156, 127, 85), (13, 23, 16, 15), (40, 77, 97, 123),
(67, 131, 178, 42), (94, 185, 70, 150), (121, 50, 151, 69), (148, 104, 43, 177),
(175, 158, 124, 96), (3, 6, 5, 10), (30, 60, 86, 118), (57, 114, 167, 37),
(54, 90, 180, 108), (84, 168, 59, 145), (111, 33, 140, 64), (138, 87, 32, 172),
(165, 141, 113, 91), (25, 17, 11, 22), (52, 71, 92, 130), (79, 125, 173, 49),
(106, 179, 65, 157), (133, 44, 146, 76), (160, 98, 38, 184), (187, 152, 119, 103),
(20, 7, 26, 14), (47, 61, 107, 122), (74, 115, 188, 41), (101, 169, 80, 149),
(128, 34, 161, 68), (155, 88, 53, 176), (182, 142, 134, 95).

**Theorem 43.** *With $v$ as in Theorem* 28 *there exists a Z-cyclic* $\mathrm{Wh}(3^3vQ)$ *for all* $Q \in T \cap T_e$.

**Proof.** Begin with the $Z$-cyclic WhFrame($3^{9v}$) of Theorem 28, inflate by $Q$ and apply Theorem 23. □

**Theorem 44.** *Theorem* 43 *is not vacuous.*

**Proof.** $\{7,11,15,19,23,31,35,39,43,47\}\subset Q$. □

**Theorem 45.** *Let $s \in S_{02}$ then there exists a Z-cyclic* Wh$(3^{2n}s)$ *for all* $n \geqslant 0$.

**Proof.** See the proof of Theorem 10. □

**Theorem 46.** *Let $s \in S_{13}$ then there exists a Z-cyclic* Wh$(3^{2n+1}s)$ *for all* $n \geqslant 0$.

**Proof.** See the proof of Theorem 14. □

**Theorem 47.** *Let $v$ be an arbitrary product of elements in $R^*$ subject to the condition that $v$ contains at least one factor $s \in S_{02}$. Then there exists a Z-cyclic* Wh$(3^{2n}v)$ *for all* $n \geqslant 0$.

**Proof.** Let $n \geqslant 0$ be fixed. Set $v_1 = v/s$. Note that $v_1 \equiv 1 \pmod 4$ and that there exists a $Z$-cyclic Wh$(v_1)$ [4,9]. Apply Theorem 7 with $P_1 = 3^{2n}s$, $P_2 = v_1$. □

## References

[1] R.J.R. Abel, S. Costa, N.J. Finizio, Directed ordered whist tournaments—existence results and some $Z$-cyclic solutions, preprint.
[2] R.J.R. Abel, G. Ge, private communication.
[3] I. Anderson, Combinatorial Designs and Tournaments, Oxford University Press, Oxford, 1997.
[4] I. Anderson, Whist tournaments, in: C.J. Colbourn, J.H. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, FL, 1996.
[5] I. Anderson, N.J. Finizio, P.A. Leonard, New product theorems for $Z$-cyclic whist tournaments, J. Combin. Theory Ser. A 88 (1999) 162–166.
[6] N.J. Finizio, A.J. Merritt, Some new $Z$-cyclic whist tournaments, Discrete Appl. Math. 101 (2000) 115–130.
[7] S. Furino, Y. Miao, J. Yin, Frames and Resolvable Designs: Uses, Constructions, and Existence, CRC Press, Boca Raton, FL, 1996.
[8] G. Ge, C.W.H. Lam, Some new triplewhist tournaments, TWh($v$), preprint.
[9] G. Ge, L. Zhu, Frame constructions for $Z$-cyclic triplewhist tournaments, Bulletin ICA 32 (2001) 53–62.