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Non-simultaneous blow-up of n components for nonlinear parabolic systems ☆

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ABSTRACT

This paper deals with non-simultaneous and simultaneous blow-up for radially symmetric solution $(u_1, u_2, \dots, u_n)$ to heat equations coupled via nonlinear boundary $\frac{\partial u_i}{\partial \eta} = u_i^{p_i} u_{i+1}^{q_{i+1}}$ $(i = 1, 2, \dots, n)$. It is proved that there exist suitable initial data such that u_i $(i \in \{1, 2, \dots, n\})$ blows up alone if and only if $q_i + 1 < p_i$. All of the classifications on the existence of only two components blowing up simultaneously are obtained. We find that different positions (different values of k, i, n) of u_{i-k} and u_i leads to quite different blow-up rates. It is interesting that different initial data lead to different blow-up phenomena even with the same requirements on exponent parameters. We also propose that $u_{i-k}, u_{i-k+1}, \dots, u_i$ $(i \in \{1, 2, \dots, n\},\ k \in \{0, 1, 2, \dots, n-1\})$ blow up simultaneously while the other ones remain bounded in different exponent regions. Moreover, the blow-up rates and blow-up sets are obtained.

1. Introduction

In this paper, we consider the following parabolic system

$$\begin{cases} (u_i)_t = \Delta u_i, \quad (x,t) \in B_R \times (0,T), \\ \dfrac{\partial u_i}{\partial \eta} = u_i^{p_i} u_{i+1}^{q_{i+1}}, \quad (x,t) \in \partial B_R \times (0,T), \\ u_i(x,0) = u_{i,0}(x), \quad i = 1,2,\dots,n,\ n \geqslant 2,\ x \in B_R, \\ u_{n+1} := u_1, \qquad p_{n+1} := p_1, \qquad q_{n+1} := q_1, \end{cases} \tag{1.1}$$

where $B_R = \{x \in \mathbf{R}^N\colon |x| < R\}$; exponents $p_i, q_i \geqslant 0$ $(i = 1, 2, \dots, n)$; $\partial/\partial\eta$ is the outer normal derivative; radially symmetric functions $u_{i,0}(x)$ $(i = 1, 2, \dots, n)$ are positive and smooth, satisfying the compatibility conditions; Let T be the blow-up time of system (1.1). The existence and uniqueness of local solutions to system (1.1) is well known (see, for example, [8]). Nonlinear parabolic system (1.1) comes from chemical reactions, heat transfer, etc., where $u_1, u_2, \dots, u_n$ represent concentrations of chemical reactants, temperatures of materials during heat propagations, etc.

Non-simultaneous and simultaneous blow-up for nonlinear parabolic systems have deserved so much attention (see [1–3,9,16,19,20,25]). If $n = 2$, system (1.1) turns into

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$$\begin{cases} (u_1)_t = \Delta u_1, \qquad (u_2)_t = \Delta u_2, \quad (x,t) \in B_R \times (0,T), \\ \dfrac{\partial u_1}{\partial \eta} = u_1^{p_1} u_2^{q_2}, \qquad \dfrac{\partial u_2}{\partial \eta} = u_2^{p_2} u_1^{q_1}, \quad (x,t) \in \partial B_R \times (0,T), \\ u_1(x,0) = u_{1,0}(x), \qquad u_2(x,0) = u_{2,0}(x), \quad x \in B_R. \end{cases} \tag{1.2}$$

For (1.2), Pinasco and Rossi [15] observed that there exist initial data such that u_1 blows up while u_2 remains bounded in bounded domain of $\mathbf{R}^N$ if and only if $q_1 + 1 < p_1$. Rossi [18], Pedersen and Lin [14], Chen [4] discussed the simultaneous blow-up rate estimates of (1.2) in B_R, respectively. For $N = R = 1$, Brändle, Quirós and Rossi [1,2] obtained that non-simultaneous blow-up happens for every initial data if $q_1 + 1 < p_1$ and $p_2 \leqslant q_2 + 1$, or $q_2 + 1 < p_2$ and $p_1 \leqslant q_1 + 1$. It is interesting that non-simultaneous blow-up and simultaneous blow-up coexist in the exponent region $q_1 + 1 < p_1$, $q_2 + 1 < p_2$.

System (1.1) with $p_i = 0$ becomes

$$\begin{cases} (u_i)_t = \Delta u_i, \quad (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial u_i}{\partial \eta} = u_{i+1}^{q_{i+1}}, \quad (x,t) \in \partial\Omega \times (0,T), \\ u_i(x,0) = u_{i,0}(x), \quad i = 1,2,\dots,n, \ n \geqslant 2, \ x \in \Omega, \\ u_{n+1} := u_1, \qquad q_{n+1} := q_1. \end{cases} \tag{1.3}$$

It is easy to check that blow-up must be simultaneous for (1.3). Pedersen and Lin [13], Wang [22] obtained the simultaneous blow-up rate estimates if $q_1 q_2 \cdots q_n > 1$.

The related discussion on blow-up solutions of parabolic systems can be seen from [5,7,10,17,21,23] and the papers therein.

By the cited papers above, one can find that non-simultaneous blow-up is possible due to $p_i \geqslant 0$. In the present paper, the solution of (1.1) is making up of n components. The non-simultaneous blow-up means that at least $i \in \{1,2,\dots,n-1\}$ components blow up simultaneously while the other ones remain bounded up to the blow-up time, which has been rarely considered before. The present paper is arranged as follows, in the next section, a necessary and sufficient condition is given on the existence of one component blowing up alone. In Section 3, we obtain all of the classifications on the existence of two components blowing up simultaneously with the other ones remaining bounded. Furthermore, the blow-up rates of u_{i-k} and u_i ($i \in \{1,2,\dots,n\}$, $k \in \{1,2,\dots,n-1\}$) are obtained. It is interesting that the representations of blow-up rates are quite different with respect to different values of n, i, and k. In Section 4, we obtain the conditions of $u_{i-k}, u_{i-k+1}, \dots, u_i$ ($i \in \{1,2,\dots,n\}$, $k \in \{0,1,2,\dots,n-1\}$) blowing up simultaneously with the others remaining bounded for every positive initial data. Moreover, the corresponding blow-up rates and sets are considered.

2. The existence of only one component blowing up

The critical blow-up exponents for (1.1) can be obtained from Rossi [17].

Theorem 2.1. *The positive solutions of system* (1.1) *blow up if and only if*

$$\max\left\{p_i - 1 \ (i = 1,2,\dots,n),\ \prod_{j=1}^{n} q_j - \prod_{j=1}^{n} (1 - p_j)\right\} > 0. \tag{2.1}$$

From now on, we assume that (2.1) always holds. Denote $\xi_i := \xi_{i+n}$ if subscript $i \leqslant 0$. The set of initial data is denoted as follows,

$$\mathbb{V}_0 = \Bigg\{ (u_{1,0}, u_{2,0}, \dots, u_{n,0}):\ u_{i,0} \geqslant \zeta > 0,\ (u_{i,0})_r \geqslant 0,\ (u_{i,0})_{rr} + \frac{N-1}{r}(u_{i,0})_r \geqslant 0, \\ r \in [0,R),\ \frac{\partial u_{i,0}(R)}{\partial \eta} = \big(u_{i,0}^{p_i} u_{i+1,0}^{q_{i+1}}\big)(R),\ 1 \leqslant i \leqslant n \Bigg\}. \tag{2.2}$$

Clearly, $U_i(t) = u_i(R,t) = \max_{(y,s)\in[0,R]\times[0,t]} u_i(y,s)$, $1 \leqslant i \leqslant n$. In the sequel, $U_i(t) \sim (T-t)^{-\beta_i}$ represents that there exist constants $C, c > 0$ such that $c(T-t)^{-\beta_i} \leqslant U_i(t) \leqslant C(T-t)^{-\beta_i}$ as t near T.

Theorem 2.2. *There exist initial data such that u_i ($i \in \{1,2,\dots,n\}$) blows up alone if and only if $q_i + 1 < p_i$.*

Corollary 2.1. *At least two components blow up simultaneously for every initial data if and only if $p_j \leqslant q_j + 1$ for all $j = 1,2,\dots,n$.*

We introduce a lemma on the upper estimate for u_i.

Lemma 2.1. *Let T be the blow-up time of system* (1.1)*. If $p_i > 1$, then*

$$U_i(t) \leqslant C_T(T-t)^{-\frac{1}{2(p_i-1)}}, \tag{2.3}$$

where $C_T = \tilde{C}(1+4C_1T^{\frac{1}{2}})^{\frac{1}{p_i-1}}$, $\tilde{C} = \tilde{C}(p_i, q_{i+1}, u_{i+1,0}(R), N, R) > 0$, $C_1 = C_1(N, R) > 0$.

Proof. Let Γ be the fundamental solution of the heat equation. By Green's identity,

$$\begin{aligned}
\frac{1}{2}U_i(t) &= \int_{B_R} \Gamma(x-y, t-z)u_i(y,z)\,dy - \int_z^t \int_{\partial B_R} U_i(\tau)\frac{\partial \Gamma}{\partial \eta}(x-y, t-\tau)\,dS_y\,d\tau \\
&\quad + \int_z^t \int_{\partial B_R} U_i^{p_i}(\tau)U_{i+1}^{q_{i+1}}(\tau)\Gamma(x-y, t-\tau)\,dS_y\,d\tau \\
&\geqslant C_2 u_{i+1,0}^{q_{i+1}}(R)\int_z^t U_i^{p_i}(\tau)(t-\tau)^{-\frac{1}{2}}\,d\tau - 2C_1T^{\frac{1}{2}}U_i(t), \quad x \in \partial B_R,\ 0<z<t<T,
\end{aligned}$$

where C_1, C_2 depend only on B_R. Set $I(t) = \int_z^t U_i^{p_i}(\tau)(T-\tau)^{-\frac{1}{2}}\,d\tau$. Then

$$I'(t) \geqslant \left(C_2 u_{i+1,0}^{q_{i+1}}(R)\right)^{p_i}\left(\frac{1}{2}+2C_1T^{\frac{1}{2}}\right)^{-p_i} I^{p_i}(t)(T-t)^{-\frac{1}{2}}.$$

Integrating the above inequality from t to T, we obtain that

$$I(t) \leqslant \left[2(p_i-1)\left(C_2^{p_i}u_{i+1,0}^{q_{i+1}}(R)\right)^{p_i}\left(\frac{1}{2}+2C_1T^{\frac{1}{2}}\right)^{-p_i}\right]^{-\frac{1}{p_i-1}}(T-t)^{-\frac{1}{2(p_i-1)}}. \tag{2.4}$$

On the other hand, for $0<z=2t-T<t<T$,

$$I(t) \geqslant \int_z^{\frac{T+z}{2}} U^{p_i}(z)(T-\tau)^{-\frac{1}{2}}\,d\tau = (2-\sqrt{2})U^{p_i}(z)(T-z)^{\frac{1}{2}}. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain the estimate (2.3) with

$$\tilde{C} = \left(2C_2u_{i+1,0}^{q_{i+1}}(R)\right)^{-\frac{1}{p_i-1}}(2-\sqrt{2})^{-\frac{1}{p_i}}\left[\sqrt{2}(p_i-1)\right]^{-\frac{1}{p_i^2-p_i}}. \qquad \square$$

Proof of Theorem 2.2. Without loss of generality, we only prove the case for $i=n$. We first prove the sufficient condition. Let $G(x,y,t,\tau)$ be Green's function of the heat equation on B_R, satisfying $\frac{\partial G}{\partial \eta}|_{\partial B_R} = 0$ (see [6,11,12]) and

$$\int_{\partial B_R} G(x,y,t,\tau)\,dS_y \leqslant \bar{C}(t-\tau)^{-\frac{1}{2}}, \tag{2.6}$$

where $\bar{C} > 0$ depends only on B_R.

Fix $u_{1,0}(R), u_{2,0}(R), \ldots, u_{n-1,0}(R)$ and then take $M_m > (2u_{m,0}(R))^{p_m}$ $(m=1,2,\ldots,n-1)$. One can choose the initial data $(u_{1,0}, u_{2,0}, \ldots, u_{n,0}) \in \mathbb{V}_0$ such that T satisfies

$$\begin{aligned}
&\left(2u_{m,0}(R) + 2\bar{C}M_{m+1}^{\frac{q_{m+1}}{p_{m+1}}}M_mT^{\frac{1}{2}}\right)^{p_m} < M_m \quad (m=1,2,\ldots,n-2), \\
&\left(2u_{n-1,0}(R) + \frac{2(p_n-1)}{p_n-1-q_n}\bar{C}M_{n-1}C_T^{q_n}T^{\frac{p_n-1-q_n}{2(p_n-1)}}\right)^{p_{n-1}} < M_{n-1},
\end{aligned}$$

where $C_T = \tilde{C}(1+4C_1T^{\frac{1}{2}})^{\frac{1}{p_n-1}}$ with $\tilde{C}$, C_1 depending only on p_n, q_1, B_R and $u_{1,0}(R)$.

By Lemma 2.1, $U_n(t) \leqslant C_T(T-t)^{-\frac{1}{2(p_n-1)}}$. Then u_{n-1} satisfies that

$$\begin{cases}
(u_{n-1})_t = \Delta u_{n-1}, & (x,t) \in B_R \times (0,T), \\
\dfrac{\partial u_{n-1}}{\partial \eta} \leqslant C_T^{q_n}(T-t)^{-\frac{q_n}{2(p_n-1)}}u_{n-1}^{p_{n-1}}, & (x,t) \in \partial B_R \times (0,T), \\
u_{n-1}(x,0) = u_{n-1,0}(x), & x \in B_R.
\end{cases} \tag{2.7}$$

Consider the auxiliary problem

$$\begin{cases} (\bar{u}_{n-1})_t = \Delta \bar{u}_{n-1}, & (x,t) \in B_R \times (0,T), \\ \dfrac{\partial \bar{u}_{n-1}}{\partial \eta} = M_{n-1} C_T^{q_n} (T-t)^{-\frac{q_n}{2(p_n-1)}}, & (x,t) \in \partial B_R \times (0,T), \\ \bar{u}_{n-1}(x,0) = \bar{u}_{n-1,0}(x), & x \in B_R, \end{cases}$$

where radially symmetric $\bar{u}_{n-1,0}$ satisfies $\frac{\partial \bar{u}_{n-1,0}}{\partial \eta}|_{\partial B_R} = M_{n-1} C_T^{q_n} T^{-\frac{q_n}{2(p_n-1)}}$, $\bar{u}_{n-1,0}(R) = 2u_{n-1,0}(R)$; $\Delta \bar{u}_{n-1,0} \geqslant 0$, $\bar{u}_{n-1,0} \geqslant u_{n-1,0}$ in B_R.

For $q_n + 1 < p_n$, by Green's identity and (2.6),

$$\bar{u}_{n-1} \leqslant 2u_{n-1,0}(R) + \frac{2(p_n-1)}{p_n-1-q_n} \bar{C} M_{n-1} C_T^{q_n} T^{\frac{p_n-1-q_n}{2(p_n-1)}} \leqslant M_{n-1}^{\frac{1}{p_{n-1}}}.$$

So $\bar{u}_{n-1}$ satisfies that

$$\begin{cases} (\bar{u}_{n-1})_t = \Delta \bar{u}_{n-1}, & (x,t) \in B_R \times (0,T), \\ \dfrac{\partial \bar{u}_{n-1}}{\partial \eta} \geqslant C_T^{q_n} (T-t)^{-\frac{q_n}{2(p_n-1)}} \bar{u}_{n-1}^{p_{n-1}}, & (x,t) \in \partial B_R \times (0,T), \\ \bar{u}_{n-1}(x,0) = \bar{u}_{n-1,0}(x), & x \in B_R. \end{cases}$$

By the comparison principle, $u_{n-1} \leqslant \bar{u}_{n-1} \leqslant M_{n-1}^{\frac{1}{p_{n-1}}}$ on $\bar{B}_R \times [0,T)$.

Introduce the following auxiliary problem

$$\begin{cases} (\bar{u}_{n-2})_t = \Delta \bar{u}_{n-2}, & (x,t) \in B_R \times (0,+\infty), \\ \dfrac{\partial \bar{u}_{n-2}}{\partial \eta} = M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} M_{n-2}, & (x,t) \in \partial B_R \times (0,+\infty), \\ \bar{u}_{n-2}(x,0) = \bar{u}_{n-2,0}(x), & x \in B_R, \end{cases} \tag{2.8}$$

where radially symmetric $\bar{u}_{n-2,0}(x)$ satisfies $\frac{\partial \bar{u}_{n-2,0}}{\partial \eta} = M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} M_{n-2}$, $\bar{u}_{n-2,0}(x) = 2u_{n-2,0}(x)$ for $x \in \partial B_R$; $\Delta \bar{u}_{n-2,0}(x) \geqslant 0$, $\bar{u}_{n-2,0}(x) \geqslant u_{n-2,0}(x)$ for $x \in B_R$. Considering the problem (2.8) in $[0,T)$, we obtain that

$$\bar{u}_{n-2} \leqslant 2u_{n-2,0}(R) + 2\bar{C} M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} M_{n-2} T^{\frac{1}{2}} \leqslant M_{n-2}^{\frac{1}{p_{n-2}}}.$$

Then $\bar{u}_{n-2}$ satisfies $\frac{\partial \bar{u}_{n-2}}{\partial \eta} \geqslant M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} \bar{u}_{n-2}^{p_{n-2}}$, $(x,t) \in \partial B_R \times (0,T)$. Due to $u_{n-1} \leqslant M_{n-1}^{\frac{1}{p_{n-1}}}$, u_{n-2} satisfies $\frac{\partial u_{n-2}}{\partial \eta} \leqslant M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} u_{n-2}^{p_{n-2}}$ for $(x,t) \in \partial B_R \times (0,T)$. By the comparison principle, $u_{n-2} \leqslant \bar{u}_{n-2} \leqslant M_{n-2}^{\frac{1}{p_{n-2}}}$ on $\overline{B}_R \times [0,T)$. The boundedness of $u_{n-3}, u_{n-4}, \ldots, u_1$ can be proved similarly. So only u_n blows up.

Now, we prove the necessary condition. Assume that $u_1 \leqslant C$. Then u_n satisfies

$$\begin{cases} (u_n)_t = \Delta u_n, & (x,t) \in B_R \times (0,T), \\ \dfrac{\partial u_n}{\partial \eta} \leqslant C^{q_1} u_n^{p_n}, & (x,t) \in \partial B_R \times (0,T), \\ u_n(x,0) = u_{n,0}(x), & x \in B_R. \end{cases} \tag{2.9}$$

By Green's identity, we have

$$U_n(t) \leqslant U_n(z) + 2\bar{C} C^{q_1} U_n^{p_n}(t)(T-z)^{\frac{1}{2}}, \quad z < t < T.$$

Take z such that $U_n(z) = U_n(t)/2$. Then $U_n(z) \geqslant c(T-z)^{-\frac{1}{2(p_n-1)}}$, $z \in (0,T)$. Also by Green's identity,

$$\frac{1}{2} U_{n-1}(t) \geqslant c \int_0^t (T-\tau)^{-\frac{q_n}{2(p_n-1)}} (t-\tau)^{-\frac{1}{2}} \, d\tau.$$

The boundedness of u_{n-1} requires that $q_n + 1 < p_n$. $\square$

It can be understood that the blow-up rate for only one component blowing up is equivalent to that of the scalar case (see [7]).

Theorem 2.3. *If only u_i $(i \in \{1,2,\ldots,n\})$ blows up, then $U_i(t) \sim (T-t)^{-\frac{1}{2(p_i-1)}}$.*

3. The existence of only two blowing up

In this section, we discuss the existence of only two components blowing up.

Theorem 3.1. *Assume $i \in \{1,2,\dots,n\}$, $k \in \{1,2,\dots,n-1\}$, and $n \geqslant 3$. If $q_i+1<p_i$ and $q_{i-k}+1<p_{i-k}$, then there exist suitable initial data such that u_{i-k}, u_i blow up simultaneously at time T while the others remain bounded up to T. Moreover,*

$$\big(U_{i-k}(t), U_i(t)\big) \sim \begin{cases} \big((T-t)^{-\frac{p_i-1-q_i}{2(p_i-1)(p_{i-k}-1)}}, (T-t)^{-\frac{1}{2(p_i-1)}}\big) & \text{for } k=1; \\ \big((T-t)^{-\frac{1}{2(p_{i-k}-1)}}, (T-t)^{-\frac{1}{2(p_i-1)}}\big) & \text{for } k \in \{2,3,\dots,n-2\}; \\ \big((T-t)^{-\frac{1}{2(p_{i-k}-1)}}, (T-t)^{-\frac{p_{i-k}-1-q_{i-k}}{2(p_{i-k}-1)(p_i-1)}}\big) & \text{for } k=n-1. \end{cases}$$

Without loss of generality, we only prove the case for $i=n$. We divide Theorem 3.1 into three propositions for $k=1$, $k\in\{2,3,\dots,n-2\}$ and $k=n-1$, respectively. At first, we deal with the case for $i=n$ and $k=1$.

Proposition 3.1. *If $q_n+1<p_n$ and $q_{n-1}+1<p_{n-1}$, then there exist suitable initial data such that u_{n-1}, u_n blow up simultaneously at time T while the others remain bounded up to T. Moreover,*

$$\big(U_{n-1}(t), U_n(t)\big) \sim \big((T-t)^{-\frac{p_n-1-q_n}{2(p_n-1)(p_{n-1}-1)}}, (T-t)^{-\frac{1}{2(p_n-1)}}\big).$$

In order to prove Proposition 3.1, we introduce a subset of $\mathbb{V}_0$ as follows:

$$\begin{aligned} \mathbb{V}_1 = \bigg\{ &\big(u_{1,0}(r), u_{2,0}(r), \dots, u_{n,0}(r)\big):\ u_{m,0}(r) = N_m + \frac{R}{2}\sqrt{M_m^2+4} - \frac{R}{2}M_m \\ &- \sqrt{R^2 - \left(\frac{1}{2}M_m\sqrt{M_m^2+4} - \frac{1}{2}M_m^2\right) r^2},\ r \in [0,R], \\ &\text{with } M_m = u_{m,0}^{p_m}(R) u_{m+1,0}^{q_{m+1}}(R),\ N_m = u_{m,0}(R)\ (m=1,2,\dots,n), \\ &\text{where } u_{1,0}(R) = \frac{R}{\lambda_1},\ u_{l,0}(R) = \frac{R}{\prod_{j=1}^{l-1}(1-\lambda_j)\lambda_l}\ (l=2,3,\dots,n-1), \\ &u_{n,0}(R) = \frac{R}{\prod_{j=1}^{n-1}(1-\lambda_j)},\ \lambda_1,\lambda_2,\dots,\lambda_{n-1} \in (0,1) \bigg\}. \end{aligned}$$

We use the following five lemmas to prove it.

Lemma 3.1. *If $q_n+1<p_n$ and $q_{n-1}+1<p_{n-1}$, then there exists $\bar{\lambda}_{n-2} \in (\frac{1}{2},1)$ such that, for any initial data satisfying that $u_{j,0}(R) = 2^j R$ $(j=1,2,\dots,n-3)$ and $u_{n-2,0}(R) = \frac{2^{n-3}R}{\bar{\lambda}_{n-2}}$ in $\mathbb{V}_1$, non-simultaneous blow-up must happen with $u_1,u_2,\dots,u_{n-2}$ remaining bounded.*

Proof. Take $M_j > (2^{j+1}R)^{p_j}$ $(j=1,2,\dots,n-2)$. Consider the following auxiliary problem

$$\begin{cases} (\underline{u}_{n-1})_t = \Delta \underline{u}_{n-1}, & (x,t) \in B_R \times (0, \underline{T}_{n-1}), \\ \dfrac{\partial \underline{u}_{n-1}}{\partial \eta} = \big(2^{n-2}R - R\big)^{q_n} \underline{u}_{n-1}^{p_{n-1}}, & (x,t) \in \partial B_R \times (0, \underline{T}_{n-1}), \\ \underline{u}_{n-1}(x,0) = \underline{u}_{n-1,0}(x), & x \in B_R, \end{cases} \tag{3.1}$$

where radially symmetric $\underline{u}_{n-1,0}(x)$ satisfies the compatibility conditions and $\frac{2^{n-3}R}{1-\lambda_{n-2}} - 2R \leqslant \underline{u}_{n-1,0}(x) \leqslant \frac{2^{n-3}R}{1-\lambda_{n-2}} - R$ with λ_{n-2} to be determined.

For problem (3.1), there must exist $\bar{\lambda}_{n-2} \in (\frac{1}{2},1)$ such that, if $\lambda_{n-2} = \bar{\lambda}_{n-2}$, then $\underline{T}_{n-1}$ satisfies

$$\begin{aligned} M_j &\geqslant \big(2^{j+1}R + 2\bar{C} M_{j+1}^{\frac{q_{j+1}}{p_{j+1}}} M_j \underline{T}_{n-1}^{\frac{1}{2}}\big)^{p_j} \quad (j=1,2,\dots,n-3), \\ M_{n-2} &\geqslant \left(2^{n-1}R + \frac{2(p_{n-1}-1)}{p_{n-1}-1-q_{n-1}} \bar{C} M_{n-2} C_{\underline{T}_{n-1}}^{q_{n-1}} \underline{T}_{n-1}^{\frac{p_{n-1}-1-q_{n-1}}{2(p_{n-1}-1)}}\right)^{p_{n-2}}. \end{aligned}$$

For any $(u_{1,0}, u_{2,0}, \dots, u_{n,0}) \in \mathbb{V}_1$ with $u_{j,0}(R) = 2^j R$ $(j=1,2,\dots,n-3)$ and $u_{n-2,0}(R) = \frac{2^{n-3}R}{\bar{\lambda}_{n-2}}$, we have

$$u_{n-1,0}(R)=\frac{2^{n-3}R}{(1-\bar\lambda_{n-2})\lambda_{n-1}}\geqslant\frac{2^{n-3}R}{1-\bar\lambda_{n-2}}\quad\text{for any } \lambda_{n-1}\in(0,1).$$

Then

$$\frac{2^{n-3}R}{1-\bar\lambda_{n-2}}-2R\leqslant\underline{u}_{n-1,0}(x)\leqslant\frac{2^{n-3}R}{1-\bar\lambda_{n-2}}-R\leqslant u_{n-1,0}(x)\leqslant\frac{2^{n-3}R}{(1-\bar\lambda_{n-2})\lambda_{n-1}}.$$

For $(u_n)_t\geqslant 0$, $u_n(x,t)\geqslant u_{n,0}(x)\geqslant 2^{n-2}R-R$. By the comparison principle, $\underline{u}_{n-1}\leqslant u_{n-1}$ and $T\leqslant\underline{T}_{n-1}$. Hence

$$M_j\geqslant\big(2^{j+1}R+2\bar C M_{j+1}^{\frac{q_{j+1}}{p_{j+1}}}M_jT^{\frac12}\big)^{p_j}\quad(j=1,2,\dots,n-3),$$
$$M_{n-2}\geqslant\left(2^{n-1}R+\frac{2(p_{n-1}-1)}{p_{n-1}-1-q_{n-1}}\bar C M_{n-2}C_T^{q_{n-1}}T^{\frac{p_{n-1}-1-q_{n-1}}{2(p_{n-1}-1)}}\right)^{p_{n-2}}.$$

Consider the second auxiliary problem

$$\begin{cases}(\bar u_{n-2})_t=\Delta\bar u_{n-2}, & (x,t)\in B_R\times(0,T),\\ \dfrac{\partial\bar u_{n-2}}{\partial\eta}=M_{n-2}C_T^{q_{n-1}}(T-t)^{-\frac{q_{n-1}}{2(p_{n-1}-1)}}, & (x,t)\in\partial B_R\times(0,T),\\ \bar u_{n-2}(x,0)=\bar u_{n-2,0}(x), & x\in B_R,\end{cases}\tag{3.2}$$

where radially symmetric $\bar u_{n-2,0}(x)$ satisfies $\frac{\partial\bar u_{n-2,0}(x)}{\partial\eta}=M_{n-2}C_T^{q_{n-1}}T^{-\frac{q_{n-1}}{2(p_{n-1}-1)}}$, $\bar u_{n-2,0}(x)=2^{n-1}R$ for $x\in\partial B_R$; $\Delta\bar u_{n-2,0}(x)\geqslant 0$, $\bar u_{n-2,0}(x)\geqslant u_{n-2,0}(x)$ for $x\in B_R$.

By Green's identity and $q_{n-1}+1<p_{n-1}$,

$$\bar u_{n-2}\leqslant 2^{n-1}R+\frac{2(p_{n-1}-1)}{p_{n-1}-1-q_{n-1}}\bar C M_{n-2}C_T^{q_{n-1}}T^{\frac{p_{n-1}-1-q_{n-1}}{2(p_{n-1}-1)}}\leqslant M_{n-2}^{\frac{1}{p_{n-2}}}.$$

So $\bar u_{n-2}$ satisfies $\frac{\partial\bar u_{n-2}}{\partial\eta}\geqslant C_T^{q_{n-1}}(T-t)^{-\frac{q_{n-1}}{2(p_{n-1}-1)}}\bar u_{n-2}^{p_{n-2}}$, $(x,t)\in\partial B_R\times(0,T)$. By Lemma 2.1 and $p_{n-1}>1$, we have $u_{n-1}\leqslant C_T(T-t)^{-\frac{1}{2(p_{n-1}-1)}}$, and hence $\frac{\partial u_{n-2}}{\partial\eta}\leqslant C_T^{q_{n-1}}(T-t)^{-\frac{q_{n-1}}{2(p_{n-1}-1)}}u_{n-2}^{p_{n-2}}$, $(x,t)\in\partial B_R\times(0,T)$. Then by the comparison principle, $u_{n-2}\leqslant\bar u_{n-2}\leqslant M_{n-2}^{\frac{1}{p_{n-2}}}$.

Introduce the third auxiliary problem

$$\begin{cases}(\bar u_{n-3})_t=\Delta\bar u_{n-3}, & (x,t)\in B_R\times(0,+\infty),\\ \dfrac{\partial\bar u_{n-3}}{\partial\eta}=M_{n-2}^{\frac{q_{n-2}}{p_{n-2}}}M_{n-3}, & (x,t)\in\partial B_R\times(0,+\infty),\\ \bar u_{n-3}(x,0)=\bar u_{n-3,0}(x), & x\in B_R,\end{cases}\tag{3.3}$$

where radially symmetric $\bar u_{n-3,0}(x)$ satisfies $\frac{\partial\bar u_{n-3,0}}{\partial\eta}=M_{n-2}^{\frac{q_{n-2}}{p_{n-2}}}M_{n-3}$, $\bar u_{n-3,0}(x)=2^{n-2}R$ for $x\in\partial B_R$; $\Delta\bar u_{n-3,0}(x)\geqslant 0$, $\bar u_{n-3,0}(x)\geqslant u_{n-3,0}(x)$ for $x\in B_R$. Considering problem (3.3) in $(0,T)$, we have

$$\bar u_{n-3}\leqslant 2^{n-2}R+2\bar C M_{n-2}^{\frac{q_{n-2}}{p_{n-2}}}M_{n-3}T^{\frac12}\leqslant M_{n-3}^{\frac{1}{p_{n-3}}}.$$

So $\bar u_{n-3}$ satisfies $\frac{\partial\bar u_{n-3}}{\partial\eta}\geqslant M_{n-2}^{\frac{q_{n-2}}{p_{n-2}}}\bar u_{n-3}^{p_{n-3}}$ for $(x,t)\in\partial B_R\times(0,T)$. For $u_{n-2}\leqslant M_{n-2}^{\frac{1}{p_{n-2}}}$, u_{n-3} satisfies $\frac{\partial u_{n-3}}{\partial\eta}\leqslant M_{n-2}^{\frac{q_{n-2}}{p_{n-2}}}u_{n-3}^{p_{n-3}}$ for $(x,t)\in\partial B_R\times(0,T)$. Then $u_{n-3}\leqslant\bar u_{n-3}\leqslant M_{n-3}^{\frac{1}{p_{n-3}}}$. The boundedness of $u_{n-4},u_{n-5},\dots,u_1$ can be proved similarly. □

Lemma 3.2. *If $q_n+1<p_n$ and $q_{n-1}+1<p_{n-1}$, then, for the fixed $\bar\lambda_{n-2}\in(\frac12,1)$ in Lemma 3.1, there exists $\lambda'_{n-1}\in(0,\frac12)$ such that non-simultaneous blow-up happens with u_{n-1} blowing up and the others remaining bounded, where the initial data satisfy that $u_{j,0}(R)=2^j$ $(j=1,2,\dots,n-3)$, $u_{n-2,0}(R)=\frac{2^{n-3}R}{\bar\lambda_{n-2}}$, $u_{n-1,0}(R)=\frac{2^{n-3}R}{(1-\bar\lambda_{n-2})\lambda'_{n-1}}$, $u_{n,0}(R)=\frac{2^{n-3}R}{(1-\bar\lambda_{n-2})(1-\lambda'_{n-1})}$ in $\mathbb{V}_1$.*

Proof. Take $M_n>(\frac{2^{n-1}R}{1-\bar\lambda_{n-2}})^{p_n}$. Introduce the following auxiliary problem

$$\begin{cases}(\bar u_n)_t=\Delta\bar u_n, & (x,t)\in B_R\times(0,+\infty),\\ \dfrac{\partial\bar u_n}{\partial\eta}=M_1^{\frac{q_1}{p_1}}M_n, & (x,t)\in\partial B_R\times(0,+\infty),\\ \bar u_n(x,0)=\bar u_{n,0}(x), & x\in B_R,\end{cases}\tag{3.4}$$

where radially symmetric $\bar{u}_{n,0}(x)$ satisfies $\frac{\partial \bar{u}_{n,0}}{\partial \eta} = M_1^{\frac{q_1}{p_1}} M_n$, $\bar{u}_{n,0}(x) = \frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}}$ for $x \in \partial B_R$; $\Delta \bar{u}_{n,0}(x) \geqslant 0$, $\bar{u}_{n,0}(x) \geqslant u_{n,0}(x)$ for $x \in B_R$.

Consider problem (3.1) with the initial data $\underline{u}_{n-1,0}$ satisfying that

$$\frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})\lambda_{n-1}} - 2R \leqslant \underline{u}_{n-1,0}(x) \leqslant \frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})\lambda_{n-1}} - R,$$

where λ_{n-1} is to be determined. There exists some $\lambda'_{n-1} \in (0, \frac{1}{2})$ such that, if $\lambda_{n-1} = \lambda'_{n-1}$, then $\underline{T}_{n-1}$ satisfies

$$M_n \geqslant \left(\frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}} + 2\bar{C} M_1^{\frac{q_1}{p_1}} M_n \underline{T}_{n-1}^{\frac{1}{2}} \right)^{p_n}.$$

Similarly to Lemma 3.1, $\underline{u}_{n-1} \leqslant u_{n-1}$ and $T \leqslant \underline{T}_{n-1}$. Hence

$$M_n \geqslant \left(\frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}} + 2\bar{C} M_1^{\frac{q_1}{p_1}} M_n T^{\frac{1}{2}} \right)^{p_n}.$$

Consider problem (3.4) in $[0, T)$. By Green's identity,

$$\bar{u}_n \leqslant \frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}} + 2\bar{C} M_1^{\frac{q_1}{p_1}} M_n T^{\frac{1}{2}} \leqslant M_n^{\frac{1}{p_n}}.$$

Then $\bar{u}_n$ satisfies $\frac{\partial \bar{u}_n}{\partial \eta} \geqslant M_1^{\frac{q_1}{p_1}} \bar{u}_n^{p_n}$ for $(x,t) \in \partial B_R \times (0,T)$. Due to $u_1 \leqslant M_1^{\frac{1}{p_1}}$, u_n satisfies $\frac{\partial u_n}{\partial \eta} \leqslant M_1^{\frac{q_1}{p_1}} u_n^{p_n}$ for $(x,t) \in \partial B_R \times (0,T)$. By the comparison principle, $u_n \leqslant \bar{u}_n \leqslant M_n^{\frac{1}{p_n}}$. So only u_{n-1} blows up. $\square$

Lemma 3.3. *If $q_n + 1 < p_n$ and $q_{n-1} + 1 < p_{n-1}$, then, for the fixed $\bar{\lambda}_{n-2} \in (\frac{1}{2}, 1)$ in Lemma 3.1, there exists $\lambda''_{n-1} \in (\frac{1}{2}, 1)$ such that non-simultaneous blow-up happens with u_n blowing up and the others remaining bounded, where the initial data satisfy that $u_{j,0}(R) = 2^j R$ $(j = 1, 2, \ldots, n-3)$, $u_{n-2,0}(R) = \frac{2^{n-3}R}{\bar{\lambda}_{n-2}}$, $u_{n-1,0}(R) = \frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})\lambda''_{n-1}}$, $u_{n,0}(R) = \frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})(1-\lambda''_{n-1})}$ in $\mathbb{V}_1$.*

Proof. Introduce the following auxiliary problem

$$\begin{cases} (\underline{u}_n)_t = \Delta \underline{u}_n, & (x,t) \in B_R \times (0, \underline{T}_n), \\ \dfrac{\partial \underline{u}_n}{\partial \eta} = R^{q_1} \underline{u}_n^{p_n}, & (x,t) \in \partial B_R \times (0, \underline{T}_n), \\ \underline{u}_n(x,0) = \underline{u}_{n,0}(x), & x \in B_R, \end{cases}$$

where radially symmetric $\underline{u}_{n,0}(x)$ satisfies the compatibility conditions and

$$\frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})(1-\lambda_{n-1})} - 2R \leqslant \underline{u}_{n,0}(x) \leqslant \frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})(1-\lambda_{n-1})} - R$$

with λ_{n-1} to be determined.

Take $M_{n-1} > (\frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}})^{p_{n-1}}$. There exists $\lambda''_{n-1} \in (\frac{1}{2}, 1)$ such that, if $\lambda_{n-1} = \lambda''_{n-1}$, then $\underline{T}_n$ satisfies that

$$M_{n-1} \geqslant \left(\frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}} + \frac{2(p_n-1)}{p_n-1-q_n} \bar{C} M_{n-1} C_{\underline{T}_n}^{q_n} \underline{T}_n^{\frac{p_n-1-q_n}{2(p_n-1)}} \right)^{p_{n-1}}.$$

Take initial data $(u_{1,0}, u_{2,0}, \ldots, u_{n,0})$ in $\mathbb{V}_1$ such that $\lambda_j = \frac{1}{2}$ $(j = 1, 2, \ldots, n-3)$, $\lambda_{n-2} = \bar{\lambda}_{n-2}$, $\lambda_{n-1} = \lambda''_{n-1}$. For $\underline{u}_{n,0}(x) \leqslant \frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})(1-\lambda''_{n-1})} - R \leqslant u_{n,0}(x)$ and $u_1(x,t) \geqslant u_{1,0}(x) \geqslant R$, u_n satisfies $\frac{\partial u_n}{\partial \eta} \geqslant R^{q_1} u_n^{p_n}$ on $\partial B_R \times (0,T)$, and hence $\underline{u}_n \leqslant u_n$ and $T \leqslant \underline{T}_n$. So

$$M_{n-1} \geqslant \left(\frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}} + \frac{2(p_n-1)}{p_n-1-q_n} \bar{C} M_{n-1} C_T^{q_n} T^{\frac{p_n-1-q_n}{2(p_n-1)}} \right)^{p_{n-1}}.$$

Consider the following auxiliary problem

$$\begin{cases} (\bar{u}_{n-1})_t = \Delta \bar{u}_{n-1}, & (x,t) \in B_R \times (0,T), \\ \dfrac{\partial \bar{u}_{n-1}}{\partial \eta} = M_{n-1} C_T^{q_n} (T-t)^{-\frac{q_n}{2(p_n-1)}}, & (x,t) \in \partial B_R \times (0,T), \\ \bar{u}_{n-1}(x,0) = \bar{u}_{n-1,0}(x), & x \in B_R, \end{cases}$$

where radially symmetric $\bar{u}_{n-1,0}(x)$ satisfies the compatibility conditions and $\bar{u}_{n-1,0}(x) = \frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}}$, $x \in \partial B_R$; $\Delta \bar{u}_{n-1,0}(x) \geqslant 0$, $\bar{u}_{n-1,0}(x) \geqslant u_{n-1,0}(x)$, $x \in B_R$.

For $q_n + 1 < p_n$ and by Green's identity,

$$\bar{u}_{n-1} \leqslant \frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}} + \frac{2(p_n-1)}{p_n-1-q_n}\bar{C}M_{n-1}C_T^{q_n}T^{\frac{p_n-1-q_n}{2(p_n-1)}} \leqslant M_{n-1}^{\frac{1}{p_{n-1}}}.$$

So $\bar{u}_{n-1}$ satisfies $\frac{\partial \bar{u}_{n-1}}{\partial \eta} \geqslant C_T^{q_n}(T-t)^{-\frac{q_n}{2(p_n-1)}}\bar{u}_{n-1}^{p_{n-1}}$, $(x,t) \in \partial B_R \times (0,T)$. For $p_n > 1$, $u_n \leqslant C_T(T-t)^{-\frac{1}{2(p_n-1)}}$. Hence u_{n-1} satisfies $\frac{\partial u_{n-1}}{\partial \eta} \leqslant C_T^{q_n}(T-t)^{-\frac{q_n}{2(p_n-1)}}u_{n-1}^{p_{n-1}}$, $(x,t) \in \partial B_R \times (0,T)$. By the comparison principle, $u_{n-1} \leqslant \bar{u}_{n-1} \leqslant M_{n-1}^{\frac{1}{p_{n-1}}}$. Then only u_n blows up. □

Lemma 3.4.

(i) *The set of initial data in $\mathbb{V}_1$ such that u_n blows up while the others remain bounded is open in L^∞-topology.*
(ii) *The set of initial data in $\mathbb{V}_1$ such that u_{n-1} blows up while the others remain bounded is open in L^∞-topology.*

Proof. Without loss of generality, we only prove case (i). Let $(u_1, u_2, \dots, u_n)$ be a solution of (1.1) with initial data $(u_{1,0}, u_{2,0}, \dots, u_{n,0})$ in $\mathbb{V}_1$ such that u_n blows up at $t = T$ while the other components remain bounded, say $0 < 2\xi \leqslant u_1, u_2, \dots, u_{n-1} \leqslant M$. It suffices to find an L^∞-neighborhood of $(u_{1,0}, u_{2,0}, \dots, u_{n,0})$ in $\mathbb{V}_1$ such that any solution $(\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n)$ of (1.1) coming from this neighborhood maintains the property that $\hat{u}_n$ blows up while the others remain bounded.

By Theorem 2.2, $q_n + 1 < p_n$. Take $S_j > (2M + 2\xi)^{p_j}$ $(j = 1, 2, \dots, n-1)$. Let $(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_n)$ be the solution of the following problem

$$\begin{cases} (\tilde{u}_j)_t = \Delta \tilde{u}_j, \quad (x,t) \in B_R \times (0, T_0), \\ \dfrac{\partial \tilde{u}_j}{\partial \eta} = \tilde{u}_j^{p_j}\tilde{u}_{j+1}^{q_{j+1}}, \quad (x,t) \in \partial B_R \times (0, T_0), \\ \tilde{u}_j(x,0) = \tilde{u}_{j,0}(x), \quad j = 1, 2, \dots, n, \ n \geqslant 2, \ x \in B_R, \\ \tilde{u}_{n+1} := \tilde{u}_1, \qquad p_{n+1} := p_1, \qquad q_{n+1} := q_1, \end{cases} \tag{3.5}$$

where radially symmetric $(\tilde{u}_{1,0}, \tilde{u}_{2,0}, \dots, \tilde{u}_{n,0}) \in \mathbb{V}_0$ is to be determined. Denote

$$\mathcal{N}(u_{1,0}, u_{2,0}, \dots, u_{n,0}) = \big\{(\tilde{u}_{1,0}, \tilde{u}_{2,0}, \dots, \tilde{u}_{n,0}) \in \mathbb{V}_0\colon \ \big\|\tilde{u}_{j,0}(x) - u_j(x, T - \varepsilon_0)\big\|_\infty < \xi, \ 1 \leqslant j \leqslant n\big\}.$$

Since $(u_1, u_2, \dots, u_n)$ blows up at time T with fixed ξ, there exists $\varepsilon_0 > 0$ such that T_0 satisfies that

$$S_j \geqslant \big(2M + 2\xi + 2\bar{C}S_{j+1}^{\frac{q_{j+1}}{p_{j+1}}}S_jT_0^{\frac{1}{2}}\big)^{p_j} \quad (j = 1, 2, \dots, n-2),$$
$$S_{n-1} \geqslant \left(2M + 2\xi + \frac{2(p_n-1)}{p_n-1-q_n}\bar{C}S_{n-1}C_{T_0}^{q_n}T_0^{\frac{p_n-1-q_n}{2(p_n-1)}}\right)^{p_{n-1}}$$

provided $(\tilde{u}_{1,0}, \tilde{u}_{2,0}, \dots, \tilde{u}_{n,0}) \in \mathcal{N}(u_{1,0}, u_{2,0}, \dots, u_{n,0})$.

Consider the auxiliary problem

$$\begin{cases} (\bar{u}_{n-1})_t = \Delta \bar{u}_{n-1}, & (x,t) \in B_R \times (0, T_0), \\ \dfrac{\partial \bar{u}_{n-1}}{\partial \eta} = S_{n-1}C_{T_0}^{q_n}(T_0 - t)^{-\frac{q_n}{2(p_n-1)}}, & (x,t) \in \partial B_R \times (0, T_0), \\ \bar{u}_{n-1}(x,0) = \bar{u}_{n-1,0}(x), & x \in B_R, \end{cases}$$

where radially symmetric $\bar{u}_{n-1,0}(x)$ satisfies $\frac{\partial \bar{u}_{n-1,0}}{\partial \eta} = S_{n-1}C_{T_0}^{q_n}T_0^{-\frac{q_n}{2(p_n-1)}}$, $\bar{u}_{n-1,0}(x) = 2\tilde{u}_{n-1,0}(x)$, $x \in \partial B_R$; $\Delta \bar{u}_{n-1,0}(x) \geqslant 0$, $\bar{u}_{n-1,0}(x) \geqslant \tilde{u}_{n-1,0}(x)$, $x \in B_R$. By Green's identity,

$$\bar{u}_{n-1} \leqslant 2M + 2\xi + \frac{2(p_n-1)}{p_n-1-q_n}\bar{C}S_{n-1}C_{T_0}^{q_n}T_0^{\frac{p_n-1-q_n}{2(p_n-1)}} \leqslant S_{n-1}^{\frac{1}{p_{n-1}}}.$$

Then $\frac{\partial \bar{u}_{n-1}}{\partial \eta} \geqslant C_{T_0}^{q_n}(T_0 - t)^{-\frac{q_n}{2(p_n-1)}}\bar{u}_{n-1}^{p_{n-1}}$, $(x,t) \in \partial B_R \times (0, T_0)$. For $p_n > 1$, $\tilde{u}_n \leqslant C_{T_0}(T_0 - t)^{-\frac{1}{2(p_n-1)}}$. So $\tilde{u}_{n-1}$ satisfies $\frac{\partial \tilde{u}_{n-1}}{\partial \eta} \leqslant C_{T_0}^{q_n}(T_0 - t)^{-\frac{q_n}{2(p_n-1)}}\tilde{u}_{n-1}^{p_n}$, $(x,t) \in \partial B_R \times (0, T_0)$. By the comparison principle, $\tilde{u}_{n-1} \leqslant \bar{u}_{n-1} \leqslant S_{n-1}^{\frac{1}{p_{n-1}}}$.

Introduce the following auxiliary problem

$$\begin{cases} (\bar{u}_{n-2})_t = \Delta \bar{u}_{n-2}, & (x,t) \in B_R \times (0,+\infty), \\ \dfrac{\partial \bar{u}_{n-2}}{\partial \eta} = S_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} S_{n-2}, & (x,t) \in \partial B_R \times (0,+\infty), \\ \bar{u}_{n-2}(x,0) = \bar{u}_{n-2,0}(x), & x \in B_R, \end{cases}$$

where $\bar{u}_{n-2,0}(x)$ satisfies the compatibility conditions and $\bar{u}_{n-2,0} = 2\tilde{u}_{n-2,0}$ on ∂B_R; $\Delta \bar{u}_{n-2,0} \geqslant 0$, $\bar{u}_{n-2,0} \geqslant \tilde{u}_{n-2,0}$ in B_R. By Green's identity, $\bar{u}_{n-2} \leqslant S_{n-2}^{\frac{1}{p_{n-2}}}$ in $B_R \times (0,T_0)$. So $\frac{\partial \bar{u}_{n-2}}{\partial \eta} \geqslant S_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} \bar{u}_{n-2}^{p_{n-2}}$, $(x,t) \in \partial B_R \times (0,T_0)$. For $\tilde{u}_{n-1} \leqslant S_{n-1}^{\frac{1}{p_{n-1}}}$, $\frac{\partial \tilde{u}_{n-2}}{\partial \eta} \leqslant S_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} \tilde{u}_{n-2}^{p_{n-2}}$, $(x,t) \in \partial B_R \times (0,T_0)$. So $\tilde{u}_{n-2} \leqslant \bar{u}_{n-2} \leqslant S_{n-2}^{\frac{1}{p_{n-2}}}$, $(x,t) \in B_R \times (0,T_0)$. The boundedness of $\tilde{u}_i$ $(i = n-3, n-4, \ldots, 1)$ can be proved similarly. So $\tilde{u}_n$ is the blow-up component.

According to the continuity on initial data for bounded solutions, there must exist a neighborhood $N(\subset \mathbb{V}_0)$ of $(u_{1,0}, u_{2,0}, \ldots, u_{n,0})$ such that every solution $(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n)$ starting from the neighborhood will enter $\mathcal{N}(u_{1,0}, u_{2,0}, \ldots, u_{n,0})$ at time $T - \varepsilon_0$, and hence keeps the property that $\hat{u}_n$ blows up while the other components remain bounded. So there must exist a neighborhood $N_1(\subset N)$ in $\mathbb{V}_1$ such that any solution coming from it blows up with $\hat{u}_n$ blowing up and the other components remaining bounded. □

Lemma 3.5. *If $q_n + 1 < p_n$, $q_{n-1} + 1 < p_{n-1}$, and u_{n-1}, u_n blow up simultaneously at time T while the others remain bounded up to T, then*

$$\big(U_{n-1}(t), U_n(t)\big) \sim \Big((T-t)^{-\frac{p_n-1-q_n}{2(p_n-1)(p_{n-1}-1)}}, (T-t)^{-\frac{1}{2(p_n-1)}}\Big).$$

Proof. Due to the boundedness of u_1 and by Green's identity, we have

$$U_n(t) \leqslant U_n(z) + CU_n^{p_n}(t)(T-z)^{\frac{1}{2}}.$$

For the blow-up property of u_n, one can take $U_n(z) = \frac{1}{2}U_n(t)$. So $U_n(z) \geqslant c(T-z)^{-\frac{1}{2(p_n-1)}}$.

Similarly to the method of Lemma 2.1, one can obtain $U_n(t) \leqslant C(T-t)^{-\frac{1}{2(p_n-1)}}$ and $U_{n-1}(t) \leqslant C(T-t)^{-\frac{p_n-1-q_n}{2(p_n-1)(p_{n-1}-1)}}$.

Combining the upper estimate of U_n with Green's identity to u_{n-1}, we have

$$U_{n-1}(t) \leqslant U_{n-1}(z) + CU_{n-1}^{p_{n-1}}(t)(T-z)^{\frac{p_n-1-q_n}{2(p_n-1)}}.$$

Take $U_{n-1}(z) = \frac{1}{2}U_{n-1}(t)$. Then $U_{n-1}(t) \geqslant c(T-t)^{-\frac{p_n-1-q_n}{2(p_n-1)(p_{n-1}-1)}}$. □

Proof of Proposition 3.1. Lemma 3.1 says that there exists $\bar{\lambda}_{n-2} \in (\frac{1}{2}, 1)$ such that any initial data in $\mathbb{V}_1$ satisfying $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-3} = \frac{1}{2}$, $\bar{\lambda}_{n-2} \in (\frac{1}{2}, 1)$ develops the non-simultaneous blow-up solution with u_j $(j = 1, 2, \ldots, n-2)$ remaining bounded. We know from Lemma 3.2 that there exists $\lambda'_{n-1} \in (0, \frac{1}{2})$ such that the solution of (1.1) with the initial data in $\mathbb{V}_1$ satisfying $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-3} = \frac{1}{2}$, $\lambda_{n-2} = \bar{\lambda}_{n-2}$ and $\lambda_{n-1} = \lambda'_{n-1}$ blows up non-simultaneously, where u_{n-1} blows up and the others remain bounded. Lemma 3.3 guarantees that there exists $\lambda''_{n-1} \in (\frac{1}{2}, 1)$ such that u_n blows up alone with the initial data in $\mathbb{V}_1$ where $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-3} = \frac{1}{2}$, $\lambda_{n-2} = \bar{\lambda}_{n-2}$ and $\lambda_{n-1} = \lambda''_{n-1}$. In addition, the sets of the initial data in $\mathbb{V}_1$ such that u_n blows up alone and that u_{n-1} blows up alone are all open by Lemma 3.4. Notice that $\mathbb{V}_1$ is connected. So there must exist initial data (suitable $\bar{\lambda}_{n-1} \in (\lambda'_{n-1}, \lambda''_{n-1})$) such that u_n and u_{n-1} blow up simultaneously while the others remain bounded.

The blow-up rates can be obtained by Lemma 3.5 directly. □

Secondly, we discuss the case for $i = n$ and $k \in \{2, 3, \ldots, n-2\}$, $n \geqslant 4$.

Proposition 3.2. *If $q_n + 1 < p_n$ and $q_{n-k} + 1 < p_{n-k}$, then there exist suitable initial data such that u_{n-k}, u_n blow up simultaneously at some time T while the others remain bounded up to T. Moreover,*

$$\big(U_{n-k}(t), U_n(t)\big) \sim \Big((T-t)^{-\frac{1}{2(p_{n-k}-1)}}, (T-t)^{-\frac{1}{2(p_n-1)}}\Big).$$

Without loss of generality, we only prove the case for $k = 2$ by the following five lemmas. Define another subset of $\mathbb{V}_0$ as follows,

$$\mathbb{V}_2=\Big\{\big(u_{1,0}(r),u_{2,0}(r),\dots,u_{n,0}(r)\big):\ u_{m,0}(r)=N_m+\frac{R}{2}\sqrt{M_m^2+4}-\frac{R}{2}M_m$$
$$-\sqrt{R^2-\left(\frac{1}{2}M_m\sqrt{M_m^2+4}-\frac{1}{2}M_m^2\right)r^2},\ r\in[0,R],$$
$$\text{with } M_m=u_{m,0}^{p_m}(R)u_{m+1,0}^{q_{m+1}}(R),\ N_m=u_{m,0}(R)\ (m=1,2,\dots,n),$$
$$\text{where } u_{1,0}(R)=\frac{R}{\lambda_1},\ u_{l,0}(R)=\frac{R}{\prod_{j=1}^{l-1}(1-\lambda_j)\lambda_l}\ (l=2,3,\dots,n-3),$$
$$u_{n-1,0}(R)=\frac{R}{\prod_{j=1}^{n-3}(1-\lambda_j)\lambda_{n-2}},\ u_{n-2,0}(R)=\frac{R}{\prod_{j=1}^{n-2}(1-\lambda_j)\lambda_{n-1}},$$
$$u_{n,0}(R)=\frac{R}{\prod_{j=1}^{n-1}(1-\lambda_j)},\ \lambda_1,\lambda_2,\dots,\lambda_{n-1}\in(0,1)\Big\}.$$

Lemma 3.6. *If $q_n+1<p_n$ and $q_{n-2}+1<p_{n-2}$, then there exists $\bar\lambda_{n-2}\in(\frac{1}{2},1)$ such that non-simultaneous blow-up happens with $u_1,u_2,\dots,u_{n-3},u_{n-1}$ remaining bounded for the initial data satisfying $u_{j,0}(R)=2^jR$ $(j=1,2,\dots,n-3)$ and $u_{n-1,0}(R)=\frac{2^{n-3}R}{\bar\lambda_{n-2}}$ in $\mathbb{V}_2$.*

Proof. Take $M_j>(2^{j+1}R)^{p_j}$ $(j=1,2,\dots,n-3)$, $M_{n-1}>(2^{n-1}R)^{p_{n-1}}$. Consider the following auxiliary problem

$$\begin{cases}(\underline{u}_{n-2})_t=\Delta\underline{u}_{n-2}, & (x,t)\in B_R\times(0,\underline{T}_{n-2}),\\ \dfrac{\partial\underline{u}_{n-2}}{\partial\eta}=\big(2^{n-3}R-R\big)^{q_{n-1}}\underline{u}_{n-2}^{p_{n-2}}, & (x,t)\in\partial B_R\times(0,\underline{T}_{n-2}),\\ \underline{u}_{n-2}(x,0)=\underline{u}_{n-2,0}(x), & x\in B_R,\end{cases}\tag{3.6}$$

where radially symmetric $\underline{u}_{n-2,0}(x)$ satisfies the compatibility conditions and $\frac{2^{n-3}R}{1-\lambda_{n-2}}-2R\leqslant\underline{u}_{n-2,0}(x)\leqslant\frac{2^{n-3}R}{1-\lambda_{n-2}}-R$ with λ_{n-2} to be determined.

For problem (3.6), there must exist $\lambda_{n-2}=\bar\lambda_{n-2}\in(\frac{1}{2},1)$ such that $\underline{T}_{n-2}$ satisfies

$$M_j\geqslant\big(2^{j+1}R+2\bar CM_{j+1}^{\frac{q_{j+1}}{p_{j+1}}}M_j\underline{T}_{n-2}^{\frac{1}{2}}\big)^{p_j}\quad(j=1,2,\dots,n-4),$$
$$M_{n-3}\geqslant\left(2^{n-2}R+\frac{2(p_{n-2}-1)}{p_{n-2}-1-q_{n-2}}\bar CM_{n-3}C_{\underline{T}_{n-2}}^{q_{n-2}}\underline{T}_{n-2}^{\frac{p_{n-2}-1-q_{n-2}}{2(p_{n-2}-1)}}\right)^{p_{n-3}},$$
$$M_{n-1}\geqslant\left(2^{n-1}R+\frac{2(p_n-1)}{p_n-1-q_n}\bar CM_{n-1}C_{\underline{T}_{n-2}}^{q_n}\underline{T}_{n-2}^{\frac{p_n-1-q_n}{2(p_n-1)}}\right)^{p_{n-1}}.$$

For any $(u_{1,0},u_{2,0},\dots,u_{n,0})\in\mathbb{V}_2$ satisfying $u_{j,0}(R)=2^jR$ $(j=1,2,\dots,n-3)$ and $u_{n-1,0}(R)=\frac{2^{n-3}R}{\bar\lambda_{n-2}}$, we have $u_{n-2,0}(R)=\frac{2^{n-3}R}{(1-\bar\lambda_{n-2})\lambda_{n-1}}\geqslant\frac{2^{n-3}R}{1-\bar\lambda_{n-2}}$ for any $\lambda_{n-1}\in(0,1)$. Then

$$\frac{2^{n-3}R}{1-\bar\lambda_{n-2}}-2R\leqslant\underline{u}_{n-2,0}(x)\leqslant\frac{2^{n-3}R}{1-\bar\lambda_{n-2}}-R\leqslant u_{n-2,0}(x)\leqslant\frac{2^{n-3}R}{(1-\bar\lambda_{n-2})\lambda_{n-1}}.$$

For $(u_{n-1})_t\geqslant0$, $u_{n-1}(x,t)\geqslant u_{n-1,0}(x)\geqslant2^{n-3}R-R$. By the comparison principle, $\underline{u}_{n-2}\leqslant u_{n-2}$ and $T\leqslant\underline{T}_{n-2}$. Hence

$$M_j\geqslant\big(2^{j+1}R+2\bar CM_{j+1}^{\frac{q_{j+1}}{p_{j+1}}}M_jT^{\frac{1}{2}}\big)^{p_j}\quad(j=1,2,\dots,n-4),$$
$$M_{n-3}\geqslant\left(2^{n-2}R+\frac{2(p_{n-2}-1)}{p_{n-2}-1-q_{n-2}}\bar CM_{n-3}C_T^{q_{n-2}}T^{\frac{p_{n-2}-1-q_{n-2}}{2(p_{n-2}-1)}}\right)^{p_{n-3}},$$
$$M_{n-1}\geqslant\left(2^{n-1}R+\frac{2(p_n-1)}{p_n-1-q_n}\bar CM_{n-1}C_T^{q_n}T^{\frac{p_n-1-q_n}{2(p_n-1)}}\right)^{p_{n-1}}.$$

Consider the second auxiliary problem

$$\begin{cases}(\bar u_{n-3})_t=\Delta\bar u_{n-3}, & (x,t)\in B_R\times(0,T),\\ \dfrac{\partial\bar u_{n-3}}{\partial\eta}=M_{n-3}C_T^{q_{n-2}}(T-t)^{-\frac{q_{n-2}}{2(p_{n-2}-1)}}, & (x,t)\in\partial B_R\times(0,T),\\ \bar u_{n-3}(x,0)=\bar u_{n-3,0}(x), & x\in B_R,\end{cases}$$

where radially symmetric $\bar{u}_{n-3,0}(x)$ satisfies $\frac{\partial \bar{u}_{n-3,0}}{\partial \eta} = M_{n-3} C_T^{q_{n-2}} T^{-\frac{q_{n-2}}{2(p_{n-2}-1)}}$, $\bar{u}_{n-3,0}(x) = 2^{n-2}R$ for $x \in \partial B_R$; $\Delta \bar{u}_{n-3,0}(x) \geqslant 0$, $\bar{u}_{n-3,0}(x) \geqslant u_{n-3,0}(x)$ for $x \in B_R$.

By Green's identity and $q_{n-2}+1 < p_{n-2}$,

$$\bar{u}_{n-3} \leqslant 2^{n-2}R + \frac{2(p_{n-2}-1)}{p_{n-2}-1-q_{n-2}} \bar{C} M_{n-3} C_T^{q_{n-2}} T^{\frac{p_{n-2}-1-q_{n-2}}{2(p_{n-2}-1)}} \leqslant M_{n-3}^{\frac{1}{p_{n-3}}}.$$

So $\bar{u}_{n-3}$ satisfies $\frac{\partial \bar{u}_{n-3}}{\partial \eta} \geqslant C_T^{q_{n-2}} (T-t)^{-\frac{q_{n-2}}{2(p_{n-2}-1)}} \bar{u}_{n-3}^{p_{n-3}}$, $(x,t) \in \partial B_R \times (0,T)$. By Lemma 2.1 and $p_{n-2} > 1$, $u_{n-2} \leqslant C_T (T-t)^{-\frac{1}{2(p_{n-2}-1)}}$, and hence $\frac{\partial u_{n-3}}{\partial \eta} \leqslant C_T^{q_{n-2}} (T-t)^{-\frac{q_{n-2}}{2(p_{n-2}-1)}} u_{n-3}^{p_{n-3}}$, $(x,t) \in \partial B_R \times (0,T)$. Then by the comparison principle, $u_{n-3} \leqslant \bar{u}_{n-3} \leqslant M_{n-3}^{\frac{1}{p_{n-3}}}$.

Similarly to the proof for u_{n-3}, we have $u_{n-1} \leqslant M_{n-1}^{\frac{1}{p_{n-1}}}$.

In order to obtain the boundedness of u_{n-4}, we introduce the third auxiliary problem

$$\begin{cases} (\bar{u}_{n-4})_t = \Delta \bar{u}_{n-4}, & (x,t) \in B_R \times (0,+\infty), \\ \dfrac{\partial \bar{u}_{n-4}}{\partial \eta} = M_{n-3}^{\frac{q_{n-3}}{p_{n-3}}} M_{n-4}, & (x,t) \in \partial B_R \times (0,+\infty), \\ \bar{u}_{n-4}(x,0) = \bar{u}_{n-4,0}(x), & x \in B_R, \end{cases}$$

where radially symmetric $\bar{u}_{n-4,0}(x)$ satisfies $\frac{\partial \bar{u}_{n-4,0}}{\partial \eta} = M_{n-3}^{\frac{q_{n-3}}{p_{n-3}}} M_{n-4}$, $\bar{u}_{n-4,0}(x) = 2^{n-3}R$ for $x \in \partial B_R$; $\Delta \bar{u}_{n-4,0}(x) \geqslant 0$, $\bar{u}_{n-4,0}(x) \geqslant u_{n-4,0}(x)$ for $x \in B_R$. By Green's identity, we have

$$\bar{u}_{n-4} \leqslant 2^{n-3}R + 2\bar{C} M_{n-3}^{\frac{q_{n-3}}{p_{n-3}}} M_{n-4} T^{\frac{1}{2}} \leqslant M_{n-4}^{\frac{1}{p_{n-4}}}.$$

So $\bar{u}_{n-4}$ satisfies $\frac{\partial \bar{u}_{n-4}}{\partial \eta} \geqslant M_{n-3}^{\frac{q_{n-3}}{p_{n-3}}} \bar{u}_{n-4}^{p_{n-4}}$ for $(x,t) \in \partial B_R \times (0,T)$. For $u_{n-3} \leqslant M_{n-3}^{\frac{1}{p_{n-3}}}$, u_{n-4} satisfies $\frac{\partial u_{n-4}}{\partial \eta} \leqslant M_{n-3}^{\frac{q_{n-3}}{p_{n-3}}} u_{n-4}^{p_{n-4}}$. By the comparison principle, $u_{n-4} \leqslant \bar{u}_{n-4} \leqslant M_{n-4}^{\frac{1}{p_{n-4}}}$. We can obtain $u_j \leqslant M_j^{\frac{1}{p_j}}$ $(j = n-5, n-6, \ldots, 1)$, similarly. □

Lemma 3.7. *If $q_n + 1 < p_n$ and $q_{n-2}+1 < p_{n-2}$, then, for the fixed $\bar{\lambda}_{n-2} \in (\frac{1}{2}, 1)$ in Lemma 3.6, there exists $\lambda'_{n-1} \in (0, \frac{1}{2})$ such that u_{n-2} blows up while the other components remain bounded for the initial data satisfying $u_{j,0}(R) = 2^j R$ $(j = 1, 2, \ldots, n-3)$, $u_{n-1,0}(R) = \frac{2^{n-3}R}{\bar{\lambda}_{n-2}}$, $u_{n-2,0}(R) = \frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})\lambda'_{n-1}}$, and $u_{n,0}(R) = \frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})(1-\lambda'_{n-1})}$ in $\mathbb{V}_2$.*

Proof. Take $M_n > (\frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}})^{p_n}$. Consider problem (3.6) with initial data $\underline{u}_{n-2,0}$ satisfying the compatibility conditions and

$$\frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})\lambda_{n-1}} - 2R < \underline{u}_{n-2,0}(x) \leqslant \frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})\lambda_{n-1}} - R,$$

where λ_{n-1} is to be determined. There exists some $\lambda'_{n-1} \in (0, \frac{1}{2})$ such that, if $\lambda_{n-1} = \lambda'_{n-1}$, $\underline{T}_{n-2}$ satisfies

$$M_n \geqslant \left(\frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}} + 2\bar{C} M_1^{\frac{q_1}{p_1}} M_n \underline{T}_{n-2}^{\frac{1}{2}} \right)^{p_n}.$$

Similarly to Lemma 3.6, $\underline{u}_{n-2} \leqslant u_{n-2}$ and $T \leqslant \underline{T}_{n-2}$. Hence

$$M_n \geqslant \left(\frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}} + 2\bar{C} M_1^{\frac{q_1}{p_1}} M_n T^{\frac{1}{2}} \right)^{p_n}.$$

Considering (3.4) in $[0,T)$, we have $\bar{u}_n \leqslant \frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}} + 2\bar{C} M_1^{\frac{q_1}{p_1}} M_n T^{\frac{1}{2}} \leqslant M_n^{\frac{1}{p_n}}$. Then $\bar{u}_n$ satisfies $\frac{\partial \bar{u}_n}{\partial \eta} \geqslant M_1^{\frac{q_1}{p_1}} \bar{u}_n^{p_n}$, $(x,t) \in \partial B_R \times (0,T)$. Due to $u_1 \leqslant M_1^{\frac{1}{p_1}}$, u_n satisfies $\frac{\partial u_n}{\partial \eta} \leqslant M_1^{\frac{q_1}{p_1}} u_n^{p_n}$, $(x,t) \in \partial B_R \times (0,T)$. By the comparison principle, $u_n \leqslant \bar{u}_n \leqslant M_n^{\frac{1}{p_n}}$. So only u_{n-2} blows up. □

Lemma 3.8. *If $q_n + 1 < p_n$ and $q_{n-2}+1 < p_{n-2}$, then, for the fixed $\bar{\lambda}_{n-2} \in (\frac{1}{2}, 1)$ in Lemma 3.1, there exists $\lambda''_{n-1} \in (\frac{1}{2}, 1)$ such that u_n blows up while the other components remain bounded for the initial data satisfying $u_{j,0}(R) = 2^j R$ $(j = 1, 2, \ldots, n-3)$, $u_{n-1,0}(R) = \frac{2^{n-3}R}{\bar{\lambda}_{n-2}}$, $u_{n-2,0}(R) = \frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})\lambda''_{n-1}}$, and $u_{n,0}(R) = \frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})(1-\lambda''_{n-1})}$ in $\mathbb{V}_2$.*

Proof. Introduce the following auxiliary problem

$$\begin{cases} (\underline{u}_n)_t = \Delta \underline{u}_n, & (x,t) \in B_R \times (0, \underline{T}_n), \\ \dfrac{\partial \underline{u}_n}{\partial \eta} = R^{q_1} \underline{u}_n^{p_n}, & (x,t) \in \partial B_R \times (0, \underline{T}_n), \\ \underline{u}_n(x,0) = \underline{u}_{n,0}(x), & x \in B_R, \end{cases}$$

where radially symmetric $\underline{u}_{n,0}(x)$ satisfies the compatibility conditions and

$$\frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})(1-\lambda_{n-1})} - 2R \leqslant \underline{u}_{n,0} \leqslant \frac{2^{n-3}R}{(1-\bar{\lambda}_{n-2})(1-\lambda_{n-1})} - R$$

with λ_{n-1} to be determined.

Choose $M_{n-2} > (\frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}})^{p_{n-2}}$. There exists $\lambda''_{n-1} \in (\frac{1}{2}, 1)$ such that, if $\lambda_{n-1} = \lambda''_{n-1}$, $\underline{T}_n$ satisfies

$$M_{n-2} \geqslant \left(\frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}} + 2\bar{C} M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} M_{n-2} \underline{T}_n^{\frac{1}{2}} \right)^{p_{n-2}}.$$

Take the initial data in $\mathbb{V}_2$ such that $\lambda_j = \frac{1}{2}$ $(j = 1, 2, \ldots, n-3)$, $\lambda_{n-2} = \bar{\lambda}_{n-2}$, $\lambda_{n-1} = \lambda''_{n-1}$. For $\underline{u}_{n,0}(x) \leqslant u_{n,0}(x)$ and $u_1(x,t) \geqslant u_{1,0}(x) \geqslant R$, we have $\underline{u}_n \leqslant u_n$ and $T \leqslant \underline{T}_n$. So

$$M_{n-2} \geqslant \left(\frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}} + 2\bar{C} M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} M_{n-2} T^{\frac{1}{2}} \right)^{p_{n-2}}.$$

Consider the following auxiliary problem

$$\begin{cases} (\bar{u}_{n-2})_t = \Delta \bar{u}_{n-2}, & (x,t) \in B_R \times (0, +\infty), \\ \dfrac{\partial \bar{u}_{n-2}}{\partial \eta} = M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} M_{n-2}, & (x,t) \in \partial B_R \times (0, +\infty), \\ \bar{u}_{n-2}(x,0) = \bar{u}_{n-2,0}(x), & x \in B_R, \end{cases}$$

where radially symmetric $\bar{u}_{n-2,0}(x)$ satisfies the compatibility conditions and $\bar{u}_{n-2,0}(R) = \frac{2^{n-1}R}{1-\bar{\lambda}_{n-2}}$; $\Delta \bar{u}_{n-2,0}(x) \geqslant 0$, $\bar{u}_{n-2,0}(x) \geqslant u_{n-2,0}(x)$ for $x \in B_R$. By Green's identity, $\bar{u}_{n-2} \leqslant M_{n-2}^{\frac{1}{p_{n-2}}}$. So $\bar{u}_{n-2}$ satisfies $\frac{\partial \bar{u}_{n-2}}{\partial \eta} \geqslant M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} \bar{u}_{n-2}^{p_{n-2}}$, $(x,t) \in \partial B_R \times (0,T)$. For $u_{n-1} \leqslant M_{n-1}^{\frac{1}{p_{n-1}}}$, u_{n-2} satisfies $\frac{\partial u_{n-2}}{\partial \eta} \leqslant M_{n-1}^{\frac{q_{n-1}}{p_{n-1}}} u_{n-2}^{p_{n-2}}$, $(x,t) \in \partial B_R \times (0,T)$. By the comparison principle, $u_{n-2} \leqslant \bar{u}_{n-2} \leqslant M_{n-2}^{\frac{1}{p_{n-2}}}$. Then only u_n blows up. □

Similarly to the proof of Lemma 3.4, we have

Lemma 3.9.

(i) *The set of initial data in $\mathbb{V}_2$ such that u_n blows up while the others remain bounded is open in L^∞-topology.*
(ii) *The set of initial data in $\mathbb{V}_2$ such that u_{n-2} blows up while the others remain bounded is open in L^∞-topology.*

Lemma 3.10. *If $q_n + 1 < p_n$, $q_{n-2} + 1 < p_{n-2}$, and u_{n-2}, u_n blow up simultaneously while the others remain bounded up to time T, then*

$$\big(U_{n-2}(t), U_n(t)\big) \sim \big((T-t)^{-\frac{1}{2(p_{n-2}-1)}}, (T-t)^{-\frac{1}{2(p_n-1)}}\big).$$

Proof. The proof is similar to the scale case [7]. We omit the detail here. □

By now, we get Proposition 3.2.

Finally, we consider the case for $i = n$ and $k = n-1$. Similarly to Proposition 3.1, we give the following proposition without proof.

Proposition 3.3. *If $q_n + 1 < p_n$ and $q_1 + 1 < p_1$, then there exist suitable initial data such that u_1, u_n blow up simultaneously at some time T while the others remain bounded up to T. Moreover,*

$$\big(U_1(t), U_n(t)\big) \sim \big((T-t)^{-\frac{1}{2(p_1-1)}}, (T-t)^{-\frac{p_1-1-q_1}{2(p_1-1)(p_n-1)}}\big).$$

At the end of this section, we give the result on $n = 2$.

Theorem 3.2. *Assume $n = 2$. If $q_1 + 1 < p_1$ and $q_2 + 1 < p_2$, then there exist suitable initial data such that u_1, u_2 blow up simultaneously at some time T. Moreover, for $N = 1$,*

$$\big(U_1(t), U_2(t)\big) \sim \Big((T-t)^{-\frac{1+q_2-p_2}{2[q_2q_1-(1-p_1)(1-p_2)]}}, (T-t)^{-\frac{1+q_1-p_1}{2[q_2q_1-(1-p_1)(1-p_2)]}}\Big).$$

Proof. Simultaneous blow-up of (u_1, u_2) can be proved similarly to the proof of Proposition 3.1. The blow-up rate estimates can be followed by Theorem 2.1 [24]. $\square$

Remark 3.1. By Theorems 3.1 and 3.2, one can check that all of the cases for the existence of the initial data such that only two components blow up simultaneously with the other ones remaining bounded are discussed (i.e., the discussion on the classification of n, i, and k is complete). Furthermore, $q_i + 1 < p_i$ and $q_{i-k} + 1 < p_{i-k}$ is the coexistent region. In fact, there exist initial data such that u_i (or u_{i-k}) blows up alone (by Theorem 2.2), and there also exist initial data such that u_{i-k} and u_i blow up simultaneously with the others remaining bounded by Theorem 3.1 ($n \geqslant 3$) and Theorem 3.2 ($n = 2$). All of the blow-up rates for (u_{i-k}, u_i) are obtained. It is interesting that the representations of blow-up rates are quite different with respect to different values of n, i, and k.

4. Non-simultaneous and simultaneous blow-up for every initial data

In this section, we will discuss the exponent regions where k ($\in \{1, 2, \dots, n\}$) components blow up while the other $(n-k)$ ones remain bounded for every initial data.

Theorem 4.1. *Fix $i \in \{1, 2, \dots, n\}$ and define $\beta_i = \frac{1}{2(p_i-1)}$. Assume $p_m \leqslant 1 < p_i$ ($m = 1, 2, \dots, i-1, i+1, \dots, n$).*

(i) *If $k \in \{0, 1, \dots, n-2\}$, $\beta_j := \frac{\frac{1}{2} - q_{j+1}\beta_{j+1}}{p_j - 1} > 0$, $p_j < 1$ ($j = i-1, i-2, \dots, i-k$), $q_{i-k}\beta_{i-k} < \frac{1}{2}$, then $u_{i-k}, u_{i-k+1}, \dots, u_i$ blow up simultaneously while the other $(n-k-1)$ components remain bounded for every initial data in $\mathbb{V}_0$. Moreover,*

$$\big(U_{i-k}(t), U_{i-k+1}(t), \dots, U_i(t)\big) \sim \big((T-t)^{-\beta_{i-k}}, (T-t)^{-\beta_{i-k+1}}, \dots, (T-t)^{-\beta_i}\big). \tag{4.1}$$

(ii) *If $k = n-1$, $\beta_j := \frac{\frac{1}{2} - q_{j+1}\beta_{j+1}}{p_j - 1}$, $p_j < 1$ ($j = i-1, i-2, \dots, i+1-n$), $\beta_j > 0$ ($j = i-1, i-2, \dots, i+2-n$), $\beta_{i+1-n} \geqslant 0$, then $u_1, u_2, \dots, u_n$ blow up simultaneously for every initial data in $\mathbb{V}_0$.*

Remark 4.1. For $n \geqslant 2$ and $N \geqslant 1$, Theorems 4.1(i) shows the exponent regions where non-simultaneous blow-up occurs with only k ($\in \{1, 2, \dots, n-1\}$) components blowing up simultaneously for every initial data, which consists with Theorem 2.4(I) of [25] ($n = 2$ and $N = 1$) for $l_{11} = l_{12} = l_{21} = l_{22} = 0$ and Theorem 1.6 of [2] ($n = 2$ and $N = 1$) for semilinear system; Case (ii) gives the result on all of the components blowing up simultaneously for every initial data, which is compatible with Theorem 2.1 for $l_{11} = l_{12} = l_{21} = l_{22} = 0$ in [25] ($n = 2$ and $N = 1$) and Theorem 1.1 for semilinear system in [2] ($n = 2$ and $N = 1$).

Without loss of generality, we prove the case $i = n$ by three lemmas. So $\beta_n = \frac{1}{2(p_n-1)}$. The first lemma deals with the case (i) for $k = 0$.

Lemma 4.1. *If $p_m \leqslant 1 < p_n$ ($m = 1, 2, \dots, n-1$) and $q_n\beta_n < \frac{1}{2}$, then only u_n blows up while the others remain bounded for every initial data in $\mathbb{V}_0$. Moreover, $U_n(t) \sim (T-t)^{-\beta_n}$.*

Proof. This proof consists of three steps.

Step 1. *u_n must be the blow-up component.* Otherwise, $u_1, u_2, \dots, u_{n-1}$ would remain bounded also for $p_m \leqslant 1$ ($m = 1, 2, \dots, n-1$). It is a contradiction.

Step 2. *$u_1, u_2, \dots, u_{n-1}$ remain bounded and $u_n \leqslant C(T-t)^{-\beta_n}$.* For $p_n > 1$, we have $u_n \leqslant C(T-t)^{-\beta_n}$ by Lemma 2.1. By Green's identity, for $0 < z < t < T$,

$$U_{n-1}(t) \leqslant U_{n-1}(z) + CU_{n-1}^{p_{n-1}}(t)(T-z)^{\frac{1}{2}-q_n\beta_n}.$$

We claim that u_{n-1} remains bounded up to blow-up time T. Otherwise, there would exist $z_j \to T$ such that $C(T-z_j)^{\frac{1}{2}-q_n\beta_n} < \frac{1}{4}$, $U_{n-1}(z_j) > 1$, $U_{n-1}(z_j) \to +\infty$ as $j \to +\infty$. Take t_j such that $U_{n-1}(z_j) = \frac{1}{2}U_{n-1}(t_j)$. We obtain a contradiction: $\frac{1}{2}U_{n-1}(t_j) < \frac{1}{4}U_{n-1}(t_j)$. Then u_m ($m = n-2, n-3, \dots, 1$) remains bounded for $p_m \leqslant 1$, recursively.

Step 3. $U_n(t) \geqslant c(T-t)^{-\beta_n}$. As u_1 remains bounded up to time T, it can be understood that the blow-up rate of u_n is equivalent to that of the scalar case [7]. □

Next, we prove case (i) for $k=1$. The other subcases $k \in \{2,3,\dots,n-2\}$ can be obtained similarly.

Lemma 4.2. *If $p_m \leqslant 1 < p_n$ $(m=1,2,\dots,n-2)$, $p_{n-1} < 1$, $\beta_{n-1} := \frac{\frac{1}{2}-q_n\beta_n}{p_{n-1}-1} > 0$, and $\frac{1}{2} - q_{n-1}\beta_{n-1} > 0$, then u_{n-1} and u_n blow up simultaneously while the other $(n-2)$ components remain bounded for every initial data in $\mathbb{V}_0$. Moreover,*

$$\big(U_{n-1}(t), U_n(t)\big) \sim \big((T-t)^{-\beta_{n-1}}, (T-t)^{-\beta_n}\big).$$

Proof. This proof is divided into four steps.

Step 1. *Both u_{n-1} and u_n are the blow-up components.* We claim that u_n is the blow-up component. If not, the other components would remain bounded for $p_m \leqslant 1$ $(m=1,2,\dots,n-2)$ and $p_{n-1} < 1$, a contradiction. We say that u_{n-1} is also the blow-up component. Otherwise, $u_1, u_2, \dots, u_{n-2}$ would remain bounded. Let $u_1 \leqslant C$. Then u_n satisfies $\frac{\partial u_n}{\partial \eta} \leqslant C^{q_1} u_n^{p_n}$ for $(x,t) \in \partial B_R \times (0,T)$. By Green's identity and (1.1), $U_n(t) \leqslant U_n(z) + C(T-z)^{\frac{1}{2}} U_n^{p_n}(t)$. Since u_n blows up, one can take z such that $2U_n(z) = U_n(t)$ for t near T. Then $U_n(z) \geqslant c(T-z)^{-\beta_n}$. So we have

$$\frac{1}{2} U_{n-1}(t) \geqslant c \int_0^t (T-\tau)^{-q_n\beta_n} (t-\tau)^{-\frac{1}{2}}\, d\tau.$$

The boundedness of U_{n-1} requires $\frac{1}{2} > q_n\beta_n$, and hence $\beta_{n-1} < 0$, which contradicts $\beta_{n-1} > 0$.

Step 2. *The upper estimates of u_{n-1} and u_n.* We know from Lemma 2.1 that $u_n \leqslant C(T-t)^{-\frac{1}{2(p_n-1)}}$. Combining Green's identity with the upper estimate for u_n, we have

$$U_{n-1}(t) \leqslant U_{n-1}(z) + C(T-z)^{\frac{1}{2}-q_n\beta_n} U_{n-1}^{p_{n-1}}(t).$$

Take z such that $U_{n-1}(z) = \frac{1}{4} U_{n-1}(t)$. Then $U_{n-1}(t) \leqslant C(T-t)^{-\beta_{n-1}}$.

Step 3. *The boundedness of $u_1, \dots, u_{n-2}$.* This part is similar to Step 2 of Lemma 4.1.

Step 4. *The lower estimates for u_{n-1}, u_n.* Assume $u_1 \leqslant C$. Then by Green's identity and (1.1), $U_n(t) \geqslant c(T-t)^{-\beta_n}$. Combining the lower estimate of U_n with (1.1), we have

$$U_{n-1}(t) \geqslant c \int_z^t U_{n-1}^{p_{n-1}}(\tau)(T-\tau)^{-\beta_n q_n - \frac{1}{2}}\, d\tau.$$

Define $J(t) = \int_z^t U_{n-1}^{p_{n-1}}(\tau)(T-\tau)^{-\beta_n q_n - \frac{1}{2}}\, d\tau$, then $J'(t) \geqslant c J^{p_{n-1}}(t)(T-t)^{-\beta_n q_n - \frac{1}{2}}$, so

$$J^{-p_{n-1}}(t) J'(t) \geqslant c(T-t)^{-\beta_n q_n - \frac{1}{2}}.$$

Integrating the above inequality from z to t and taking $z = 2t - T$, we have $U_{n-1}^{1-p_{n-1}}(t) \geqslant c J^{1-p_{n-1}}(t) \geqslant c(T-t)^{\frac{1}{2}-\beta_n q_n}$, and hence $U_{n-1}(t) \geqslant c(T-t)^{-\beta_{n-1}}$. □

Then we prove case (ii).

Lemma 4.3. *If $\beta_j := \frac{\frac{1}{2}-q_{j+1}\beta_{j+1}}{p_j-1}$, $p_j < 1 < p_n$ $(j=1,2,\dots,n-1)$, and $\beta_1 \geqslant 0$, $\beta_j > 0$ $(j=2,3,\dots,n-1)$, then $u_1, u_2, \dots, u_n$ blow up simultaneously for every initial data in $\mathbb{V}_0$.*

Proof. Due to $p_n > 1$, the solution of (1.1) must blow up for every initial data. We claim that u_n is the blow-up component. Otherwise, $u_{n-1}, u_{n-2}, \dots, u_1$ would remain bounded. Next, we prove that u_{n-1} also blow up. If not, $u_{n-2}, u_{n-3}, \dots, u_1$ would be bounded up to blow-up time T. Let $u_1 \leqslant C$. It is easy to get from Green's identity that $U_n(t) \geqslant c(T-t)^{-\beta_n}$. Combining the lower estimate of U_n with Green's identity, we have $U_{n-1}(t) \geqslant c\int_0^t (T-\tau)^{-\frac{1}{2}-q_n\beta_n}\, d\tau$. The boundedness of U_{n-1} requires that $\frac{1}{2} > q_n\beta_n$, so $\beta_{n-1} < 0$, a contradiction. Then u_{n-1} must blow up. By Step 4 of Lemma 4.2, we have $U_{n-1}(t) \geqslant c(T-t)^{-\beta_{n-1}}$. By the similar method, we obtain that u_m must be the blow-up component and $U_m(t) \geqslant c(T-t)^{-\beta_m}$ $(m = n-2, n-3, \dots, 2)$. For $\beta_1 \geqslant 0$, u_1 also blows up at time T, similarly. That means $u_1, u_2, \dots, u_n$ must blow up simultaneously. □

So Theorem 4.1 is proved.

Remark 4.2. As for $\beta_{i-k}=0$ $(k\geqslant 1)$ in Theorem 4.1(i), we can also obtain that u_{i-k}, $u_{i-k+1},\dots,u_i$ blow up simultaneously while the other $(n-k-1)$ components remain bounded for every initial data in $\mathbb{V}_0$, but fail to obtain (4.1) here. In fact, without loss of generality, we only prove $i=n$ and $k=1$. We can easily obtain that both u_{n-1} and u_n are the blow-up components by changing $\beta_{n-1}>0$ to $\beta_{n-1}=0$ in the last line of Step 1 in Lemma 4.2. And then by Green's identity, one can obtain $U_{n-1}(t)\leqslant C(\ln\frac{1}{T-t})^{\frac{1}{1-p_n}}$. Then

$$U_{n-2}(t)\leqslant U_{n-2}(z)+C^*U_{n-2}^{p_{n-2}}(t)\int_z^t\left(\ln\frac{1}{T-\tau}\right)^{\frac{q_{n-1}}{1-p_n}}(t-\tau)^{-\frac{1}{2}}\,d\tau. \tag{4.2}$$

We claim u_{n-2} remains bounded. Otherwise, there would exist z_j such that $z_j\to T$, $1<U_{n-2}(z_j)\to+\infty$ as $j\to+\infty$, $(\ln\frac{1}{T-\tau})^{\frac{q_{n-1}}{1-p_n}}\leqslant(T-\tau)^{-\frac{1}{4}}$ for $\tau\in(z_j,T)$, $4C^*(T-z_j)^{\frac{1}{4}}<\frac{1}{4}$. Take t_j such that $U_{n-2}(t_j)=2U_{n-2}(z_j)$. Then (4.2) turns into $\frac{1}{2}U_{n-2}(t_j)\leqslant\frac{1}{4}U_{n-2}(t_j)$, a contradiction. So u_{n-2} remains bounded. Due to $p_m\leqslant 1$ $(m=n-3,n-4,\dots,1)$, we can obtain the boundedness of $u_{n-3},u_{n-4},\dots,u_1$ recursively.

Theorem 4.1(i) gives the results on $k+1$ $(k\in\{0,1,\dots,n-2\})$ components blowing up while the other $(n-k-1)$ ones remaining bounded for every initial data. In the following theorem, if we restrict $k\in\{0,1,\dots,n-3\}$, then p_{i+1} can be extended from $p_{i+1}\leqslant 1$ to $1<p_{i+1}\leqslant q_{i+1}+1$.

Theorem 4.2. *Fix $i\in\{1,2,\dots,n\}$ and define $\beta_i=\frac{1}{2(p_i-1)}$. Assume $p_m\leqslant 1<p_i$ $(m=1,2,\dots,i-1,i+2,\dots,n)$ and $1<p_{i+1}\leqslant q_{i+1}+1$. If $k\in\{0,1,\dots,n-3\}$, $\beta_j:=\frac{\frac{1}{2}-q_{j+1}\beta_{j+1}}{p_j-1}>0$, $p_j<1$ $(j=i-1,i-2,\dots,i-k)$, $q_{i-k}\beta_{i-k}<\frac{1}{2}$, then $u_{i-k},u_{i-k+1},\dots,u_i$ blow up simultaneously while the other $(n-k-1)$ ones remain bounded for every initial data in $\mathbb{V}_0$. Moreover,*

$$\big(U_{i-k}(t),U_{i-k+1}(t),\dots,U_i(t)\big)\sim\big((T-t)^{-\beta_{i-k}},(T-t)^{-\beta_{i-k+1}},\dots,(T-t)^{-\beta_i}\big).$$

We use two lemmas to prove it. Without loss of generality, we only give the proof for $i=n-1$. So $\beta_{n-1}=\frac{1}{2(p_{n-1}-1)}$. First, we deal with the subcase $k=0$.

Lemma 4.4. *Assume $p_m\leqslant 1<p_{n-1}$ $(m=1,2,\dots,n-2)$. If $q_{n-1}\beta_{n-1}<\frac{1}{2}$ and $1<p_n\leqslant q_n+1$, then only u_{n-1} blows up for every initial data in $\mathbb{V}_0$. Moreover, $U_{n-1}(t)\sim(T-t)^{-\beta_{n-1}}$.*

Proof. Firstly, we will show that non-simultaneous blow-up happens with $u_1,u_2,\dots,u_{n-2}$ remaining bounded up to blow-up time T for every initial data in $\mathbb{V}_0$. One can prove that $U_{n-1}(t)\leqslant C(T-t)^{-\beta_{n-1}}$. Also by Green's identity,

$$U_{n-2}(t)\leqslant U_{n-2}(z)+C^*U_{n-2}^{p_{n-2}}(t)(T-z)^{\frac{1}{2}-q_{n-1}\beta_{n-1}}.$$

We claim that u_{n-2} is bounded up to time T. If not, there would exist $z_j\to T$ such that $U_{n-2}(z_j)>1$, $U_{n-2}(z_j)\to+\infty$ as $j\to+\infty$, and $C^*(T-z_j)^{\frac{1}{2}-q_{n-1}\beta_{n-1}}<\frac{1}{4}$. Take t_j such that $U_{n-2}(t_j)=2U_{n-2}(z_j)$. So $\frac{1}{2}U_{n-2}(t_j)<\frac{1}{4}U_{n-2}(t_j)$, a contradiction. Considering $p_m\leqslant 1$ $(m=1,2,\dots,n-3)$, one can obtain the boundedness of $u_{n-3},u_{n-4},\dots,u_1$, recursively.

Secondly, we will prove that u_n also remains bounded up to time T. Assume that u_n blows up at T. By the boundedness of u_1, we obtain $U_n(t)\geqslant C_1(T-t)^{-\frac{1}{2(p_n-1)}}$. So u_{n-1} satisfies that $\frac{\partial u_{n-1}}{\partial\eta}\geqslant C_1^{q_n}(T-t)^{-\frac{q_n}{2(p_n-1)}}u_{n-1}^{p_{n-1}}$, $(x,t)\in\partial B_R\times(0,T)$.

Consider the auxiliary problem

$$\begin{cases}(\underline{u}_{n-1})_t=\Delta\underline{u}_{n-1}, & (x,t)\in B_R\times(0,\underline{T}_{n-1}),\\ \dfrac{\partial\underline{u}_{n-1}}{\partial\eta}=C_1^{q_n}(T-t)^{-\frac{q_n}{2(p_n-1)}}\underline{u}_{n-1}^{p_{n-1}}, & (x,t)\in\partial B_R\times(0,\underline{T}_{n-1}),\\ \underline{u}_{n-1}(x,0)=\underline{u}_{n-1,0}(x), & x\in B_R,\end{cases} \tag{4.3}$$

where $\underline{u}_{n-1,0}(x)$ satisfies the compatibility conditions and $\Delta\underline{u}_{n-1,0}(x)\geqslant 0$, $\underline{u}_{n-1,0}(x)\leqslant u_{n-1,0}(x)$ for $x\in B_R$. Then we have $u_{n-1}\geqslant\underline{u}_{n-1}$ and $T\leqslant\underline{T}_{n-1}$ by the comparison principle. But problem (4.3) means $\underline{T}_{n-1}\leqslant T$. Hence $\underline{T}_{n-1}=T$. By Green's identity,

$$\underline{U}_{n-1}(t)\geqslant c\int_0^t\frac{\underline{U}_{n-1}^{p_{n-1}}(\tau)}{(T-\tau)^{\frac{1}{2}+\frac{q_n}{2(p_n-1)}}}\,d\tau=cW(t). \tag{4.4}$$

It is easy to see that $W(t)$ blows up at time T. By (4.4),

$$W^{-p_{n-1}}(t)W'(t)\geqslant c(T-t)^{-(\frac{1}{2}+\frac{q_n}{2(p_n-1)})}.$$

Integrating the above inequality from $\frac{T}{2}$ to t, we obtain that

$$\frac{1}{p_{n-1}-1}\left(W^{1-p_{n-1}}\left(\frac{T}{2}\right)-W^{1-p_{n-1}}(t)\right)\geqslant c\int_{\frac{T}{2}}^{t}(T-\tau)^{-(\frac{1}{2}+\frac{q_n}{2(p_n-1)})}\,d\tau=I(t). \tag{4.5}$$

For $p_n\leqslant q_n+1$, $I(t)\to+\infty$ as $t\to T$. It is a contradiction to the boundedness of the left part of (4.5). So u_n still remains bounded up to time T. Then only u_{n-1} blows up. The blow-up rate estimates can also be followed by the scale case (see [7]). □

Second, we consider subcase $k=1$. The other subcases of k can be proved similarly.

Lemma 4.5. *If $p_m\leqslant 1<p_{n-1}$ $(m=1,2,\dots,n-3)$, $p_{n-2}<1$, $1<p_n\leqslant q_n+1$, $\beta_{n-2}:=\frac{\frac{1}{2}-q_{n-1}\beta_{n-1}}{p_{n-2}-1}>0$, $q_{n-2}\beta_{n-2}<\frac{1}{2}$, then u_{n-2} and u_{n-1} blow up simultaneously while the other $(n-2)$ components remain bounded for every initial data in $\mathbb{V}_0$. Moreover,*

$$\big(U_{n-2}(t),U_{n-1}(t)\big)\sim\big((T-t)^{-\beta_{n-2}},(T-t)^{-\beta_{n-1}}\big).$$

Proof. We claim that whether u_{n-2} is bounded or not, u_1 always remains bounded. If u_{n-2} is bounded up to time T, then $u_{n-3},u_{n-4},\dots,u_1$ are bounded also. Assume that u_{n-2} blows up at time T. Since $p_{n-1}>1$, we have $U_{n-1}(t)\leqslant C(T-t)^{-\beta_{n-1}}$. Combining Green's identity with the upper estimate of u_{n-1}, we have

$$U_{n-2}(t)\leqslant U_{n-2}(z)+CU_{n-2}^{p_{n-2}}(t)(T-t)^{\frac{1}{2}-q_{n-1}\beta_{n-1}},\quad 0<z<t<T.$$

Take z such that $U_{n-2}(z)=\frac{1}{4}U_{n-2}(t)$. Then $U_{n-2}(t)\leqslant C(T-t)^{-\beta_{n-2}}$. By Green's identity,

$$U_{n-3}(t)\leqslant U_{n-3}(z)+CU_{n-3}^{p_{n-3}}(t)(T-z)^{\frac{1}{2}-q_{n-2}\beta_{n-2}}.$$

Similarly to Step 2 of Lemma 4.1, we obtain that u_{n-3} is bounded. Then $u_{n-4},u_{n-5},\dots,u_1$ are bounded for $p_{n-4},p_{n-5},\dots,p_1\leqslant 1$.

By the similar method used in Lemma 4.4, one can check that u_n also remains bounded up to time T. It is easy to see that u_{n-1} is the blow-up component. In fact, if u_{n-1} remains bounded up to time T, then u_{n-2} will be bounded also for $p_{n-2}<1$, a contradiction with at least one component blowing up. By the method of Lemma 4.2, we obtain the blow-up property of u_{n-2} and the blow-up rates of u_{n-2} and u_{n-1}. □

In the following, we show another result on n $(\geqslant 2)$ components blowing up simultaneously.

Theorem 4.3. *If $p_1,p_2,\dots,p_n\leqslant 1$ and $\prod_{j=1}^n q_j-\prod_{j=1}^n(1-p_j)>0$, then $u_1,u_2,\dots,u_n$ blow up simultaneously for every initial data.*

Proof. Without loss of generality, assume that u_n would remain bounded up to the blow-up time T. Then the others would be bounded also for $p_i\leqslant 1$ $(1\leqslant i\leqslant n-1)$. Due to $\prod_{j=1}^n q_j-\prod_{j=1}^n(1-p_j)>0$, it contradicts to Theorem 2.1. □

Similarly to Theorem 4.1 of [7] or Theorem 4.8 of [9], we have the following result.

Theorem 4.4. *If u_i blows up with $U_i(t)\leqslant C(T-t)^{-\alpha}$ for any $i\in\{1,2,\dots,n\}$, then the blow-up only can occur on the boundary.*

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