Positivity and Discrete Spectra for Differential Operators

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Received July 24, 1980

1. INTRODUCTION

We shall characterize, in terms of certain decompositions of the coefficients of the formally symmetric differential expression

$$L y = \sum_{j=0}^{n} (-1)^j (p_j y^{(j)})^{(j)},$$

the property that the minimal operator generated by $L$ on an open interval $I$ is positive definite. We shall then apply this to give several different sets of conditions sufficient for self-adjoint operators generated by $L$ on a weighted Hilbert space $L^2_w(a, b)$ to be bounded below with discrete spectrum. Here $b$ may be either a finite or infinite singularity. We assume throughout that the coefficient $p_j$ in (1.1) is a real-valued function on $I$ with with $p_j^{(j-1)}$ locally absolutely continuous and that $p_n > 0$.

The characterization of positivity appears as part (4) of Theorem 2.1. It takes the form of $n$ inequalities involving decompositions of the coefficients and is a direct generalization of Theorem 2.1(c) of Read [13].

In Section 3 the link with discreteness of the spectrum is outlined, and the characterization is applied to give a criterion for discreteness of the spectrum of some fourth-order operators with nonnegative coefficients in which further assumptions about each coefficient are made only on a sequence of disjoint intervals (Theorem 3.6). This criterion comes very close to including an example of Hinton and Lewis [8] where the spectrum is not discrete.

Our principal result on discreteness of the spectrum is Theorem 5.1. It is somewhat in the spirit of the well-known result of Molchanov [10] (see also Glazman [7, p. 90]) for two-term expressions in that the bottom coefficient $p_0$ is assumed to approach infinity on the average as $x$ approaches $b$, but differs in allowing intermediate terms and in allowing much more irregular behavior by all coefficients.

In Section 6 we use an extension of Theorem 2 of Hinton and Lewis [8] to modify Theorem 5.1 so that any coefficient $p_k$, $k = 0, 1, ..., n - 1$, may play
The role of a large term assigned to $p_0$ in Theorem 5.1. In this form the result contains that of Eastham [5] on discreteness of the spectrum, Theorem 3 of Hinton and Lewis [8], and an extension of this theorem due to Müller–Pfeiffer [11]. We also give an example to show how the theorem may be applied to an expression with several large positive coefficients, none of which is large enough for the theorem to be directly applicable.

A recent paper of Hinton and Lewis [9] contains several results on discreteness of the spectrum in which the emphasis is on expressions with a large leading term. One of these is related to the well-known second-order results of Friedrichs [6] and Berkowitz [2]. Theorem 6.1 covers many of the same expressions as the results in [9], but neither includes nor is included in them.

Finally, we establish in Section 7 a somewhat different condition for discreteness of the spectrum in which it is the product $p_0p_n$ which is assumed to be large.

We shall use the symbols $K_1$, $K_2$, ... to denote constants in what follows. They do not represent the same constant in successive appearances unless this is explicitly stated.

2. Characterization of Positivity

We begin by developing an explicit formula for the coefficients of the formally symmetric expression $M^+M$ in terms of those of $M$. In the following two lemmas we shall use the symbol $[\cdot \cdot \cdot]$ applied to an index to denote the greatest integer function. (Elsewhere in the paper they are simply brackets.) We use $D$ throughout to denote the differentiation operator $d/dx$.

**Lemma 2.1.** If $f$ is $r$ times differentiable, then

$$D^r f + (-1)^r fD^r = \sum_{s=0}^{[r/2]} c_{rs} D^{\frac{r - 2s}{2}}.$$  \hspace{1cm} (2.1)

where

$$c_{rs} = \binom{r-s}{s} + \binom{r-s-1}{r-1}.$$  \hspace{1cm} (2.2)

Here we adopt the convention $\binom{r}{-1} = 0$.

**Proof.** We need only show that the $c_{rs}$ satisfy (2.2). We have

$$D^r f + (-1)^r fD^r$$

$$= D(D^{r-2} f + (-1)^{r-2} fD^{r-2}) D + D^{r-1} f' + (-1)^{r-1} f'D^{r-1}$$
Thus \( c_{r0} = c_{r-1,0} = \cdots = c_{10} = 1 \) and, using the fact that \( \lfloor r/2 \rfloor > \lfloor (r - 1)/2 \rfloor \) for \( r \) even,

\[
c_{2j,j} = c_{2(j-1),j-1} = \cdots = c_{21} = 2
\]

for each \( j \geq 1 \). Also

\[
c_{rs} = c_{r-1,s} + c_{r-2,s-1}, \quad 1 \leq s \leq \lfloor (r - 1)/2 \rfloor.
\]

It is easily verified that only a single system \( c_{rs} \) can satisfy these equations and that those given by (2.2) do have this property.

We next use this to identify the coefficients of \( M^+ M \), where as usual \( M^+ \) is the formal adjoint of \( M \), that is, \( M = \sum_{i=0}^n a_i D^i \) implies \( M^+ = \sum_{i=0}^n (-1)^i D^i a_i \). We adopt some notation in the statement of Lemma 2.2 which will be convenient for Theorem 2.3.

**Lemma 2.2.** Let \( a_i, i = 0, 1, \ldots, n \), be real valued with \( a_i^{(n-1)} \) locally absolutely continuous. If \( M = \sum_{i=0}^n a_i D^i \) and \( M^+ M = \sum_{k=0}^n (-1)^k D^k p_k D^k \), then

\[
p_k = (-1)^{n-k} (a_k a_n)^{(n-k)} + F_k((-1)^n a_0 a_n, (-1)^{n-1} a_1 a_n, \ldots, -a_{n-1} a_n, a_n),
\]

where \( F_k \) is defined by

\[
f_k^{(n-k)} + F_k(f_0, \ldots, f_n) = \sum_{i=k}^n \sum_{j=k}^n \sum_{l=k-1}^n c_{ijk}(f_i f_j / f_n)^{(l+j-2k)}
\]

and

\[
c_{ijk} = (-1)^{i+k} c_{i-j, k-j}, \quad j \geq 0
\]

\[
= 0, \quad j < 0.
\]

**Proof.** We have

\[
M^+ M = \sum_{k=0}^n (-1)^k D^k a_k^2 D^k + \sum_{i>j}^n ((-1)^i D^i a_i a_j D^j + (-1)^j D^j a_i a_j D^i).
\]
Now

\[ (-1)^i D' a_i a_j D^j + (-1)^j D' a_i a_j D^i \]

\[ = (-1)^j D^i (D^{i-j} a_i a_j + (-1)^{i-j} a_i a_j D^{i-j}) D^j \]

\[ = (-1)^j \sum_{k=j}^{[(i+j)/2]} c_{i-j, k-j} D^k (a_i a_j)^{(i+j-2k)} D^k. \]

One of these terms will contribute to \( p_k \) if and only if \( k + 1 \leq i \leq n \) and \( 2k - i \leq j \leq k \). Thus

\[ (-1)^k p_k = \sum_{i=k}^{n} \sum_{j=2k-1}^{k} (-1)^i c_{i-j, k-j} (a_i a_j)^{(i+j-2k)}. \]

Since the functions \( f_i \) are related to the \( a_i \) by \( f_i f_j / f_n = (-1)^{i+j} a_i a_j \), the lemma is proved.

Now we are ready to characterize factorization of the form \( \ell = M^+ M \) for a formally symmetric differential expression \( \ell \) in terms of properties of the coefficients of \( \ell \).

**Theorem 2.3.** Let \( \ell = \sum_{k=0}^{n} (-1)^k D^k p_k D^k \) on an open interval \( I \), where each \( p_k \) is locally absolutely continuous and \( p_n > 0 \). Then the following are equivalent.

1. \( \int_I f^2 (\ell f) > 0 \) for all \( f \in C_0^\infty(I), f \neq 0 \).
2. There is \( M = \sum_{i=0}^{n} a_i D^i \) such that \( \ell = M^+ M \).
3. There is \( M_0 = \sum_{i=0}^{n} b_i D^i \) such that \( \ell - M_0^+ M_0 = \sum_{k=0}^{n-1} (-1)^k D^k t_k D^k \) with \( t_k \) nonnegative, \( k = 0, 1, ..., n - 1 \).
4. There are functions \( r_k \) and \( S_k, k = 0, 1, ..., n - 1 \), with \( r_k \) and \( S_k \) locally absolutely continuous on \( I \) such that

\[ p_k = r_k + S_k^{(n-k)}, \quad (2.3) \]

and

\[ r_k - F_k(S_0, ..., S_{n-1}, p_n) \geq 0, \quad k = 0, 1, ..., n - 1. \quad (2.4) \]

**Remark 2.4.** For \( n = 2 \) inequalities (2.4) take the form

\[ r_1 - S_{1/2}^2 + 2S_0 \geq 0, \]

\[ r_0 - S_{1/2}^2 - (S_0 S_{1/2})' \geq 0. \quad (2.5) \]

**Proof.** It is well known that (1) and (2) are equivalent. See, for instance, Coppel [4, pp. 78–80]. It is trivial that (2) implies (3), and clear after an
integration by parts that (3) implies (1). If \( M_0 \) satisfies (3), define
\[ S_k = (-1)^{n-k} b_k \] and
\[ r_k = p_k - S_k^{(n-k)}. \] Then (2.4) follows immediately from Lemma 2.2. Suppose, finally, that functions \( r_k \) and \( S_k \) exist satisfying (2.3) and (2.4). Set
\[ M_0 = (-1)^n \frac{1}{2} D^n + (-1)^{n-1} S_{n-1} p_n^{-1/2} D_{n-1} + \cdots + S_0 p_0^{-1/2}. \]
Then \( M_0 = \sum_{k=0}^n b_i D^i \) with \( S_i = (-1)^{i-1} b_i b_n \) so that
\[ M_0^+ M_0 = \sum_{k=0}^n (-1)^k D^k q_k D^k \]
with \( q_k = S_k(r_k) + F_k(S_0, \ldots, S_{n-1} p_n) \) for \( k = 0, 1, \ldots, n - 1 \) and \( q_n = b_n^2 = p_n \).
Thus \( L - M_0^+ M_0 \) has nonnegative coefficients and order at most \( 2n - 2 \). This completes the proof.

**Remark 2.5.** The point of (3) in the characterization above and especially its formulation in (4) in terms of the decomposition (2.3) and the inequalities (2.4) is that such decompositions can frequently be constructed explicitly for wide classes of differential expressions. It is this method that will be exploited below to obtain criteria for discreteness of the spectrum.

### 3. The Essential Spectrum

Let \( w \) be a positive, locally Lebesgue integrable function on an interval \( (a, b) \), \( -\infty < a < b \leq \infty \). We shall denote by \( L^2_w(a, b) \), or simply \( L^2_w \), the Hilbert space with inner product
\[ (f, g)_w = \int_a^b \bar{f} g w. \]
The differential expression \( L \) defined by (1.1) is regular at \( a \). It can be used to define operators \( T \) on linear subspaces \( D \) of \( L^2_w \) by \( Tf = w^{-1} L f \) for all \( f \) in \( D \). We shall be particularly concerned here with the **minimal operator** \( T_0 \). We recall that the graph of \( T_0 \) is the closure in \( L^2_w \times L^2_w \) of the graph of the operator \( T \) whose domain consists of all \( C^\infty \) functions \( f \) with compact support in the interior of \( [a, b) \). We denote the domain of \( T_0 \) by \( D(T_0) \). For more details, see Naimark [12, Sect. 17].

The **essential spectrum**, \( \sigma_e(T) \), of a differential operator \( T \) consists of those complex numbers \( \lambda \) for which the range of \( T - \lambda I \) is not closed in \( L^2_w \). The essential spectrum of the minimal operator \( T_0 \) coincides with that of each of its self-adjoint extensions. Thus the spectrum of each self-adjoint extension of \( T_0 \) is discrete if and only if the essential spectrum of \( T_0 \) is empty. Since we
are concerned here with properties dependent only on the differential expression and not on the boundary conditions used to define self-adjoint operators, we shall phrase our results in terms of the minimal operator $T_0$.

The connection between Theorem 2.3 and criteria for the absence of the essential spectrum is provided by the following result, which is essentially Theorem 28 of Glazman [7].

\textbf{Theorem 3.1.} $T_0$ is bounded below with empty essential spectrum if and only if for each $K > 0$ there is $a \leq c < b$ such that

\[ (T_0 f, f)_w \geq K(f, f)_w \quad (3.1) \]

for each $f$ in $D(T_0)$ whose support is contained in $(c, b)$.

This may be translated into the context of (4) of Theorem 2.3 as follows.

\textbf{Theorem 3.2.} $T_n$ is bounded below with empty essential spectrum if and only if for each $K > 0$ there is $c, a \leq c < b$, and a decomposition $p_k = r_k + S_k^{(n-k)}$ on $(c, b)$ for $k = 0, 1, \ldots, n-1$ such that

\[ r_k - F_k(S_0, \ldots, S_{n-1}, p_n) \geq 0, \quad k = 1, \ldots, n-1, \]

\[ w^{-1}[r_0 - F_0(S_0, \ldots, S_{n-1}, p_n)] \geq K. \]

\textbf{Proof.} By Theorem 2.3 the inequalities are equivalent to the property that

\[ \int_c^b f(L_K f) > 0 \]

for $f$ in $C^\infty_0(c, b)$, where $L_K f = Lf - Kwf$. This in turn is equivalent to (3.1).

\textbf{Corollary 3.3.} $T_0$ is bounded below with empty essential spectrum if there is a decomposition $p_k = r_k + S_k^{(n-k)}$ on a subinterval $(c, b)$ of $[a, b)$ such that

\[ r_k - F_k(S_0, \ldots, S_{n-1}, p_n) \geq 0, \quad k = 1, \ldots, n-1, \]

\[ w^{-1}[r_0 - F_0(S_0, \ldots, S_{n-1}, p_n)] \rightarrow \infty \quad \text{as} \quad x \rightarrow b. \]

\textbf{Remark 3.4.} For $n = 2$ these inequalities take the form

\[ r_1 - S_1^2/p_2 + 2S_0 \geq 0, \]

\[ w^{-1}[r_0 - S_0^2/p_2 - (S_0S_1/p_2)^2] \rightarrow \infty \quad \text{as} \quad x \rightarrow b. \quad (3.2) \]

We give two applications of inequalities (3.2) which illustrate somewhat different types of decompositions.
Example 3.5. Let \( L = D^2 x^2 D^2 - D(1 + x^3 \sin x^4) D + x^2 + x^5 \sin x^4 \) and \( w = 1 \) on \([1, \infty)\). If \( S'_0(x) = x^3 \sin x^4 \), then \( S' = O(1) \). Also, there is a function \( S_0 \) such that \( S'_0 = O(x^{-1}) \), \( S_0' = O(x^2) \), and \( S''_0 = x^3 \sin x^4 \). Thus, with \( r_1 = 1 \) and \( r_0 = x^2 \), we have that
\[
 r_1 - S_1^2/p_2 + 2S_0 \to 1, 
\]
and, since \((S_0S_1/p_2)' = O(1)\), that
\[
 r_0 - S_0^2/p_2 - (S_0S_1/p_2)' \geq x^2 - K \to \infty. 
\]
We discuss an example similar to the above as Example 4.8.

As a second illustration of the use of inequalities (3.2), we show that the minimal operator associated with
\[
 L = D^4 - Dp_1 D + p_0
\]
on \([a, b)\) with \( w = 1 \) has empty essential spectrum in a situation in which \( p_0 \) and \( p_1 \) may each vanish on a sequence of intervals.

Theorem 3.6 Let \( \{a_k\}, \{b_k\}, \{c_k\} \) be sequences in \([a, b)\) with
\( a < a_k < b_k < c_k < a_{k+1} < b \) for each \( k \), and \( a_k \to b \). Set
\[
 m_k = \min \{p_0(x) : b_k \leq x \leq c_k\}, 
 n_k = \min \{p_0(x) : c_k \leq x \leq a_{k+1}\}, 
 M_k = \min \{p_1(x) : a_k \leq x \leq b_k\}, 
 N_k = \min \{p_1(x) : b_k \leq x \leq c_k\}. 
\]
Suppose that \( p_0 \) and \( p_1 \) are nonnegative, and that each of the following sequences approaches \( \infty \) as \( k \to \infty \):

(i) \( m_k, n_k, m_k(c_k - b_k)/(b_k - a_k) \).

(ii) \( M_k/(b_k - a_k)^2, N_k/(b_k - a_k)^2, N_k(c_k - b_k)/(b_k - a_k) \).

Then \( T_0 \) is bounded below and has empty essential spectrum.

Remark 3.7. Note that no assumption beyond nonnegativity is made about \( p_1 \) on \((c_k, a_{k+1})\) or about \( p_0 \) on \((a_k, b_k)\). In particular, if the length of each interval is taken to be 2, then an example may be constructed where each of \( p_0 \) and \( p_1 \) vanishes on a sequence of intervals of length 1. Moreover, in this case the assumption is simply that \( p_1 \to \infty \) and \( p_0 \to \infty \) on the complement of these intervals so that this growth may be arbitrarily slow.

Proof. Set
\[
 K_k = (1/2) \min \{M_k^{1/2}, N_k^{1/2}, N_k(c_k - b_k), m_k(c_k - b_k)\}/(b_k - a_k). 
\]
Then $K_k \to \infty$ and $L_k = K_k(b_k - a_k)/(c_k - b_k)$ satisfies

$$L_k \leq (1/2) \min\{m_k, N_k\}.$$  

Define $S_0 = 1$ (so $r_0 = p_0$) and define $S_1$ by

$$S_1(x) = \begin{cases} 
-K_k(x - a_k), & a_k \leq x \leq b_k \\
-L_k(x - b_k), & b_k \leq x \leq c_k \\
0, & c_k \leq x \leq a_{k+1}.
\end{cases}$$

Now $r_1 = p_1 - S_1'$ so

$$r_1 - S_1^2 + 2S_0 \geq \begin{cases} 
p_1 + 2 + K_k - K_k^2(b_k - a_k)^2 & M_k = \frac{M_k}{4} \\
p_1 + 2 - L_k - K_k^2(b_k - a_k)^2 & N_k = \frac{N_k}{2} - \frac{N_k}{4} \geq 0.
\end{cases}$$

Also

$$r_0 - S_0^2 - (S_0 S_1) \geq \begin{cases} 
p_0 - 1 + K_k & K_k - 1 \\
p_0 - 1 - L_k & m_k - m_k/2 \to \infty \\
p_0 - 1 & n_k
\end{cases}$$

as $x \to b$. Thus the result follows from Corollary 3.3.

**Remark 3.8.** Hinton and Lewis [8, p. 345] have given a class of examples in which the essential spectrum is not empty, although for some members of the class the hypotheses of Theorem 3.6 are very nearly satisfied. A special case of their construction is as follows. Let $x_k = k^2$, $k = 3, 4, \ldots$, and define $p_0$ and $p_1$ to be nonnegative $C^\infty$ functions such that

$$p_1(x) = \begin{cases} 
1, & x \in [k^2, k^2 + 1] \\
x^3, & x \notin [k^2 - (1/2), k^2 + 2]
\end{cases}$$

and

$$p_0(x) = \begin{cases} 
x^{1/2}, & x \in [k^2 - 1, k^2 + 2] \\
0, & x \notin [k^2 - 2, k^2 + 3].
\end{cases}$$

Take $a_k = k^2 + 2$, $b_k = (k + 1)^2 - 1$, $c_k = (k + 1)^2 - 1/2$. Then in the notation of Theorem 3.6, $k < m_k < n_k < k + 1$, $M_k = (k^2 + 2)^2$, $N_k = [(k + 1)^2 - 2]^3$. Thus all hypotheses of Theorem 3.6 are satisfied except that

$$k/4(k - 1) < m_k(c_k - b_k)/(b_k - a_k) < (k + 1)/4(k - 1).$$
It requires only a slight modification of the example for this hypothesis to be satisfied also and hence for the essential spectrum to become empty. In particular, it would suffice to make \( p_0 \) increase slightly faster on the intervals \([k^2 - 1, k^2 + 2]\) (e.g., \( p_0(x) = x^a \) for some \( a > 1/2 \)), or to make the \( x_k \) slightly closer together (e.g., \( x_k = k^\beta \) for some \( \beta < 2 \)), or even just to make the lengths of the "overlap" intervals \([b_k, c_k] \) where both \( p_0 \) and \( p_1 \) are large approach infinity with \( k \).

4. SOME LEMMAS AND EXAMPLES

In this section we prove some lemmas which will be required for Theorem 5.1. Two of these, Lemmas 4.3 and 4.5, relate decompositions of the type required in (4) of Theorem 2.3 to average growth properties of the function being decomposed. As an application of the lemmas we show that the minimal operator generated by \((-1)^n D^{2n} + q\), where \( q \) is the function introduced in Read [13] with \( \sum_{n} q \to -\infty \), is bounded below with empty essential spectrum on \([1, \infty)\).

DEFINITION 4.1. Let \( f \) and \( g \) be positive functions on an interval \( I \) with \( x + g(x) \in I \) whenever \( x \in I \). We shall write \( f \in B(g) \) if there is a constant \( K \) such that
\[
K^{-1} < \frac{f(y)}{f(x)} < K \quad \text{whenever} \quad x \leq y \leq x + g(x).
\]

LEMMA 4.2. If \( g \) is a positive, left continuous function on \([a, \infty)\) and \( g \in B(g) \), then \( g(x) = O(x) \) as \( x \to \infty \).

Proof. Fix any \( x_1 \in [a, \infty) \) and define an increasing sequence \( \{x_n\} \) by \( x_{n+1} = x_n + g(x_n) \). Then \( x_n \to \infty \), since if \( x_n \to z \), then \( g(x_n) < z - x_n \) for all \( n \) and the left continuity of \( g \) would imply \( g(z) = 0 \). Thus, some \( x_n \) is positive. If \( n > N \), then
\[
g(x_{n+1})/x_{n+1} \leq Kg(x_n)/(x_n + g(x_n)) \leq K.
\]
If \( y \in [x_n, x_{n+1}) \) for \( n > N + 1 \), then
\[
g(y)/y \leq Kg(x_n)/x_n \leq K^2.
\]

LEMMA 4.3. Let \( f \) and \( g \) be positive functions on \([a, b)\) with \( g \) left continuous and with \( f \) and \( g \) in \( B(g) \). Then the following conditions are equivalent for a locally integrable function \( F \).

(a) \( |f g(x)|^{-1} \int_{x}^{y} F \geq -K \) whenever \( x \leq y \leq x + g(x) \),
(b) \( F = r + S' \) where \( r \geq -K_2 f \) and \( |S| \leq K_3 f g \) as \( x \to b \).
Proof. If the decomposition in (b) exists, then

\[
\left[ f g(x) \right]^{-1} \int_x^y (r + S') \geq -K_2 \frac{K(g(y)/g(x)) + \left[ f g(x) \right]^{-1} [S(y) - S(x)]}{g(x)} \geq -K_1 K_2^2 - K_1 (1 + K),
\]

where $K$ is larger of the constants associated with $f$ and $g$ as in Definition 4.1.

Conversely, if (a) holds, define a sequence $\{x_k\}$ by $x_1 = a$ and

\[
x_{k+1} = \min \left\{ x_k + g(x_k), \inf \left\{ x > x_k : \int_{x_k}^x F \geq K_1 f g(x_k) \right\} \right\}.
\]

It follows as in the previous lemma from the left continuity of $g$ that $x_k \to b$ as $k \to \infty$. Define $r$ on $[x_k, x_{k+1}]$ to be the constant function

\[
r(x) = \left( x_{k+1} - x_k \right)^{-1} \int_{x_k}^{x_{k+1}} F.
\]

Then $r(x) \geq -K_1 f g(x_k) \geq -K_1 f g(x_k) \geq -K_1 f g(x)$ since either $x_{k+1} - x_k = g(x_k)$ or $r \geq 0$.

Set $S(x) = \int_a^x (F - r)$. Then $F = r + S'$ and $S(x_k) = 0$ for each $k$. Thus for $x \in [x_k, x_{k+1})$, $S(x) = \int_{x_k}^x (F - r) = K_1 f g(x_k) + K_1 f(x_k)(x - x_k) \leq 2K_1 K f g(x)$. Similarly, $S(x) \geq -2K_1 f g(x_k) \geq -2K_1 f g(x)$. Thus $r$ and $S$ have the required properties.

Remark 4.4. Note that although the function $r$ defined using (a) in the preceding lemma is a step function, it may easily be converted into a $C^\infty$ function by altering it on sufficiently small neighborhoods of the $x_k$ in such a way that the modified $S$ differs from the one defined above by an arbitrarily small amount.

Lemma 4.5. When the conditions of Lemma 4.3 hold, the following additional properties are also equivalent.

(c) $[f g(x)]^{-1} \int_x^{x + \theta e(x)} F \to \infty$ as $x \to b$ for each $\theta > 0$.

(d) The function $r$ from the decomposition (b) satisfies $r/f \to \infty$ as $x \to b$.

Proof. Given (a) and (c), define the sequence $\{x_k\}$ and the decomposition $F = r + S'$ as in the proof of Lemma 4.3. Note that for any $\theta > 0$,

\[
\int_{x_k}^{y_k} F > K_1 f g(x_k)
\]

for all large $k$, where $y_k = x_k + \theta g(x_k)$. For such $k$, $\int_{x_k}^{y_k} F = K_1 f g(x_k)$ and
$x_{k+1} < x_k + \theta g(x_k)$, so \( r(x) > (K_1/\theta)f(x_k) \geq (K_1/\theta K)f(x) \). Since \( \theta \) can be made arbitrarily small, \( r/f \to \infty \).

For the other direction, when \( r/f > A \),

\[
[f g(x)]^{-1} \int_{x_k}^{x_{k+1}} F \geq \theta A/K - K_3(1 + K).
\]

For any fixed \( \theta \), this becomes arbitrarily large with \( A \).

**Lemma 4.6.** Let \( g \) and \( H \) be positive functions on \([a, b)\) with \( g \) left continuous and with \( g \) and \( H \) in \( B(g) \). Suppose \( f = F' \) with \( F = O(gH) \) as \( x \to b \). Then for each positive integer \( m \),

\[
f = f_m + F_m^{(m)},
\]

where \( f_m = O(H) \) and \( F_m^{(j)} = O(g^{m-j}H) \), \( j = 0, 1, \ldots, m - 1 \).

**Proof.** The conclusion holds by assumption for \( m = 1 \) with \( F_1 = F \), and \( f_1 = 0 \). Suppose that it holds for \( m \). Let \( x_1 = a \), and for \( k \geq 1 \), let \( x_{k+1} = x_k + g(x_k) \). Fix a nonnegative function \( \varphi \in C_0^\infty(0, 1) \) with \( \int_0^1 \varphi = 1 \) and define

\[
\Phi_m(x) = \sum_{k=0}^{\infty} c_k \varphi((x - x_k)/(x_{k+1} - x_k)), \quad (4.1)
\]

where

\[
c_k = (x_{k+1} - x_k)^{-1} \int_{x_k}^{x_{k+1}} F_m^*.
\]

Only one term of this series is different from zero for each \( x \). Note that

\[
\int_{x_k}^{x_{k+1}} \Phi_m = \int_{x_k}^{x_{k+1}} F_m.
\]

and that \( c_k \leq \max\{|F_m(x)|: x_k \leq x \leq x_{k+1}\} \).

Now set

\[
F_{m+1}(x) = \int_a^x (F_m - \Phi_m).
\]

Then \( F_{m+1}(x_k) = 0 \) for each \( k \), so that on \([x_k, x_{k+1})\)

\[
F_{m+1}(x) = \int_{x_k}^x (F_m - \Phi_m) \leq K_1(x - x_k) \max\{g^{m+1}H(y): x_k \leq y \leq x_{k+1}\} < K_2 g^{m+1}H(x).
\]
Here we have used the fact that, for instance, \( H(y) \leq KH(x) \leq K^2H(x) \) for any \( x \) and \( y \) in \( [x_k, x_{k+1}) \).

Moreover, \( \Phi^{(j)}(x) = \sum_{k=0}^{\infty} c_k \left| g(x_k) \right|^{-j} \phi^{(j)}((x-x_k)/(x_{k+1}-x_k)) \) so that on \( [x_k, x_{k+1}) \),

\[
F_{m+1}^{(j)}(x) = |F_{m}^{(j-1)}(x)| + |\Phi_{m}^{(j-1)}(x)| \\
\leq K_1 g^{m-j+1} H(x) + K_2 c_k \left| g(x_k) \right|^{1-j} \\
\leq K_3 g^{m-j+1} H(x).
\]

Finally, \( f_{m+1} - f_m = F_{m+1}^{(m+1)} - F_m^{(m)} = \Phi_m^{(m)} \) so that on \( [x_k, x_{k+1}) \),

\[
|f_{m+1}(x) - f_m(x)| \leq K c_k \left| g(x_k) \right|^{-m} \leq K_4 H(x).
\]

**Lemma 4.7.** If \( L = D^2p_1D^2 - Dp_1D + p_0 \) on \( (a, b) \), where \( p_1 = r_1 + S_1' \), \( p_0 = r_0 + S_0'' \), and if \( P \) is a locally integrable function with \( P \leq p_2 \), then for \( f \in C_0^\infty(a, b) \) and any \( 0 < \varepsilon < 1 \),

\[
\int_a^b (Lf) \geq \int_a^b \left( r_1 - (1-\varepsilon)^{-1} S_1'/P + 2S_0 \right) |f''|^2 + \frac{\varepsilon}{\varepsilon-1} \int_a^b \left( r_0 - \varepsilon^{-1} S_0'/P \right) |f|^2. \tag{4.2}
\]

**Proof.** Integrating by parts yields for \( f \in C_0^\infty(a, b) \),

\[
\int_a^b (Lf) = \int_a^b p_2 |f''|^2 + \int_a^b (r_1 + S_1') |f'|^2 + \int_a^b (r_0 + S_0'') |f|^2.
\]

Moreover, \( \int_a^b S_1' |f'|^2 = -2 \text{ Re } \int_a^b S_1 f' \bar{f}' \) so that

\[
\int_a^b \left( (1-\varepsilon) P |f''|^2 + S_1' |f'|^2 \right) \geq -(1-\varepsilon)^{-1} \int_a^b \left( S_1'/P \right) |f'|^2.
\]

Here we have used the inequality \( c |\alpha|^2 - 2 \text{ Re } \alpha \beta \geq |\beta|^2/c \). Similarly,

\[
\int_a^b S_0'' |f|^2 = - \int_a^b S_0 (f\bar{f}' + f'\bar{f}) + 2 \int_a^b S_0 |f'|^2 + 2 \text{ Re } \int_a^b S_0 f' \bar{f}
\]

and

\[
\varepsilon \int_a^b P |f''|^2 + 2 \text{ Re } \int_a^b S_0 f' \bar{f} \geq -\varepsilon^{-1} \int_a^b \left( S_0'/P \right) |f|^2.
\]

Adding these inequalities gives (4.2).

It follows from this lemma that the minimal operator \( T_0 \) is positive definite if

\[
r_1 - (1-\varepsilon)^{-1} S_1'/P + 2S_0 \geq 0
\]
and

\[ r_0 - \varepsilon^{-1}S_0^2/p \geq 0. \]

These inequalities are sometimes more convenient to use than (2.5). Consider, for instance, the following more awkward variant of Example 3.4.

**EXAMPLE 4.8.** Let \( L = D^2x^2(2 + \cos x)D^2 - D(-x^2 + x^4)D + x^2 + x^5 \sin x^4 \). Set \( P(x) = x^2 \), choose \( S_1 \) so that \( S_1'(x) = x^3 \sin x^4 \) and \( S_1(x) = O(1) \) and \( S_{00} \) so that \( S_{00} = O(x^{-1}) \). Take \( S_0 = (2/3)x^2 + S_{uu} \). Then \( r_1 - x^2, r_0 - x^2 - 4/3 \). Thus, with \( \varepsilon = 1/2 \),

\[ r_1 - 2S_0^2/P + 2S_1 = -x^2 + (4/3)x^2 + O(1) \geq 0 \]

and

\[ r_0 - 2S_0^2/P = x^2 - (8/9)x^2 + O(1) \rightarrow \infty \]
as \( x \rightarrow \infty \).

The fourth-order inequalities of Lemma 4.7 can also be used to investigate 2\( n \)-th-order expressions. We shall use this technique to prove Theorem 5.1. We use it here to generalize Example 3.9 of Read [13].

**EXAMPLE 4.9.** Let \( \{m_k\} \) be a sequence such that \( m_k \rightarrow \infty, m_{k+1}/m_k \rightarrow \infty \). Set \( a_k = m_k/m_{k+1} \). We may assume \( a_k < 1 \) for all \( k \). Now let \( L = (-1)^nD^{2n} + q \) where \( q \) is defined on \([1, \infty)\) by

\[
q(x) = \frac{m_k}{1 - a_k}, \quad x \in [k, k + 1 - a_k) \\
\quad = -\frac{2m_k}{a_k}, \quad x \in [k + 1 - a_k, k + 1).
\]

Then

\[
\int_k^{k+1} q = -m_k \rightarrow -\infty
\]
as \( k \rightarrow \infty \), but we shall show that the minimal operator generated by \( L \) is bounded below and has empty essential spectrum in \( L^2(1, \infty) \). Define functions \( r \) and \( S \) by

\[
r = q, \quad k + 3a_{k-1} \leq x < k + 1 - a_k = m_{k+1}/4, \quad k + 1 - a_k \leq x < k + 1 + 3a_k,
\]

and

\[
S(x) = \int_1^x (q - r).
\]
Then $S = 0$ on each interval $[k + 3a_{k-1}, k + 1 - a_k]$ and $|S(x)| \leq 3m_k$ on $[k + 1 - a_k, k + 1 + 3a_k]$. Thus
\[
\left| \int_{k+a}^{k+2-a} S \right| = \left| \int_{k+1-a}^{k+1+3a_k} S \right| \leq 12m_k a_k \leq 12m_k/m_{k+1} \rightarrow 0.
\]
Define $\Phi$ as in (4.1) with $x_k = k + 1 - a_k$ and
\[
S_0(x) = 2 + \int_1^x (S - \Phi).
\] (4.3)

Then $1 \leq S_0 \leq 3$ eventually and $q = r_0 + S_0''$ where $r_0 = r + \Phi'$. Since $\Phi' \to 0$, $r_0 - 2S_0^2 \to \infty$. Also, with $r_i = S_i = 0$, $r_i - 2S_i^2 + 2S_i = 2S_i \geq 2$ eventually. Thus the conclusion follows for $n = 2$ from Lemma 4.7.

For the general case, write $L = -DL_1D + L_2$ where $L_1$ is the constant coefficient expression $L_1 = (-1)^{n-1} D^{2(n-1)} + D^2 + 2c$ and $L_2 = D^4 + 2cD^2 + q$. It is clear that the minimal operator generated by $-DL_1D$ is positive definite if $c$ is sufficiently large. For this $c$ the argument just given for $n = 2$ shows that the minimal operator generated by $L_2$ has empty essential spectrum if we set $r_i = -2c$ and modify (4.3) to
\[
S_0(x) = c + 1 + \int_1^x (S - \Phi).
\]

5. THE MAIN THEOREM

We are now ready to state and prove our extension of Molchanov's criterion for discreteness of the spectrum. We formulate it on an interval $[a, b)$ where $b \leq \infty$, so both finite and infinite singularities are included.

**Theorem 5.1.** Let $g$, $h$, and $p$ be positive functions on $[a, b)$ with $g$ left continuous and $x + g(x) < b$ for $x < b$, and with $g$, $h$, and $p$ in $B(g)$. Suppose that

(i) $p^{2n} \leq K_1 p_n$, $h^{2n} \geq K_2 w$, $gh \leq K_3 p$,

(ii) $h$ and $p$ are $C^2$ functions with

\[
h^{(j)} = O(h^{j+1}/p^j), \quad (h/p)^{(j)} = O((h/p)^{j+1}), \quad j = 1, 2.
\]

Let $L = \sum_{j=0}^n (-1)^j D^j p_j D^j$, $n \geq 2$. If

(a) $\left[ gh^{2(n-j)} p^{2j}(x) \right]^{-1} \int_x^y p_j \geq -K$ whenever $x \leq y \leq x + g(x), \quad j = 0, 1, \ldots, n - 1,$

then $L$ is a Sturm-Liouville operator.
(b) \[ gh^{2n}(x) \left| \int_{x}^{x+\theta(x)} p_{0} \, dx \right| \rightarrow \infty \text{ for each } \theta > 0, \]
then \( T_{0} \) is bounded below and has empty essential spectrum.

Remark 5.2. Hypothesis (ii) implies that \( p \) cannot be too much larger than \( h \). More precisely, \((h/p) > -K(h/p)^{2}\) implies \((p/h) \leq K\) and so \( p/h = O(x) \) when \( b = \infty \). We have also seen in Lemma 4.2 that \( g \in B(g) \) implies \( g = O(x) \) when \( b = \infty \).

Proof: By (a) and Lemma 4.3, \( p_{j} \) can be decomposed as \( p_{j} = r_{j} + S_{j} \), where \( r_{j} \geq -Kh^{2(n-j)}p^{2j} \) and \(|S_{j}| \leq 2Kgh^{2(n-j)}p^{2j} \leq 2KK_{3}h^{2(n-j)-1}p^{2j+1} \) for each \( j = 0, 1, \ldots, n - 1 \). Lemma 4.5 and (b) imply that \( r_{0} \) has the stronger property that \( r_{0}/h^{2n} \rightarrow c \) as \( x \rightarrow b \). Moreover, by Lemma 4.6 we may modify the decomposition so that \( p_{0} = r_{0} + S_{0} \) with \( r_{0}/h^{2n} \rightarrow c \) and \(|S_{0}| \leq Kg^{2n}h^{2} \leq Kh^{2n-2}p^{2} \). (This \( r_{0} \) may differ from the one originally defined by a function which is \( O(h^{2n}) \).) Here \( K \) is chosen large enough so that the same constant suffices for all \( j - 0, 1, \ldots, n - 1 \).

Also, it follows from (ii) that there is a constant \( J \) so that
\[
|(h^{2(n-j)}p^{2j})^{n}| \leq Jh^{2(n-j+1)}p^{2(n-j)} \quad \text{for } j = 1, 2, \ldots, n - 1.
\]

We may assume that both \( K \) and \( J \) are greater than 1.

Now define two sequences \( K(j), \ j = 1, 2, \ldots, n - 1 \), and \( J(j), \ j = 0, 1, \ldots, n - 2 \), of positive constants so that \( K(n - 1) = 1 \) and if \( K(j + 1) \) has already been defined,
\[
J(j) = K + K^{2} + 2K(j + 1), \quad j \geq 1,
\]
\[
K(j) = J(j)[2J(j) + J],
\]
\[
J(0) = 3K + K^{2} + 2K(1).
\]

We can rewrite \( L \) in the form
\[
L = \sum_{j=0}^{n-2} (-1)^{j} D^{j}L_{j}D^{j},
\]
where the coefficients of the fourth-order expressions
\[
L_{j} = D^{2}q_{j2}D^{2} - Dq_{j1}D + q_{j0}
\]
are defined by
\[
q_{j2} = K(j + 2) h^{2(n-j-2)}p^{2(j+2)}, \quad j = 0, 1, \ldots, n - 3
\]
\[
= p_{n}, \quad j = n - 2.
\]
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\[ q_{j+1} = p_{j+1} - 2K(j + 1) h^{2(n-j-1)} j^{2(j+1)}, \quad j \neq 0, n - 2 \]

\[ q_{j} = p_{j} - K(j + 1) h^{2(n-j-1)} j^{2(j+1)}, \quad j = 0, n - 2, \]

\[ q_{0} = K(j) h^{2(n-j)} j^{2j}, \quad j = 1, 2, \ldots, n - 2 \]

\[ = p_{0}, \quad j = 0. \]

We shall show that each \( L_{j} \) is positive definite and that the minimal operator, \( T_{00} \), generated by \( L_{0} \) in \( L_{0}^{-2}(a, b) \) has the property that for each positive \( A \), there is \( c \) such that \( (T_{00}, f, f)_{w} \geq A(f, f)_{w} \) for all \( f \in C_{0}^{\infty}(c, \infty) \). This will suffice to establish the theorem, since then \( (T_{00}, f, f)_{w} \geq (T_{00}, f, f)_{w} \geq A(f, f)_{w} \) for \( f \in C_{0}^{\infty}(c, \infty) \) and the result follows from Theorem 3.1.

To establish that \( L_{j} \) is positive definite, it is enough, according to Lemma 4.7, to construct decompositions

\[ q_{j+1} = r_{j+1} + S_{j+1} \]

\[ q_{0} = r_{0} + S_{0} \]

such that

\[ r_{j+1} - 2S_{j+1}^{2} / q_{j+2} + 2S_{j+2} \geq 0, \quad (5.1) \]

\[ r_{0} - 2S_{0}^{2} / q_{2} \geq 0. \quad (5.2) \]

For the stronger property of \( L_{0} \) we need to strengthen (5.2) for \( j = 0 \) as in Corollary 3.3 to

\[ w^{-1}[r_{0} - 2S_{0}^{2} / q_{0}2] \rightarrow \infty. \quad (5.3) \]

We can define decompositions for \( L_{j}, j = 1, 2, \ldots, n - 2 \), as follows.

\[ r_{j+1} = r_{j+1} - 2K(j + 1) h^{2(n-j-1)} j^{2(j+1)}, \]

\[ S_{j+1} = S_{j+1}, \]

\[ r_{0} = K(j) h^{2(n-j)} j^{2j} - J(j)(h^{2(n-j-1)} j^{2(j+1)}), \]

\[ S_{0} = J(j) h^{2(n-j)} j^{2(j+1)}. \]

Here \( r_{j} \) and \( S_{j} \) are from the decompositions of \( p_{j} \) obtained at the beginning of the proof. The constant factor of 2 should be omitted from the definition of \( r_{j+1} \) for \( j = n - 2 \).

Now

\[ r_{j+1} - 2S_{j+1}^{2} / q_{j+2} + 2S_{j+2} \]

\[ \geq h^{2(n-j-1)} j^{2(j+1)}[2J(j) - K - 2K(j + 1) - K^{2} / K(j + 2)] \]

\[ \geq 0. \]
and

\[ r_{j0} - 2S_{j0}^2/q_{j0} \geq h^{2(n-1)p^2}[K(j) - J(j)J - 2|J(j)|^2/K(j + 2)]. \]

Here we have used that \( K(j) \geq 1 \) for all \( j \).

For \( L_0 \) we proceed in a similar way. We define

\[ r_{01} = r_1 - K(1) \frac{h^{2(n-1)p^2}}{\mu}, \]
\[ S_{01} = S_1, \]
\[ r_{00} = r_0 - J(0)(h^{2(n-1)p^2})^n, \]
\[ S_{00} = J(0) h^{2(n-1)p^2} + S_0. \]

We recall that \( p_0 = r_0 + S_0^0 \) with \( r_0/h^{2n} \to \infty \) and \( |S_0| \leq Kh^{2(n-1)p^2} \). Thus

\[ r_{01} - S_{01}^2/q_{02} + 2S_{00} \geq h^{2(n-1)p^2}[-K - K(1) - K^2/K(2) + 2J(0) - 2K] \geq 0 \]

and

\[ w^{-1}[r_{00} - S_{00}^2/q_{02}] \geq w^{-1}[r_0 - Bh^{2n}] \]
\[ = w^{-1}h^{2n}[r_0/h^{2n} - B] \]
\[ \to \infty. \]

Here \( B = J(0)J + (J(0) + K)^2/K(2) \). That the right side does approach \( \infty \) follows from the facts that \( r_0/h^{2n} \to \infty \) and \( h^{2n} \geq K_2w \). This completes the proof of the theorem.

**Remark 5.3.** When \( w = p_n = p = g = h = 1 \) and \( b = \infty \), conditions (a) and (b) become respectively

(a) \( \int_x^y p_j \geq -K \) whenever \( x \leq y \leq x + 1 \),

(b) \( \int_{x+\theta} x p_0 \to \infty \) as \( x \to b \) for each \( \theta > 0 \).

Condition (b) was shown by Molchanov [10] (see also Glazman [7, p. 901]) to imply that the minimal operator generated by the two-term expression \((-1)^nD^{2n} + p_0\) is bounded below with empty essential spectrum provided that \( p_0 \) is bounded below. For second-order expressions, Brink [3] showed that the condition that \( p_0 \) be bounded below could be replaced by the weaker (a). This special case already strengthens Molchanov’s result, not only by introducing the Brinck condition for \( p_0 \) but also by permitting intermediate terms which also satisfy the Brinck condition. This special case also contains Theorem 3.8 of Müller–Pfeiffer [11] which does allow intermediate terms.
Remark 5.4. An important difference between Theorem 5.1 and Molchanov's theorem is that one may regard the function \( h^{2n} \) as an estimate for the average rate of growth of \( p_0 \) via condition (b). This is formulated more precisely in terms of a decomposition at the beginning of the proof. Then condition (a) expresses, in terms of \( h \) and the lower bound \( p^{2n} \) for the leading coefficient, the amount of irregular behavior which the coefficients \( p_0, p_1, ..., p_{n-1} \) may have without disturbing the property that \( T_0 \) has empty essential spectrum. In particular, as \( h \) and \( p \) increase, the latitude permitted the coefficients \( p_j \) also increases.

To give a concrete example of this phenomenon, suppose \( w = 1, p_n(x) = c_n x^{a_n}, p_0(x) = c_0 x^{a_0} \) for some constants \( c_0, c_n, a_0, a_n \), with \( c_0, c_n, a_0 \) positive. Then (i), (ii), and (b) will be satisfied when \( h(x) = x^\beta \) for any \( \beta < a_0/2n \), \( p(x) = x^\alpha \) for \( \alpha = \min\{a_n/2n, \beta + 1\} \), and \( g(x) = x^{a - \beta} \). Then (a) will also be satisfied provided \( p_j > -c_j x^{a_j} \) for \( j = 1, 2, ..., n - 1 \), where

\[
\alpha_j < [(n - j) a_0 + 2j \min\{a_n, a_0 + 2n\}] / 2n.
\]

It should be emphasized, however, that condition (a) does not require this sort of pointwise lower bound on the \( p_j \). For instance, one may add a term \( x^2 \sin x^5 \) to \( p_j \) without disturbing (a) or (b) provided

\[
\gamma - \delta + 1 < \alpha_j.
\]

Note that for a given \( a_0 \) and \( a_n \), \( \gamma \) may be arbitrarily large, provided \( \delta \) is also sufficiently large.

Remark 5.5. There is another difference between Theorem 5.1 and Molchanov's result, namely, that in his result (b) is both necessary and sufficient for the essential spectrum to be empty within a certain class of expressions. There are two reasons why (b) of Theorem 5.1 cannot be expected to be necessary within the class of expressions satisfying (a). The first is that an operator satisfying (a) may have empty essential spectrum because one of the intermediate coefficients \( p_j \) is large even though \( p_0 \) is not. This possibility will be studied in the next section. Thus in order for (b) to be a necessary condition, the intermediate coefficients would have to be subjected to rather restrictive upper bounds. Such theorems can be proved (see, for instance, Theorem 3.10 of Müller-Pfeiffer [11]), but they apply to a much more restricted class of expressions than Theorem 5.1.

A second, and even more fundamental, reason why (b) cannot be necessary as well as sufficient arises whenever \( h^{2n}/w \to \infty \) as \( x \to b \). We have already noted that as \( h \) increases, condition (a) become less restrictive, but (b) requires faster average growth by \( p_0 \). In particular, given such an \( h \), a two-term operator \((-1)^n D^{2n} + p_0 \) where \( p_0/w \to \infty \) but more slowly than
h^{2n}/w will certainly satisfy (a) and have empty essential spectrum, but will not satisfy (b).

The only use made of (a) in the proof of Theorem 5.1 is to obtain a decomposition $p_j = r_j + S'_j$ where $r_j \geq -Kh^{2(n-j)}p^{2j}$ and $S_j = O(h^{2(n-j)-1}p^{2j+1})$. Thus we have the following variant of that result, where we write $p_j^-$ for the negative part of $p_j$: $p_j^- = (1/2)(|p_j| - p_j)$.

**Corollary 5.6.** The conclusion of Theorem 5.1 holds if in place of (a) for $j = 1, 2, ..., n - 1$, we assume

$$(a') \quad \int_a^x p_j^- \leq Kh^{2(n-j)-1}(x)p^{2j+1}(x), \quad j = 1, 2, ..., n - 1.$$ 

This substitution may also be made for $j = 0$ if $h$ and $p$ are in $B(p/h)$.

**Proof.** The decomposition $p_j = p_j^+ - p_j^-$ has the necessary properties for $j = 1, 2, ..., n - 1$. For $j = 0$, it follows from Lemma 4.6 if $h$ and $p$ are in $B(p/h)$ that $-p_0^- = f_0 + S_0''$ where $f_0 = O(h^{2n})$ and $S_0 = O(h^{2(n-1)p^2})$. Then $p_0 - (p_0^+ + f_0) + S_0''$ is a decomposition with the properties required for the proof given above for Theorem 5.1.

### 6. Large Intermediate Terms

In this section we will modify Theorem 5.1 so that the large coefficient need not be $p_0$ but may be any $p_k$, $k = 0, 1, ..., n - 1$. The modified result appears as Theorem 6.6. Such a $p_k$ must not only satisfy conditions like those for $p_0$ in Theorem 5.1; it must also be large enough not only to dominate those $p_j$ for $j < k$, but also to compensate for the fact that it is the coefficient of $D^{2k}$. It is easy to see, for instance, that the minimal operator generated by $(-1)^k D^k x^\alpha D^k$ on $[1, \infty)$ with $w = 1$ has empty essential spectrum if and only if $\alpha > 2k$.

It is convenient to begin by formulating a suitable set of additional requirements for $p_k$ separately. The result is due to Hinton and Lewis [8] in the special case $S_j = 0$, $j = 0, 1, ..., n - 1$, for the interval $[a, \infty)$. The case $n = 1$ with $w_1 = \int_a^x w$ extends Theorem 2 of Amos and Everitt [1].

**Theorem 6.1.** Suppose $p_n^{-1} \in L^1(a, b)$. If there is a sequence $\{w_j\}$, $j = 1, ..., n$, of positive differentiable functions on $(a, b)$ such that

1. $w'_i \geq cw$ and $w'_{i+1} \geq cw_i'/w'_i$, $j = 1, 2, ..., n$,
2. $w_n(x) \int_a^x p_n^{-1} \to 0$ as $x \to b$,

and if there is a decomposition $p_j = r_j + S'_j$, $j = 0, 1, ..., n - 1$, such that

3. $r_j \geq -c_j w'_i$, and $|S'_j| \leq d_j w_{i+1}$, $j = 0, 1, ..., n - 1$,

then $T_0$ is bounded below and has empty essential spectrum.
Proof. We proceed by induction on $n$. For $n = 1$, set

$$T(x) = -[w_1(x)]^{1/2} \left( \int_x^b p_1^{-1} \right)^{-1/2}.$$ 

Then, with the notation $\varphi(x) = w_1(x) \int_x^b p_1^{-1}$,

$$p_0 = r_0 + w'_1/2\varphi^{1/2} + \varphi^{1/2}/2p_1 \left( \int_x^b p_1^{-1} \right)^2 + S'_0 + T',$n

where $S = S_0 + T$. Thus

$$w^{-1}(r - S^2/p_1) \geq (w'_1/w)(1/2\varphi^{1/2} - K) + w^2_1(1/2 - C\varphi^{3/2} - 2\varphi^{1/2})/w_1 \varphi^{3/2}.$$ 

Here the second term is eventually positive and the first approaches infinity as $x \to b$. Thus the result follows for $n = 1$ from the second-order case of Corollary 3.3. (See Corollary 3.7 of Read [13].)

Assuming the result for $n - 1$, write

$$L = (-1)^{n-1} D^{n-1} + L_1 + L_2,$$ 

and

$$L_1 = -Dp_n D + p_{n-1} - \Phi w_{n-1}^2/w_{n-1},$$ 

$$L_2 = (-1)^{n-1} D^{n-1} \Phi w_{n-1}^2/w_{n-1} D^{n-1} + \sum_{j=0}^{n-2} (-1)^j D/p_j D.$$ 

Here $\Phi = \varphi^{-1/2}$.

$$\lim_{x \to b} w_{n-1}(x) \int_x^b w'_{n-1}/\Phi w_{n-1} = \lim_{x \to b} \left( \int_x^b w'_{n-1}/\Phi w_{n-1}^2 \right) = 0$$ 

since $\Phi(x) \to \infty$. It then follows from the induction hypothesis that the minimal operator generated by $L_1$ has empty essential spectrum. Thus it suffices to show that the minimal operator generated by $L_1$ is positive. This follows from an argument similar to that given for $n = 1$ with now $T(x) = -w_n(x)^{1/2} [Dp_n^{-1}]^{-1/2}$ since $\Phi w_{n-1}^2/w_{n-1} = o(w_n^2 \varphi^{-1/2}).$

Condition (3) is certainly satisfied if the negative part $p_j^-$ of $p_j$ does not grow too rapidly. More precisely,

**Corollary 6.2.** If there is a sequence $\{w_j\}$ satisfying (1) and (2) and if

$$(3') \int_a^x p_j^- \leq Kw_{j+1}, j = 0, 1, ..., n - 1,$$ 

then $T_0$ is bounded below and has empty essential spectrum.

*Proof.* Take $r_j = p_j^+$, $S_j = -\int_a^x p_j^-, j = 0, 1, ..., n - 1.$
As with Corollary 5.6, (3') is more restrictive than (3) since it does not allow cancellation between $p_j^+$ and $p_j^-$. It is convenient for our main result to reformulate Theorem 6.1 as follows.

**Definition 6.3.** Let $w$ be a positive, locally integrable function on $[a, b)$. A sequence $h_0, ..., h_n$ of positive, locally integrable functions on $[a, b)$ is $w$-admissible if $h_0 \geq cw$ and each $h_j$, $j \leq n - 1$, has an antiderivative $H_j$ such that $h_{j+1} \geq c H_j^2/h_j$.

Thus, for instance, on $[1, \infty)$ the sequence $h_j(x) = x^{2j+\alpha}$ is $x^\alpha$-admissible.

**Remark 6.4.** One way of constructing a $w$-admissible sequence "from the top" is, given $h_k$ with $h_k^{-1} \in L^1(a, b)$, to define

$$h_{k-1}(x) = 1/h_k(x) \left( \int_x^b h_k^{-1} \right)^2.$$ 

If $h_k(x) = x^\alpha$ on $[1, \infty)$, then $h_{k-1}(x) = (a - 1)^2 x^{\alpha - 2}$. This process can be continued as long as $h_k^{-1} \in L^1$ and $h_k \geq cw$.

**Theorem 6.5.** Let $h_0, ..., h_n$ be a $w$-admissible sequence on $[a, b)$ and let $g_0, ..., g_{n-1}$ be a sequence of positive, left continuous functions such that $g_j h_j = O(H_j)$ and such that $g_j, h_j \in B(g_j)$, $j = 0, 1, ..., n - 1$.

If $p_n = h_n \varphi$ for some $\varphi(x) \to \infty$ as $x \to b$ and

$$[g_j h_j(x)]^{-1} \int_x^y p_j \geq -K, x \leq y \leq x + g_j(x), \quad j = 0, 1, ..., n - 1,$$

then $T_0$ is bounded below and has empty essential spectrum.

**Proof.** Set $w_{j+1} = H_j, j = 0, 1, ..., n - 1$. Then $p_n^{-1} \leq c h_{n-1}/\varphi H_{n-1}^2$ so that $p_n^{-1} \in L^1(a, b)$. Moreover,

$$\lim_{x \to b} w_n(x) \int_x^b p_n^{-1} = \lim_{x \to b} \left( \int_x^b p_n^{-1} \right) H_n^{-1}$$

$$= \lim_{x \to b} H_{n-1}^2/h_{n-1} h_n \varphi = 0.$$ 

By Lemma 4.2 each $p_j, j \leq n - 1$, can be decomposed as $p_j = r_j + S_j$ with $r_j \geq -c_j h_j \leq -c_j w_{j+1}$ and $|S_j| \leq d_j g_j h_j \leq K d_j w_{j+1}$. Thus the result follows from Theorem 6.1.

Now we are ready to prove the principal result of this section.

**Theorem 6.6.** Let $k$ be given, $0 \leq k \leq n - 1$. Let $g, h$, and $p$ be positive
functions on \([a, b)\) with \(g\) left continuous and \(x + g(x) < b\) for \(x < b\). Suppose that \(g\), \(h\), and \(p\) are in \(B(g)\) and that

(i) \(p^{2(n-k)} \leq K_1 p_n\) and \(gh \leq K_2 p\),
(ii) \(h\) and \(p\) are \(C^2\) functions with

\[ h^{(j)} = O(h^{j+1}/p^j), \quad (h/p)^{(j)} = O((h/p)^{j+1}), \quad j = 1, 2. \]

Further, let \(h_0, \ldots, h_k\) be a \(w\)-admissible sequence such that \(h_k \leq h^{2(n-k)}\) and let \(g_0, \ldots, g_k\) be positive left continuous functions with \(g_j h_j = O(H_j)\) and \(g_j\), \(h_j \in B(g_j), j = 0, \ldots, k\), and such that

(iii) \(g_k^{2(n-k)} h_k = O(p^{2(n-k)}).\)

If

(a) \(\left[ g h^{2(n-j)} p^{2/(n-k)}(x) \right]^{-1} \int_x^y p_j \geq -K_j, x \leq y \leq x + g(x), k < j < n,\)

(b) \(\left[ g_j h_j(x) \right]^{-1} \int_x^y p_j \geq -K_j, x \leq y \leq x + g_j(x), 0 \leq j \leq k,\)

(c) \(\left[ g_k h_k(x) \right]^{-1} \int_x^{x + \theta_{k}\alpha} p_k \rightarrow \infty \text{ as } x \rightarrow b \text{ for each } \theta > 0,\)

then \(T_0\) is bounded below and has empty essential spectrum.

Remark 6.7. When \(b = \infty\) and \(w = 1\) the special case \(p(x) = x^{\alpha n/(2(n-k))}, h(x) = x^{\alpha k/(2(n-k))}, h_j(x) = x^{2j}, g_k(x) = x^\beta, \beta = [\alpha - k(2 - \alpha)]/2\) extends Theorem 3.9 of Müller–Pfeiffer [11]. Actually if the hypotheses are satisfied for these values of \(p\), \(h\), \(h_j\), and \(g_k\) and some \(g\) and \(g_j, j < k\), then they will also be satisfied when \(h(x) = x^{k/(n-k)}\) so that the pointwise lower bounds \(p_j(x) \geq -c_j x^{\alpha j}, k < j < n\) given in Theorem 3.9 may be relaxed when \(\alpha < 2\) to \(p_j(x) \geq -c_j x^{\alpha j}, \gamma_j = \alpha_j + (2 - \alpha) k(n - j)/(n - k).\)

A slightly more general illustration of Theorem 6.6, generalizing that in Remark 5.4 is as follows. Suppose \(p_n(x) = c_n x^{\alpha_n}, p_k(x) = c_k x^{\alpha_k}\), where \(c_k\) and \(c_n\) are positive and \(\alpha_k > 2k\). Then we may set \(h(x) = x^\beta\) for any \(\beta\) satisfying \(k \leq \beta(n - k) \leq \alpha_k/2,\) \(p(x) = x^{\alpha},\) and \(g(x) = x^{\alpha - \beta}\). Here \(\alpha = \min\{\alpha_n/2(n-k), \beta + 1\}\). Next we may define a \(1\)-admissible sequence by \(h_j(x) = x^{2(\beta(n-k) -(k-j))}\) and set \(g_j(x) = \min\{x, x^{2(\alpha - \beta(n-k))}\}, g_j(x) = x\) for \(j < k\). The result is that \(T_0\) has empty essential spectrum if \(p_j(x) \geq -c_j x^{\alpha_j}\), where

\[ \alpha_j < \frac{\alpha_k(n-j) + 2(j-k) min\{\alpha_n, \alpha_k + 2(n-k)\}}{2(n-k)}, \quad k < j \]

\[ < \alpha_k - 2(k-j), \quad j < k. \]

In this notation the result of Müller–Pfeiffer is the special case \(\alpha \leq n/(n-k)\) and \(\beta = k/(n-k)\).

Proof. From (c) and Lemma 4.3 we have that \(p_k = r_k + S'_k\) where \(r_k/h_k \rightarrow \infty\) and \(S_k = O(g_k h_k)\) as \(x \rightarrow b\). If \(k < n - 1\), then we may modify the decomposition so that \(p_k = r_k + S'_k\) where still \(r_k/h_k \rightarrow \infty\) and \(S_k = O(g_k h_k).\)
Now write \( L = (-1)^k D^k L_1 D^k + L_2 \) where

\[
L_1 = \sum_{j=1}^{n-k} (-1)^j D^j p_{j+k} D^j + (1/2) r_k + S''_k,
\]

\[
L_2 = (-1)^k D^k(1/2) r_k D^k + \sum_{j=0}^{n-1} (-1)^j D^j p_j D^j.
\]

If \( k = n - 1 \), then \( L_1 \) is the second-order expression

\[
L_1 = -Dp_n D + (1/2) r_{n-1} + S'_{n-1}.
\]

If \( k < n - 1 \), then \( L_1 \) satisfies the hypotheses of Theorem 5.1 with the same \( g, h, \) and \( p \). If \( k = n - 1 \), then \( L_1 \) satisfies the hypotheses of Theorem 4.1 of Read [13] with \( gh^2 \) playing the role of the function \( h \) of that theorem. Thus \( (-1)^k D^k L_1 D^k \) is positive. Since \( L_2 \) satisfies the hypotheses of Theorem 6.5, the theorem is proved.

It is not difficult to further modify Theorem 6.6 so that it applies to an expression which has several large positive coefficients, no one of which is sufficiently large to play the role of \( p_k \) in Theorem 6.6. The proof involves decomposing \( L \) into a sum of expressions and applying Theorem 6.6 separately to each term. Instead of formulating such a result, however, we content ourselves with illustrating the application of this technique to a relatively simple example.

**Example 6.8.** On \([1, \infty)\) with \( w = 1 \) set

\[
L = D^4(2 + \cos x) D^4 - D^3(\gamma_4 + \gamma_8 \sin \gamma_6) D^3 + D^2(-\gamma_5 + \gamma_7 \sin \gamma') D^2
- D(\gamma_7 + \gamma_8 \sin \gamma_4) D - \gamma_4 + \gamma_8 \sin \gamma'.
\]

Here \( p_1 \) and \( p_3 \) are large and positive on the average, but it is not difficult to see that the hypothesis on \( p_2 \) in Theorem 6.6 cannot be satisfied with either \( k = 3 \) or \( k = 1 \). To deal with \( L \) we write \( L = -D^3 I_1, D^3 + I_2 \), where

\[
L_1 = -D(2 + \cos x) D + (1/2) \gamma_4 + \gamma_8 \sin \gamma_6,
\]

\[
L_2 = -D^3(1/2) \gamma_4 D^3 + D^2(-\gamma_5 + \gamma_7 \sin \gamma') D^2
\]

\[
- D(\gamma_7 + \gamma_8 \sin \gamma_4) D - \gamma_4 + \gamma_8 \sin \gamma'.
\]

To see that \( L_1 \) is positive, set \( n = 1, k = 0, p = g = h = g_0 = 1 \), and \( h_0(x) = x^{7/2} \). Then \( a \) is vacuous and \( (b) \) and \( (c) \) are satisfied since the function \( x^a \sin x^b \) has a primitive which is \( O(x^a-\beta+1) \) or, in this case, \( O(x^3) = o(h_0) \).
For \( L_2 \) we set \( n = 3, \ k = 1, \ p(x) = x, \ h(x) = x^{3/2}, \ g(x) = x^{-1/2}, \ g_1(x) = x^{-1/2}, \ h_1(x) = x^8, \ h_0(x) = x^4, \) and \( g_0(x) = 1. \) Then (a) becomes
\[
x^{-9/2} \int_x^y (-x^5 + x^7 \sin x^4) \geq -K_2, \quad x \leq y \leq x + x^{-1/2}.
\]
Similarly, (b) for \( j = 1 \) and 0 becomes respectively
\[
x^{-11/2} \int_x^y (x^7 + x^8 \sin x^4) \geq -K_1, \quad x \leq y \leq x + x^{-1/2},
\]
\[
x^{-4} \int_x^y (-x^4 + x^8 \sin x^4) \geq -K_0, \quad x \leq y \leq x + 1.
\]
Since these inequalities and (c) all hold, the minimal operator generated by \( L \) does have the empty essential spectrum.

7. A PRODUCT CRITERION

In this final section we use Corollary 3.3 and Theorem 5.1 to establish a somewhat different criterion for the essential spectrum to be empty. The idea is to allow negative intermediate terms whose size depends only on the product \( p_0 p_n. \) Thus these two coefficients may have quite irregular behavior individually, of a type which would prevent the direct application of Theorem 5.1. We shall restrict ourselves to the case \( w = 1 \) and to the interval \([a, \infty)\).

**Theorem 7.1.** Suppose \( p_0 \to \infty \) and \( p_0 p_n/x^{2n} \to \infty \) as \( x \to \infty. \) If \( p_j \geq 0 \) for \( j > N = \lfloor n/2 \rfloor, \) \( p_N \geq -Kx^{2N} \) and
\[
x^{-2j+1} \int_x^y p_j \geq -K_j, \quad x \leq y \leq x + 1, \quad j = 1, ..., N - 1,
\]
then \( T_0 \) is bounded below and has empty essential spectrum.

**Proof:** We decompose \( L \) as \( L = L_1 + L_2 \) by defining \( L_1 = \sum_{k=0}^n (-1)^j D^j q_j D^j, \) where \( q_j = p_j \) for \( j > N \) and
\[
q_j = p_N - x^{2N}, \quad j = N
\]
\[
= (-1)^{N-k+1} (K + 1)(x^{2N-j})(2N-2j), \quad j = 1, ..., N - 1
\]
\[
= p_0/2, \quad j = 0.
\]
Then \( L_2 = \sum_{k=0}^{N} (-1)^k D^j u_j D^j \) is of order \( 2N \) and
\[
\begin{align*}
u_j &= x^{2N}, & j &= N \\
&= p_j + (-1)^{N-k} (K + 1) (x^{2N})^{(2N-2k)}, & j &= 1, \ldots, N-1 \\
&= p_0/2, & j &= 0.
\end{align*}
\]

It follows from Theorem 5.1 with \( p(x) = g(x) = x \), \( h(x) = 1 \) that the minimal operator generated by \( L_2 \) has empty essential spectrum. To see that the same is true of the minimal operator generated by \( L_1 \), set \( S_0(x) = (-1)^t C(K + 1) x^n \), where \( t = 1 \) if \( N \) is even, \( t = 0 \) if \( N \) is odd, \( C = 1 \) if \( n \) is even, and \( C = n^{-1} \) if \( n \) is odd. Set \( S_j = 0 \) for \( j = 1, \ldots, n-1 \). Then \( r_0 = p_0/2 + A \) for some constant \( A \) and \( r_j = q_j \) otherwise. From Lemma 2.2 we see that the inequalities in Corollary 3.3 now assume the form,
\[
p_j > 0, \quad j > N,
\]
\[
(-1)^{N-j+1} (K + 1) (x^{2N})^{(2N-2j)} + (-1)^{j+1} s_{N-j} 
\]
\[
p_0/2 + A - C^2 x^{2n}/p_n \to \infty.
\]

The first group of inequalities are part of the hypotheses, and the expressions in the second group are equal to 0 by the choice of \( S_0 \). Finally, \( x^{2n}/p_n = o(p_0) \) so that the last expression does approach infinity with \( x \). This completes the proof.

If we assume in addition that \( p_n \) is bounded below, then we can allow all the intermediate terms to be negative.

**Theorem 7.2.** Suppose \( p_0 \to \infty \), \( p_n \geq c > 0 \), and \( p_0 p_n / x^{2n} \to \infty \) as \( x \to \infty \). If
\[
x^{-\alpha_j} \int_x^y p_j \geq -K_j, \quad x \leq y \leq x + x^{-N/(n-N)}, \quad j = N, \ldots, n-1,
\]
where \( \alpha_j = 2n - 2j - 1 N/(n-N) \) and
\[
x^{-2j+2} \int_x^y p_j \geq -K_j, \quad x \leq y \leq x + 1, \quad j = 1, \ldots, N-1,
\]
then \( T_0 \) is bounded below and has empty essential spectrum.

**Proof.** We may assume \( c \geq 1 \). Suppose first \( n = 2 \). Then \( \alpha_1 = 1 \), so \( p_1 = r_1 + S_1 \) with \( r_1(x) \geq -K_1 x^2 \), \( |S_1(x)| \leq K_2 x \). Set \( S_0 = (K_1 + K_2) x^2 \) and \( r_0 = p_0 + S_0 \). Then
\[
r_1 - S_1^2/p_2 + 2S_0 \geq (K_1 + K_2)^2 x^2 \geq 0
\]
and

\[ r_0 - S_0^2/p_1 \to \infty \]

since \( x^4/p_1 = o(r_0) \).

For \( n > 2 \), we write \( L = L_1 + L_2 + (-1)^N D^N L_3 D^N \). Here \( L_3 = \sum_{j=0}^{n-N} (-1)^j D^j v_j D^j \) with

\[
\begin{align*}
  v_j &= p_n/2, & j &= n - N \\
     &= p_{j+N}, & j &= 1, \ldots, n - N - 1 \\
     &= p_N + K x^j, & j &= 0.
\end{align*}
\]

We can show that \( L_3 \) is positive if \( K \) is sufficiently large by writing \( L_3 \) as a sum \( \sum (-1)^j D^j L_{3j} D^j \) where each \( L_{3j} \) is a fourth-order expression as in the proof of Theorem 5.1 and taking \( p(x) = 1, h(x) = g(x)^{-1} = x^{N/2-N} \). Then \( L_1 = (-1)^n D^n p_n/2 D^n + \sum_{j=0}^{N} (-1)^j D^j q_j D^j \) with

\[
\begin{align*}
  q_j &= -(K + 1) x^{2N}, & j &= N \\
      &= (-1)^{N-j+1} (K + 1)(x^{2N})^{(2N-2j)}, & j &= 1, \ldots, N - 1 \\
      &= p_0/2, & j &= 0.
\end{align*}
\]

is also positive by the same argument as in the proof of Theorem 7.1. The minimal operator generated by \( L_2 \), defined as in the proof of Theorem 7.1, again has empty essential spectrum by Theorem 5.1 with \( h(x) = 1, p(x) = g(x) = x \). This completes the proof.

ACKNOWLEDGMENT

The author is grateful to the Econometric Institute of the University of Groningen for the opportunity to spend 1978–79 and 1979–80 in Groningen, where this research was done.

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