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A globalisation of the Gelfand duality theorem

Bernhard Banaschewski[a], Christopher J. Mulvey[b,*]

[a] *Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, L8S 4K1, Canada*
[b] *Department of Mathematics, University of Sussex, Falmer, Brighton, BN1 9RF, United Kingdom*

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Abstract

In this paper we bring together results from a series of previous papers to prove the constructive version of the Gelfand duality theorem in any Grothendieck topos $\mathbb{E}$, obtaining a dual equivalence between the category of commutative C*-algebras and the category of compact, completely regular locales in the topos $\mathbb{E}$.

MSC: 18F99; 46L99

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1. Introduction

In this paper, we establish that Gelfand duality holds between the category of commutative C*-algebras and the category of compact, completely regular locales in any Grothendieck topos. It should be remarked immediately that this result represents the final step in a chain of preliminary papers that have appeared over a period of time. Indeed, the work contained in these papers was originally presented at the International Meeting on Categorical Topology held in Ottawa in 1980, of which the details were published in a widely circulated preprint (Banaschewski–Mulvey [4]) several years later,

* Corresponding author.
E-mail addresses: marcouxp@mcmail.cis.mcmaster.ca (B. Banaschewski), c.j.mulvey@sussex.ac.uk (C.J. Mulvey).

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the length of which made immediate publication difficult. In a sequence of papers that followed (Banaschewski–Mulvey [3,5–7]), many of the results that provide the natural components from which Gelfand duality is derived were published independently. With some preliminaries to recall the conceptual framework within which the result is set and to make this paper readable without continual reference to its predecessors, these are finally here assembled to prove the Gelfand duality theorem.

The Gelfand duality which is proved consists in the main of two results: firstly, that any commutative C*-algebra A is canonically isometrically *-isomorphic to the commutative C*-algebra $\mathbb{C}(\text{Max } A)$ of continuous complex functions on the compact, completely regular locale Max A that is its maximal spectrum; and secondly, that any compact, completely regular locale M is canonically isomorphic to the maximal spectrum Max $\mathbb{C}(M)$ of its commutative C*-algebra of continuous complex functions. Evidently, each of these assertions has to be set within the constructive context of the Grothendieck topos within which we are working, with which the preliminary sections will be concerned. The second of these results was effectively established earlier in considering one approach to the construction of the Stone–Čech compactification of a locale [3], together with earlier work on the maximal spectrum of a commutative C*-algebra [19,20], although the details adapted to the present situation will be given again here.

The constructivisation of the Gelfand theorem establishing the existence of the isometric *-isomorphism

$$A \to \mathbb{C}(\text{Max } A)$$

from any commutative C*-algebra A to that of continuous complex functions on its maximal spectrum Max A occupies the principal part of the paper, building on the results established in the sequence of preliminary papers. Its conceptual context is, nevertheless, quite straightforward, and is worth outlining at this point before becoming involved with the detail of the proof. The first thing to note is that the existence of the Gelfand homomorphism is an immediate consequence of the construction, or more properly, one of the constructions, of the spectrum Max A of the commutative C*-algebra A, as is the case classically. The form of the theory of multiplicative linear functionals on A that provides this construction canonically assigns to each element $a \in A$ a continuous complex function

$$\hat{a} : \text{Max } A \to \mathbb{C}$$

on the spectrum of A. Equally, this construction of the spectrum yields that the locale obtained is compact and completely regular, by inheritance from the locale of bounded linear functionals on A which is compact and completely regular by the constructive form of Alaoglu's theorem (Mulvey–Pelletier [26,27]).

To obtain that the Gelfand representation is an isometric *-isomorphism requires establishing that the spectrum Max A may be constructed equivalently by introducing directly a theory of the maximal spectrum, constructivising its classical introduction as the topological space of maximal ideals of the commutative C*-algebra. That this classically may be identified with the space of multiplicative linear functionals is a consequence of the Gelfand–Mazur theorem, which shows that each maximal ideal of a commutative C*-algebra is the kernel of a multiplicative linear functional. The constructive form of this result is exactly the equivalence of the theory of multiplicative linear functionals with that of the maximal spectrum of the commutative C*-algebra, a result established in any Grothendieck topos in one of the preliminary papers (Banaschewski–Mulvey [6,7]).

Considering the commutative C*-algebra $\mathbb{C}(\text{Max } A)$ as the global sections of the sheaf of continuous complex functions on the compact, completely regular locale Max A, it may be shown first, by working with the theories defining the spectrum, that the Gelfand representation

$$A \to \mathbb{C}(\text{Max } A)$$

is necessarily isometric in any Grothendieck topos. By a further consequence of the constructive form of the Gelfand–Mazur theorem it may be shown that finite partitions of unity exist subordinate to any covering of the spectrum, in turn allowing the constructive form of the Stone–Weierstrass theorem, established in another of the preliminary papers (Banaschewski–Mulvey [5]), to be applied to show that the image of the Gelfand representation is exactly the commutative C*-algebra $\mathbb{C}(\text{Max } A)$, thereby establishing the Gelfand theorem.

Finally, it may be remarked that the papers establishing the Gelfand–Mazur theorem and the Stone–Weierstrass theorem have as a common and consistent theme that these results concerning commutative C*-algebras at the end of the day come down to establishing certain facts about the topology of the complex numbers in a constructive context. In the case of the Gelfand–Mazur theorem, this is just that within any bounded region of the complex plane the topology defined by rational open rectangles coincides with that defined by open rational codiscs. In the case of the Stone–Weierstrass theorem, it is that the complex rationals are dense in the complex numbers, and hence that any closed subset that contains them is necessarily the space of complex numbers itself. For the way in which these observations translate into the theorems asserted, the interested reader is referred to the papers concerned (Banaschewski–Mulvey [5–7]).

It may further be remarked that, although the proofs involved in the Gelfand–Mazur theorem and in the isometricity of the Gelfand representation depend on the topos in which we are working being a Grothendieck topos, allowing Barr coverings to be used, these were subsequently shown by Mulvey and Vermeulen to admit constructive proofs, along the lines of those outlined in the case of the Hahn–Banach theorem in [25]. The tragic death of Japie Vermeulen has introduced a further delay into the publication of these results, to which it is hoped to return. In the meantime, a constructive proof of the Gelfand theorem in the real, rather than complex, case, in which many of the lattice-theoretic aspects are dealt with explicitly in the axiomatisation introduced, rather than implicitly in the complex structure, has been obtained by Coquand [10].

2. Preliminaries

In this section, we recall the principal concepts with which we shall be concerned, the commutative C*-algebras and the compact, completely regular locales between which we shall establish the Gelfand duality theorem in a Grothendieck topos $\mathbb{E}$. For a more detailed introduction to these ideas, we refer to the earlier papers (Banaschewski–Mulvey [2,3,5–7], Mulvey [18,23]), which also provide an extensive discussion of the motivation behind them and, in particular, concerning the constructive context of the Grothendieck topos within which we shall be working. For the moment, we recall the critical observation that this context is one in which in general neither the Axiom of Choice nor the Law of the Excluded Middle holds, requiring the concepts concerned to be adapted thoughtfully to this situation [18]. Nevertheless, the fact that we may work within a topos, albeit constructively, in a way that is similar to that in which we work classically will be reflected in referring throughout to the objects constructed as sets, even though this may be far from being the case [15].

To begin with, noting that the concepts of norm and of completeness with respect to a norm need to be made appropriate to this constructive context, we recall the following:

Definition 2.1. By a *commutative* C*-*algebra* A in a Grothendieck topos $\mathbb{E}$ is meant a commutative Banach *-algebra in $\mathbb{E}$ satisfying the condition that:

$$a \in N(q) \;\leftrightarrow\; aa^* \in N(q^2)$$

for each $a \in A$ and each positive rational q.

It should be recalled that by a *-*algebra* A is meant an algebra over the field of complex rational numbers, together with an involution satisfying the conditions that:

(1) $(a+b)^* = a^* + b^*$;
(2) $(\alpha a)^* = \bar{\alpha} a^*$;
(3) $(ab)^* = b^* a^*$;
(4) $a^{**} = a$,

for each $a, b \in A$ and each complex rational α. The *-algebra A is said to be *seminormed* provided that there is given a mapping

$$N : \mathbb{Q}^+_{\mathbb{E}} \to \Omega^A$$

from the positive rationals in $\mathbb{E}$ to the set of subsets of A satisfying the conditions that:

(1) $\exists q \in \mathbb{Q}^+_{\mathbb{E}} a \in N(q)$;
(2) $a \in N(q) \wedge a' \in N(q') \to a + a' \in N(q + q')$;
(3) $a \in N(q') \to \alpha a \in N(qq')$ whenever $\alpha \in N(q)$;
(4) $0 \in N(q)$;
(5) $a \in N(q) \wedge a' \in N(q') \to aa' \in N(qq')$;
(6) $a \in N(q) \to a^* \in N(q)$;
(7) $1 \in N(q)$ whenever $q > 1$;
(8) $a \in N(q) \leftrightarrow \exists q' < q \;\; a \in N(q')$,

for all $a, a' \in A$, for each complex rational α, and for all positive rationals q, q'. Evidently these conditions express the properties of a seminorm in terms of the open balls which it classically would determine. In passing, it should be noted once again that although we have referred to a mapping from the positive rationals to the set of subsets of A, this construction is to be interpreted within the context of the Grothendieck topos $\mathbb{E}$, and hence involves the object Ω^A of subobjects of A.

By a *Cauchy approximation* on a seminormed *-algebra A is meant a mapping

$$C : \mathbb{N}_{\mathbb{E}} \to \Omega^A$$

which satisfies the following conditions:

(1) $\forall n \in \mathbb{N}_{\mathbb{E}} \exists a \in A \; a \in C_n$;
(2) $\forall k \in \mathbb{N}_{\mathbb{E}} \exists m \in \mathbb{N}_{\mathbb{E}} \forall n, n' \geq m \; a \in C_n \wedge a' \in C_{n'} \to a - a' \in N(1/k)$.

Intuitively, this describes a sequence of subsets of A from which a Cauchy sequence could arbitrarily be chosen, if the Axiom of Countable Choice were available to do the choosing. A Cauchy approximation C on a seminormed *-algebra A is said to be *convergent* to an element $b \in A$ provided that

$$\forall k \in \mathbb{N}_{\mathbb{E}} \exists m \in \mathbb{N}_{\mathbb{E}} \forall n \geq m \; a \in C_n \to a - b \in N(1/k),$$

and the seminormed *-algebra A is said to be *complete* provided that for each Cauchy approximation C on the algebra A there exists a unique element $b \in A$ to which C converges. Of course, the uniqueness incorporated in this definition then implies that the seminorm on A satisfies the condition that:

$$(\forall q \in \mathbb{Q}^+_{\mathbb{E}} a \in N(q)) \to a = 0$$

for each $a \in A$, which describes the property of the seminorm on A of actually being a *norm*. A commutative seminormed *-algebra A is then said to be a *commutative Banach *-algebra* provided that it is complete, to which then is added the condition

$$a \in N(q) \; \leftrightarrow \; aa^* \in N(q^2),$$

for each $a \in A$ and each positive rational q, that describes the characterising property of a commutative C*-algebra in terms of the open balls of A.

As a fundamental example of a commutative C*-algebra in the topos $\mathbb{E}$ we have the algebra $\mathbb{C}_{\mathbb{E}}$ of complex numbers in $\mathbb{E}$, together with the norm defined by setting

$$N(q) = \{a \in \mathbb{C}_{\mathbb{E}} \mid |a| < q\}$$

for each positive rational q. It may be remarked that, although a seminormed *-algebra A is required only to be an algebra over the complex rationals in the topos $\mathbb{E}$, by completeness any Banach *-algebra is necessarily also an algebra over the algebra $\mathbb{C}_{\mathbb{E}}$ of complex numbers in $\mathbb{E}$ in a canonical way induced by its structure as an algebra over the field of complex rationals in $\mathbb{E}$. In particular, any commutative C*-algebra A in the Grothendieck topos $\mathbb{E}$ is an algebra over that of the complex numbers in $\mathbb{E}$.

The other notion with which we shall be concerned is that of a *compact, completely regular locale L* in a Grothendieck topos $\mathbb{E}$, which is recalled in the following:

Definition 2.2. By a *locale L* in a Grothendieck topos $\mathbb{E}$ is meant a complete lattice in $\mathbb{E}$ satisfying the condition that

$$u \wedge \bigvee S = \bigvee \{u \wedge v \in L \mid v \in S\}$$

for any $u \in L$ and any subset S of L, in which $\wedge$ denotes binary meet and $\bigvee$ arbitrary join.

In any locale L the *rather below* relation is that defined by writing $v \lhd u$ whenever there exists $w \in L$ such that

$$v \wedge w = 0_L \quad \text{and} \quad w \vee u = 1_L$$

(or equivalently $u \vee v^* = 1_L$, in which $v^* \in L$ denotes the pseudo-complement

$$v^* = \bigvee\{w \in L \mid v \wedge w = 0_L\}$$

of $v \in L$). Moreover, the *completely below* relation is then defined by writing $v \lhd\!\lhd u$ whenever there exists a family of elements $v_q \in L$, indexed by the rationals $0 \leq q \leq 1$, for which

$$v_0 = v\,, \ \ v_p \lhd v_q \text{ whenever } p < q\,, \text{ and } v_1 = u.$$

A locale L is then said to be *compact* provided that any open covering of L admits a finite subcovering, that is, for any subset S of L such that $\bigvee S = 1_L$, the unit, that is, top element, of L, there exists a finite subset T of S for which $\bigvee T = 1_L$, and *completely regular* provided that

$$u = \bigvee\{v \in L \mid v \lhd\!\lhd u\}$$

for any $u \in L$, in which $\lhd\!\lhd$ denotes the completely below relation.

The compact, completely regular locales in a Grothendieck topos $\mathbb{E}$ are linked functorially to the commutative C*-algebras in $\mathbb{E}$ by assigning to each such locale M the algebra $\mathbb{C}(M)$ of continuous complex-valued functions on M, in a sense that we now make precise by recalling first the following:

Definition 2.3. By a *map of locales*

$$\varphi : L \to M$$

in a Grothendieck topos $\mathbb{E}$ is meant a mapping

$$\varphi^* : M \to L\,,$$

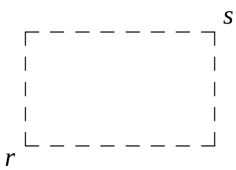
referred to as the *inverse image homomorphism*, which preserves finite meets and arbitrary joins.

In the case of the locales of open subsets of topological spaces, the notion of a map of locales coincides with that of a continuous mapping between the topological spaces concerned. In the constructive context of a Grothendieck topos, considering locales, rather than topological spaces, allows the development of analytical topology to proceed in a way that is recognisably that to which one is accustomed. For instance, we have already remarked that in the commutative C*-algebra $\mathbb{C}_{\mathbb{E}}$ of complex numbers in the Grothendieck topos $\mathbb{E}$ it is not in general the case that the closed unit disc is compact. In a sense, this may be viewed as a perfectly reasonable consequence of the constructivity of the context, expressing a fundamental deficiency in the concept of topological space, as compared with the naturality of that of locale. Instead, one should consider the locale of complex numbers in the topos $\mathbb{E}$, and with it a concept of continuous complex function on a locale, defined in ways that we now recall [6]:

Definition 2.4. By the *propositional geometric theory* of complex numbers in $\mathbb{E}$ is meant that obtained by introducing for each pair (r, s) of rational complex numbers in $\mathbb{E}$ a primitive proposition

$$z \in (r, s),$$

intuitively representing the assertion that the complex number z being described lies in the complex rational open rectangle



together with the following axioms:

(C1) $z \in (r, s) \vdash \mathit{false}$ whenever $(r, s) \leq 0$;
(C2) $\mathit{true} \vdash \bigvee_{(r,s)} z \in (r, s)$;
(C3) $z \in (r, s) \vdash z \in (p, q) \vee z \in (p', q')$ whenever $(r, s) \lhd (p, q) \vee (p', q')$;
(C4) $z \in (p, q) \wedge z \in (p', q') \vdash z \in (r, s)$ whenever $(p, q) \wedge (p', q') \lhd (r, s)$;
(C5) $z \in (r, s) \dashv\vdash \bigvee_{(r',s') \lhd (r,s)} z \in \left(r', s'\right)$,

in which the conditions involved refer to the open subsets of the complex rational plane, defined in algebraic terms.

Then, by the *locale* $\mathbb{C}$ *of complex numbers* in $\mathbb{E}$ is meant that given by the Lindenbaum algebra of this theory, that is to say, the locale obtained by taking the propositions of the theory, obtained by taking arbitrary disjunctions of finite conjunctions of the primitive propositions, modulo provable equivalence in the theory, partially ordered by provable entailment in the theory. Alternatively, this is just the locale in $\mathbb{E}$ obtained by taking the primitive propositions as generators, and the axioms, with $\vdash$ interpreted as $\leq$, as relations. The locale $\mathbb{C}$ will also be referred to as the *complex plane* in the topos $\mathbb{E}$.

It may be remarked finally that the technique that we have used for constructing the locale of complex numbers, namely that of considering the propositional geometric theory of its classical points, namely the complex numbers, is one which has been used extensively [3,6,7,14,19,20,25–27] to develop analytical and algebraic ideas within the constructive context of a topos. In particular, it may be remarked immediately that the sublocale of the locale of complex numbers in the Grothendieck topos $\mathbb{E}$ obtained by taking those complex numbers of modulus ≤ 1, effected simply by adding the axiom

(U) $z \in (r, s) \vdash \mathit{false}$

whenever (r, s) lies strictly outside the unit disc in the complex rational plane, is always a compact, completely regular locale. An explicit proof of this compactness, albeit in the case of the closed unit square in the complex plane, rather than the closed unit disc, and of the complete regularity which holds for the complex plane itself, may be found in [6].

On a final notational point: although we have written the primitive propositions of the theory of the complex plane in the form

$$z \in (r, s)$$

for each complex rational open rectangle, we shall from here onwards frequently write simply

$$(r, s)$$

for the element of the locale $\mathbb{C}$ which it determines, referring to it as an open subset of the complex plane $\mathbb{C}$.

Finally, it may be recalled that by a *point* of a locale L in the Grothendieck topos $\mathbb{E}$ is meant a map of locales

$$x : \mathbf{1}_{\mathbb{E}} \to L$$

from the locale $\mathbf{1}_{\mathbb{E}}$ of which the underlying lattice is the topology $\Omega_{\mathbb{E}}$ of the one-point space in the topos $\mathbb{E}$. In particular, taking the topological space of points of the complex plane $\mathbb{C}$ in the topos $\mathbb{E}$ yields exactly the algebra $\mathbb{C}_{\mathbb{E}}$ of complex numbers in the topos $\mathbb{E}$, while that of the sublocale that is the closed unit disc of the complex plane yields exactly the subspace given by the closed unit disc of $\mathbb{C}_{\mathbb{E}}$. It may be noted that whilst the sublocale given by the closed unit disc of $\mathbb{C}$ is always a compact, completely regular locale, the subspace given by the closed unit disc of $\mathbb{C}_{\mathbb{E}}$ is in general not a compact topological space, exemplifying the necessity of the consideration of locales if mathematics is to develop as one expects.

An observation that may be considered converse to this is that the unit disc of the locale $\mathbb{C}$ is exactly the dual locale of the underlying seminormed space of the commutative C*-algebra $\mathbb{C}_{\mathbb{E}}$, in the sense described by Mulvey–Pelletier [26,27]. Indeed, this relationship between the complex numbers $\mathbb{C}_{\mathbb{E}}$ as a commutative C*-algebra and the complex plane $\mathbb{C}$ as a locale will later be seen to be pivotal to the discussion of Gelfand duality.

3. The spectrum of a commutative C*-algebra

Classically, the spectrum of a commutative C*-algebra A may be constructed in either of two ways, equivalent by the Gelfand–Mazur theorem. The first of these is by considering the topological space of multiplicative linear functionals on A. In the constructive context of a Grothendieck topos $\mathbb{E}$, the spectrum in this form is obtained by considering the propositional geometric theory of multiplicative linear functionals on A, obtained by adapting that of linear functionals of norm ≤ 1 on the seminormed space A introduced by Mulvey–Pelletier [26,27] in a way that we now recall:

Given a commutative C*-algebra A in a Grothendieck topos $\mathbb{E}$, consider firstly the propositional geometric theory $\mathbb{F}\mathrm{n}\ A$ in the topos $\mathbb{E}$, determined by introducing for each $a \in A$ and each rational open rectangle (r, s) in the complex plane a proposition

$$a \in (r, s)$$

together with the following axioms:

(M1) $true \vdash 0 \in (r, s)$ whenever $0 \in (r, s)$, and
$0 \in (r, s) \vdash false$ otherwise;
(M2) $a \in (r, s) \vdash ta \in (tr, ts)$ whenever $t > 0$, and
$a \in (r, s) \vdash ia \in i\,(r, s)$;
(M3) $a \in (r, s) \wedge a' \in (r', s') \vdash a + a' \in (r + r', s + s')$;
(M4) $true \vdash a \in N\,(1)$ whenever $a \in N\,(1)$;
(M5) $a \in (r, s) \vdash a \in (p, q) \vee a \in (p', q')$ whenever $(r, s) \lhd (p, q) \vee (p', q')$;
(M6) $a \in (r, s) \dashv\vdash \bigvee_{(r',s') \lhd (r,s)} a \in \left(r', s'\right)$.

It may be noted that in the axiom (M2) the symbol i denotes the imaginary unit, and in the axiom (M4) the expression $a \in N(1)$ within the entailment is used to denote the disjunction $\bigvee_{(r,s) \lhd N(1)} a \in (r, s)$ of propositions indexed by those rational open rectangles (r, s) that are rather below the open disc $N\,(1)$ of radius 1 centred on the origin in the complex plane, a convention which will shortly be extended to other open subsets of the complex plane.

Denote by Fn A the Lindenbaum locale of this theory, that is, the locale of all propositions derived from the primitive propositions by applying finite conjunctions and arbitrary disjunctions, ordered by provable entailment in the theory, modulo provable equivalence. This locale, introduced in Mulvey–Pelletier [26,27], is the constructive equivalent of the unit ball of the dual of the seminormed space A in the weak* topology. The axioms of the theory $\mathbb{F}$n A describe the conditions required to deduce that any model of the theory, and hence any point of the locale which it determines, is exactly a linear functional of norm ≤ 1 on the seminormed space A.

Now consider the theory $\mathbb{MF}$n A in the topos $\mathbb{E}$ obtained by adjoining to those of the theory $\mathbb{F}$n A the following additional axioms:

(M7) $true \vdash 1 \in (r, s)$ whenever $1 \in (r, s)$, and
$1 \in (r, s) \vdash false$ otherwise;
(M8) $a \in (r, s) \vdash a^* \in \overline{(r, s)}$;
(M9) $aa' \in (r, s) \vdash \bigvee_i\ a \in (p_i, q_i) \wedge a' \in (p'_i, q'_i)$
whenever $\bigvee_i (p_i, q_i) \times (p'_i, q'_i) = \mu^*(r, s)$,

which together require that the linear functional is indeed multiplicative. It may be noted that in the axiom (M8) the bar denotes complex conjugation and in the axiom (M9) the expression $\mu^*\,(r, s)$ denotes the inverse image of the rational open rectangle (r, s) under the map μ of locales determined by multiplication in the locale of complex numbers in the topos $\mathbb{E}$. Then, the locale MFn A is defined to be the locale obtained from this theory, by ordering its propositions by provable entailment in the theory, modulo provable equivalence. The locale MFn A is then that which is said to be the *spectrum of the commutative* C*-*algebra* A.

By construction of the theory, the points of the spectrum MFn A of a commutative C*-algebra A are exactly the multiplicative linear functionals on A, since these are the models of the theory. However, the existence of these points will depend on the particular properties of the commutative C*-algebra A and of the Grothendieck topos $\mathbb{E}$ in which it lives. The spectrum MFn A, however, enjoys, as a locale, the properties that one would expect. In particular, we note the following:

Theorem 3.1. *For any commutative* C^**-algebra* A *in a Grothendieck topos* $\mathbb{E}$*, the spectrum* MFn A *is a compact, completely regular locale.*

Although the details of the proof may be found elsewhere [6], it will be helpful to recall a couple of matters arising within it. Firstly, writing for any $a \in A$ and open subset U of the complex plane

$$a \in U$$

for the proposition $\bigvee_{(r,s) \lhd U} a \in (r, s)$, it may be shown that for any open subsets U, U' and any family of open subsets $(U_i)_{i \in I}$ of the complex plane

$$a \in U \ \wedge \ a \in U' \ \vdash\dashv \ a \in U \wedge U', \text{ and}$$
$$\bigvee_i a \in U_i \ \vdash\dashv \ a \in \bigvee_i U_i$$

are provable within the theory of the spectrum, an observation that will be referred to as the *continuity principle*. Noting that *true* $\vdash a \in \mathbb{C}$ and $a \in \emptyset \vdash$ *false* are also provable in the theory, it follows that

$$a \in (r', s') \ \lhd \ a \in (r, s)$$

in the locale MFn A whenever $(r', s') \ \lhd \ (r, s)$ in the complex plane, yielding the complete regularity of the spectrum by the axiom (M6). On the other hand, the compactness of the spectrum is proved by showing that it is a closed sublocale of the dual locale Fn A of the seminormed space A, which is compact by the constructive form of Alaoglu's theorem proved by Mulvey–Pelletier [27].

In the next section, it will be seen that the continuity principle recalled above is the aspect of the spectrum of a commutative C^*-algebra A that provides the Gelfand representation of A. However, to establish the isometricity of the representation we shall need to identify the spectrum with the locale obtained from another theory. Classically, the Gelfand–Mazur theorem states that every maximal ideal of a commutative C^*-algebra A is the kernel of a unique multiplicative linear functional on A. Constructively, this is interpreted by introducing the propositional geometric theory which describes, albeit in a slightly roundabout way, the maximal ideals of a commutative C^*-algebra A. The constructive form of the Gelfand–Mazur theorem is then obtained by showing that the canonical interpretation of this theory in the theory of multiplicative linear functionals on A, corresponding classically to assigning to each multiplicative linear functional its kernel, determines an equivalence between the theories, and hence a canonical isomorphism

$$\text{MFn } A \to \text{Max } A$$

to the locale Max A obtained from the theory. It is the particular form of the theory of the maximal spectrum Max A that then allows us to work with the Gelfand representation obtained to show that it is an isometric *-isomorphism.

Given a commutative C^*-algebra A in a Grothendieck topos $\mathbb{E}$, consider [3,7,19] the propositional geometric theory $\mathbb{M}$ax A in the topos $\mathbb{E}$, determined by introducing for each $a \in A$ and each non-negative rational q a proposition

$$a \in A(q)$$

together with the following axioms:

(A1) $\mathit{true} \vdash 1 \in A(q)$ whenever $q < 1$;
(A2) $a \in A(q) \vdash \mathit{false}$ whenever $a \in N(q)$;
(A3) $a \in A(q) \vdash a^* \in A(q)$;
(A4) $a + b \in A(r + s) \vdash a \in A(r) \vee b \in A(s)$;
(A5) $a \in A(r) \wedge b \in A(s) \vdash ab \in A(rs)$;
(A6) $ab \in A(rs) \vdash a \in A(r) \vee b \in A(s)$;
(A7) $a \in A(r) \wedge b \in A(s) \vdash aa^* + bb^* \in A\left(r^2 + s^2\right)$;
(A8) $a \in A(q) \dashv\vdash \bigvee_{q' > q} a \in A(q')$.

Then denote by Max A, the maximal spectrum of the commutative C*-algebra A, the Lindenbaum locale of this theory, that is, the locale of propositions derived from the primitive propositions of the theory by applying finite conjunctions and arbitrary disjunctions, ordered by provable entailment in the theory, modulo provable equivalence.

By way of motivation, it may be recalled that in any commutative C*-algebra A the maximal ideals are exactly the prime ideals that are closed with respect to the norm. Because we are working constructively these are naturally axiomatised in terms of their complements: the theory described is thus more properly that of an *open prime* of A, whose complement is then a maximal ideal of A. The primitive proposition

$$a \in A(q)$$

is therefore to be interpreted as asserting that the element $a \in A$ is to be assigned a coseminorm that is $> q$, yielding contrapositively a seminorm on A of which the kernel is a maximal ideal, for a more detailed discussion of which the reader is referred to [7].

Anticipating the assertion of the Gelfand–Mazur theorem, asserting constructively that the quotient algebra determined by this seminorm is in fact the commutative C*-algebra $\mathbb{C}_{\mathbb{E}}$ of complex numbers in the Grothendieck topos $\mathbb{E}$, the proposition

$$a \in A(q)$$

may therefore be considered to assert that the element $a \in A$ will be mapped under the quotient homomorphism into the complement $A(q)$ in the complex plane of the closed disc of radius q, motivating the interpretation of the theory $\mathbb{M}$ax A in the theory $\mathbb{MF}$n A of multiplicative linear functionals on A which we now outline. Again, for a more detailed discussion the reader is referred to [7].

So, consider the interpretation of the theory $\mathbb{M}$ax A in the theory $\mathbb{MF}$n A obtained by assigning to the primitive proposition

$$a \in A(q)$$

the proposition

$$\bigvee_{(r,s) \lhd A(q)} a \in (r, s)$$

for each $a \in A$ and non-negative rational q. Observe in passing that, by the notational convention introduced in discussing the preceding theorem, this proposition is exactly that asserting that the element $a \in A$ is mapped into the open subset $A(q)$ of the complex plane described above. In other words, this interpretation is intuitively a canonical one of the theory of the maximal spectrum in that of the spectrum of the commutative C*-algebra A. Indeed, it may be verified that this interpretation validates the axioms of $\mathbb{M}$ax A in the spectrum of A, and hence determines a map of locales

$$\text{MFn } A \to \text{Max } A$$

of which the inverse image homomorphism is that induced by the assignment. Concerning this map of locales one then has the following constructive form of the Gelfand–Mazur theorem:

Theorem 3.2. *For any commutative* C*-*algebra A in a Grothendieck topos $\mathbb{E}$, the canonical map*

$$\text{MFn } A \to \text{Max } A$$

is an isomorphism of compact, completely regular locales in the topos $\mathbb{E}$.

Again, the reader is referred to the earlier paper [7] for a detailed discussion of the proof. It will suffice here to note that the proof given there depends on the existence of a Barr covering for any Grothendieck topos, and hence is constructive only to that extent. Subsequently a constructive proof has been outlined in work of Mulvey and Vermeulen, depending on the observation concerning the geometry of the complex plane on which the present proof also depends, namely that, in any bounded region of the complex plane, the open subsets obtained by translation of those of the form

$$A(q)$$

into open codiscs centred on any complex rational point within the region also form a subbasis for the topology of the complex plane. It is the geometric content of this argument, involving the lattice structure on the self-adjoint elements of the commutative C*-algebra A, that may be found in the earlier paper [7], together with a proof, independently of that given by the existence of this isomorphism, of the fact that the maximal spectrum Max A is a compact, completely regular locale.

For the moment, we note only that henceforth we shall identify the maximal spectrum with the spectrum introduced earlier, denoting it throughout the remainder of the paper by

$$\text{Max } A.$$

It will be seen in what follows that Gelfand duality depends critically on the equivalence of these descriptions of the spectrum, allowing the aspects which evolve out of the lattice structure on the self-adjoint elements of a commutative C*-algebra through the description of the maximal spectrum to interact with the Gelfand representation which arises out of the earlier description of the spectrum and the continuity principle to which it gave rise. However, it may be observed once again before proceeding that this interaction itself expresses the fundamental geometric fact that the topology of the complex plane may be determined equivalently by nearness, in terms of the open subsets (r, s), and by awayness, in terms of translates of the open subsets $A(q)$ (cf. [9]).

4. The Gelfand representation

Given a compact, completely regular locale M in the Grothendieck topos $\mathbb{E}$, denote by $\mathbb{C}(M)$ the set of maps of locales

$$\alpha : M \to \mathbb{C}$$

from the locale M to the locale $\mathbb{C}$ of complex numbers in $\mathbb{E}$. Because $\mathbb{C}$ is straightforwardly seen to be a commutative *-algebra in the category of locales, it follows that $\mathbb{C}(M)$ is also a commutative *-algebra. Define a seminorm on $\mathbb{C}(M)$ by assigning to each positive rational q the subset

$$N(q) = \{\alpha \in \mathbb{C}(M) \mid 1_M \leq \alpha^*(N(q))\},$$

obtained by taking those continuous complex functions for which the inverse image of the open subset $N(q)$ of the complex plane is the top element of the locale M. It may be verified straightforwardly that this makes $\mathbb{C}(M)$ into a commutative seminormed *-algebra in the topos $\mathbb{E}$. Noting that $\mathbb{C}(M)$ is exactly the global sections of the sheaf $\mathbb{C}_M$ of complex numbers (Mulvey [18]) in the topos of sheaves on the compact, completely regular locale M over $\mathbb{E}$, it follows that $\mathbb{C}(M)$ is complete, by the completeness of $\mathbb{C}_M$ in the topos of sheaves on M. Hence, $\mathbb{C}(M)$ is a commutative Banach *-algebra in $\mathbb{E}$, which can then be verified straightforwardly to be a commutative C*-algebra in $\mathbb{E}$.

Now, for any commutative C*-algebra A consider the commutative C*-algebra $\mathbb{C}(\text{Max } A)$ of continuous complex functions on its spectrum Max A. For each $a \in A$, define a map of locales

$$\hat{a} : \text{Max } A \to \mathbb{C}$$

by assigning to each rational open rectangle (r, s) of the locale $\mathbb{C}$ the proposition

$$a \in (r, s)$$

of the spectrum of the commutative C*-algebra A. Observe that this indeed determines a map of locales, for, by the continuity principle,

$$a \in U \wedge a \in U' \vdash\dashv a \in U \wedge U',$$
$$\bigvee_i a \in U_i \vdash\dashv a \in \bigvee_i U_i, \text{ and}$$
$$\mathit{true} \vdash a \in \mathbb{C}$$

are provable in the theory of the spectrum of A.

Moreover, the mapping

$$\hat{} : A \to \mathbb{C}\,(\text{Max } A)$$

which assigns to each $a \in A$ its Gelfand transform is indeed a map of seminormed *-algebras in the topos $\mathbb{E}$. For the algebraic operations of the involutive algebra A, one observes firstly that the axiomatization of the theory of Max A allows one to prove straightforwardly that zero, identity, involution, scalar multiplication and multiplication are preserved by the Gelfand representation. Given $a, a' \in A$, one sees, for instance, that the Gelfand transform of their product is equal to the product of their transforms, by noting that for each rational open rectangle (r, s) of the complex plane, its inverse image in Max A under the transform of $aa' \in A$ is given by the proposition

$$aa' \in (r, s).$$

Now, by the axiom (M9) of the theory of Max A, one has that

$$aa' \in (r, s) \dashv\vdash \bigvee a \in (p, q) \wedge a' \in (p', q'),$$

in which the disjunction is taken over all rational open rectangles for which

$$(p, q) \times (p', q') \lhd \mu^* (r, s),$$

with respect to the multiplication map

$$\mu : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

of the locale $\mathbb{C}$. However, this disjunction is exactly that describing the inverse image of (r, s) under the product of the Gelfand transforms of $a, a' \in A$. Hence, the Gelfand representation is a multiplicative homomorphism. That the other algebraic operations of A, with the exception of addition, are preserved follows similarly.

Algebraically, it remains only to show that the representation is an additive homomorphism. For this, a little more subtlety is needed, because the axiomatisation of additivity in the theory of Max A is rather less explicit than that of multiplication. The axiom which one has, which is to say (M3), implies that

$$a \in (p, q) \wedge a' \in (p', q') \vdash a + a' \in (r, s)$$

whenever

$$(p, q) \times (p', q') \lhd \alpha^* (r, s),$$

with respect to the addition map

$$\alpha : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

of the locale $\mathbb{C}$. However, it is now necessary to show that taking the disjunction over all pairs $(p, q), (p', q')$ of rational open rectangles satisfying this condition with respect to (r, s) allows this entailment to become a provable equivalence in the theory.

Given $a, a' \in A$, we remark firstly that there exist positive rationals t, t' for which $a \in N(t)$, $a' \in N(t')$ in the seminormed *-algebra A. Applying axioms (M2) and (M4) of the theory of Max A, it follows that

$$\mathit{true} \vdash a \in N(t),$$

and similarly

$$\mathit{true} \vdash a' \in N(t').$$

Observing that these open discs in the complex plane lie inside open squares, and that these open squares may be subdivided into smaller open squares of arbitrary mesh, one may prove that

$$\mathit{true} \vdash \bigvee a \in (p, q),$$

in which the disjunction is taken over open rectangles $(p, q) \lhd N(t)$ of size less than any preassigned amount. Similarly, one has that

$$true \vdash \bigvee a' \in (p', q')$$

in which each $(p', q') \lhd N(t')$ has size less than the required amount.

Now, given any rational open rectangle (r, s), one has that

$$a + a' \in (r, s) \vdash \bigvee_{(r',s') \lhd (r,s)} a + a' \in \left(r', s'\right)$$

is provable, by axiom (M8). Taking any $\left(r', s'\right) \lhd (r, s)$, we observe that one may choose $\varepsilon > 0$ to be such that any rational open rectangle (u, v) of length and breadth less than ε will be disjoint from (r', s'), unless one has that

$$(u, v) \lhd (r, s).$$

Intuitively, one chooses $\varepsilon > 0$ to be the size of the gap between (r', s') and (r, s), and observes that the rather below relation amongst open rectangles in the rational complex plane is describable algebraically in terms of the rationals, and hence is decidable.

Choose now to consider only rational open rectangles (p, q), (p', q') of one half this size, and observe that

$$true \vdash \bigvee a \in (p, q) \wedge a' \in (p', q')$$

is provable in the theory, in view of the above remarks. Hence,

$$a + a' \in \left(r', s'\right) \vdash \bigvee a \in (p, q) \wedge a' \in (p', q'),$$

taken over all these rational open rectangles. However, for any rectangles $(p, q), (p', q')$, one has that

$$(p, q) \times (p', q') \lhd \alpha^* (r, s)$$

is equivalent to

$$(p + p', q + q') \lhd (r, s).$$

Unless this condition is satisfied, it will follow that the rectangle obtained will be disjoint from (r', s'), by the choice of the size of these rectangles. Now observe that

$$a \in (p, q) \wedge a' \in (p', q') \vdash a + a' \in (p + p', q + q')$$

by the axiom (M3), and that

$$a + a' \in \left(r', s'\right) \wedge a + a' \in (p + p', q + q') \vdash false$$

in the case that the rectangle $(p + p', q + q')$ is disjoint from (r', s'). However, this will be the case unless one see that

$$(p, q) \times (p', q') \lhd \alpha^* (r, s) ,$$

by the remarks above. Hence, the entailment

$$a + a' \in \left(r', s'\right) \vdash \bigvee a \in (p, q) \wedge a' \in (p', q')$$

remains provable when the disjunction is taken over only those rectangles which satisfy this condition. Applying axiom (M6), one obtains that

$$a + a' \in (r, s) \vdash \bigvee a \in (p, q) \wedge a' \in (p', q'),$$

which completes the proof of the additivity of the Gelfand representation. One therefore has that

$$\hat{} : A \to \mathbb{C}\,(\text{Max } A)$$

is indeed a map of commutative *-algebras in the topos $\mathbb{E}$.

Observe now that the seminorm on the commutative C*-algebra $\mathbb{C}\,(\text{Max } A)$ is defined by taking the open ball $\hat{N}(q)$ of radius q to consist of all maps

$$\alpha : \text{Max } A \to \mathbb{C}$$

of locales, for which

$$1_{\text{Max } A} \leq \alpha^*(N(q)),$$

in which $N(q)$ denotes the open disc of radius q in the complex plane. Then it follows that one has that

$$\hat{a} \in \hat{N}(q) \quad \text{if, and only if,} \quad \textit{true} \vdash a \in N(q)$$

is provable in the theory of the spectrum Max A for any $a \in A$ and any positive rational q. Noting that in the theory $\mathbb{M}$Fn A one has that

$$\textit{true} \vdash a \in N(q)$$

is provable whenever $a \in N(q)$, by axioms (M2) and (M3), we see that:

$$a \in N(q) \quad \text{implies} \quad \hat{a} \in \hat{N}(q).$$

Hence, the *Gelfand representation*

$$\hat{} : A \to \mathbb{C}\,(\text{Max } A)$$

is a map of seminormed *-algebras, and so of commutative C*-algebras, in the topos $\mathbb{E}$, concerning which may be proved the following:

Theorem 4.1. *For any commutative* C*-*algebra* A *in a Grothendieck topos* $\mathbb{E}$*, the Gelfand representation*

$$\hat{} : A \to \mathbb{C}\,(\text{Max } A)$$

*is an isometric *-homomorphism.*

Proof. Evidently it remains only to prove the converse of the preceding remark, namely that

$$\textit{true} \vdash a \in N(q) \quad \text{implies} \quad a \in N(q),$$

for any $a \in A$ and any positive rational q, establishing the isometricity of the representation. To prove this assertion, we consider a Barr covering [8,15]

$$\gamma : \mathbb{B} \to \mathbb{E}$$

of the Grothendieck topos $\mathbb{E}$ by a Grothendieck topos $\mathbb{B}$ in which the Axiom of Choice, and hence the Law of the Excluded Middle, is satisfied. For further details of this kind of argument, in particular the observation that the inverse image $\gamma^* A$ of the commutative C*-algebra A is then only a commutative pre-C*-algebra in the topos $\mathbb{B}$, we refer the reader to the earlier papers [6,7]. The argument then runs in the following way: assume that

$$\mathit{true} \vdash a \in N(q)$$

is provable in the theory $\mathbb{M}\mathrm{ax}\ A$. Then certainly

$$\mathit{true} \vdash \gamma^* a \in N(q)$$

is provable in the theory $\mathbb{M}\mathrm{ax}\ \gamma^* A$ determined by the inverse image of the seminormed *-algebra A. Consider now the canonical homomorphism

$$\gamma^* A \to B$$

of the seminormed *-algebra $\gamma^* A$ into its completion, which is then a commutative C*-algebra in $\mathbb{B}$. The canonical homomorphism induces a map

$$\mathrm{Max}\ B \to \mathrm{Max}\ \gamma^* A$$

of locales, along which one concludes that

$$\mathit{true} \vdash \gamma^* a \in N(q)$$

is provable in the theory $\mathbb{M}\mathrm{ax}\ B$.

For the commutative C*-algebra B in the topos $\mathbb{B}$, in which the Axiom of Choice is satisfied, one has that $\mathrm{Max}\ B$ is exactly the lattice of open subsets of the spectrum of B in the classical sense. Hence, one may conclude that $\gamma^* a \in N(q)$ in the commutative C*-algebra B, by the isometricity of the Gelfand representation in that context. However, the canonical mapping from $\gamma^* A$ into its completion B is isometric, so that $\gamma^* a \in N(q)$ in the seminormed *-algebra $\gamma^* A$. Hence, one has that $a \in N(q)$ in the commutative C*-algebra A, because the seminormed structure on the inverse image $\gamma^* A$ is the inverse image of that on A. The condition that

$$\mathit{true} \vdash a \in N(q) \quad \text{implies} \quad a \in N(q)$$

is therefore satisfied, so that the Gelfand representation

$$\hat{}\ : A \to \mathbb{C}\,(\mathrm{Max}\ A)$$

is necessarily isometric, which completes the proof. □

Hence, the Gelfand representation yields an isometric *-isomorphism from A to a closed *-subalgebra of the commutative C*-algebra $\mathbb{C}\,(\mathrm{Max}\ A)$ of continuous complex functions on the compact, completely regular locale $\mathrm{Max}\ A$, concerning which we need now to make a number of observations. Before doing so, recall that, in proving the Gelfand–Mazur theorem that the canonical map

$$\text{MFn } A \to \text{Max } A$$

is an isomorphism of locales in the Grothendieck topos $\mathbb{E}$, it was observed [3,7] that the propositional geometric theory of the maximal spectrum could also be expressed in a way that made it more exactly that of closed prime ideals of the commutative C*-algebra A. To make this explicit, denote for each $a \in A$ by

$$a \in P$$

the proposition $a \in A(0)$ in the theory of Max A, noting that once the notational conventions following the proof of the Gelfand–Mazur theorem have been established this notation equivalently expresses that the multiplicative linear functional corresponding to any maximal ideal intuitively maps the element $a \in A$ into the open subset P of the complex plane obtained by removing zero.

Applying the axioms of the theory of Max A it is straightforward to verify that the following conditions relating these propositions are satisfied:

(P1) $\mathit{true} \vdash 1 \in P$;
(P2) $0 \in P \vdash \mathit{false}$;
(P3) $a + b \in P \vdash a \in P \vee b \in P$;
(P4) $ab \in P \dashv\vdash a \in P \wedge b \in P$,

which are exactly the axioms of the theory of the prime spectrum Spec A of the commutative ring A, together with an additional axiom

$$\text{(I)} \quad a \in P \dashv\vdash \bigvee_q a \in A(q),$$

where the disjunction is taken over the positive rationals q, relating these propositions to those already considered, and intuitively requiring the prime ideal described to be closed.

In consequence, there is a canonical interpretation of the theory of Spec A in the theory of Max A, and hence a canonical map of locales

$$\text{Max } A \to \text{Spec } A$$

which embeds the maximal spectrum of the commutative C*-algebra A as a sublocale of the prime spectrum of the commutative ring A, for which it may be shown that there exists a map of locales

$$\text{Spec } A \to \text{Max } A$$

giving a retraction, intuitively equivalent to assigning to each prime ideal of A the maximal ideal that is its closure.

Assigning to each element $a \in A$ the element

$$|a| = \left(aa^*\right)^{\frac{1}{2}},$$

that is its absolute value, allows one to show that for any non-negative rational q

$$a \in A(q) \dashv\vdash |a| \in A(q)$$

is provable in the theory. Moreover, for any element of the positive cone of the commutative C*-algebra A, one has that

$$a \in A(q) \dashv\vdash (a - q1)^+ \in P$$

is provable in the theory. As a consequence, the theory of the maximal spectrum Max A may be expressed entirely in terms of propositions of the form

$$a \in P$$

for each element $a \in A$. Indeed, since, by virtue of the axiom (P4), any finite conjunction of these propositions is again a proposition of this form, it follows that any open subset of the locale Max A is obtained as a join of those corresponding to primitive propositions of the form $a \in P$.

Now, denoting for each $a \in A$ the open subset of Max A obtained by taking the inverse image under the Gelfand transform of $a \in A$ of the open subset P of the complex plane by

$$D(a),$$

one may recall the following:

Definition 4.1. A subalgebra A of the commutative C*-algebra $\mathbb{C}(M)$ is said to *separate* the compact, completely regular locale M provided that each open set U of the locale may be expressed in the form

$$U = \bigvee D(a)$$

taken over those elements $a \in A$ for which $D(a)$ is contained in U.

By the observations above, together with the constructive version of the Stone–Weierstrass theorem (Banaschewski–Mulvey [5]) to which they lead, we therefore have the following:

Corollary 4.2. *For any commutative* C*-*algebra A in a Grothendieck topos $\mathbb{E}$, the Gelfand representation*

$$\hat{} : A \to \mathbb{C}\,(\mathrm{Max}\ A)$$

*is an isometric *-isomorphism from A to the commutative* C*-*algebra* $\mathbb{C}\,(\mathrm{Max}\ A)$ *of continuous complex functions on the spectrum* Max A.

Proof. Consider the inverse image along the Gelfand transform

$$\hat{a} : \mathrm{Max}\ A \to \mathbb{C}$$

of the open subset P of the complex plane obtained by removing zero. Since this open subset is the join of those rational open rectangles (r, s) which do not contain zero, it follows that the inverse image of P is exactly the join of the inverse images of these rectangles. However, by the Gelfand–Mazur Theorem the join of the inverse images $a \in (r, s)$ of these rational open rectangles is the open subset determined by the proposition $a \in P$ of the theory Max A, while the inverse image of the open subset P is by definition the open subset $D(a)$. Hence, $D(a)$ is the open subset determined by the proposition $a \in P$ of the theory $\mathbb{M}$ax A.

But in the theory $\mathbb{M}$ax A, it has already been proved that any proposition is provably equivalent to the disjunction of those propositions $a \in P$ which entail it. Thus, each open subset U of Max A is the join of those open subsets which are contained in it. So, the closed C*-subalgebra that is the image of the isometric *-homomorphism

$$\hat{} : A \to \mathbb{C}\,(\text{Max } A)$$

separates the compact, completely regular locale Max A. Hence, by the Stone–Weierstrass theorem proved constructively in [5], this closed C*-subalgebra is exactly the commutative C*-algebra $\mathbb{C}\,(\text{Max } A)$, which completes the proof. □

5. Gelfand duality

To each commutative C*-algebra A in the topos $\mathbb{E}$ there has been assigned a compact, completely regular locale Max A in $\mathbb{E}$. Consider now a map

$$\varphi : A \to B$$

of commutative C*-algebras, which is to say a *-homomorphism with the property that

$$a \in N(q) \to \varphi\,(a) \in N(q)$$

for each $a \in A$ and each positive rational q. Define a map

$$\text{Max}\,\varphi : \text{Max } B \to \text{Max } A$$

of locales, by assigning to each proposition

$$a \in (r, s)$$

of the theory of Max A the proposition

$$\varphi\,(a) \in (r, s)$$

of the theory of Max B. Observe that it follows immediately from the fact that $\varphi : A \to B$ is a seminormed *-homomorphism that each axiom of the theory of Max A is validated in the locale Max B under this interpretation. Moreover, it is immediate that the assignment is functorial on the category of commutative C*-algebras, yielding a functor

Commutative C*-algebras → Compact, completely regular locales$^{\text{op}}$.

To each compact, completely regular locale M in the topos $\mathbb{E}$ there has been assigned a commutative C*-algebra $\mathbb{C}(M)$ in $\mathbb{E}$. Consider a map

$$\psi : L \to M$$

of compact, completely regular locales, and define a map

$$\mathbb{C}(\psi) : \mathbb{C}(M) \to \mathbb{C}(L)$$

of commutative C*-algebras, by mapping each continuous complex function

$$\alpha : M \to \mathbb{C}$$

to that defined on L by composition with the map $\psi : L \to M$ of locales. Because the commutative C*-algebra $\mathbb{C}(M)$ inherits its algebraic structure from the locale $\mathbb{C}$, it is immediate that this is a map of commutative *-algebras. Recalling that the seminorm on $\mathbb{C}(M)$ is defined by requiring that

$$\alpha \in N(q) \quad \text{if, and only if,} \quad 1_M \leq \alpha^* N(q)$$

it is immediate also that one has a map of seminormed *-algebras. Moreover, one sees that the assignment is evidently functorial on the category of compact, completely regular locales, yielding a functor:

$$\textbf{Compact, completely regular locales}^{\text{op}} \to \textbf{Commutative C*-algebras}.$$

Concerning these functors one has the following:

Theorem 5.1. *In any Grothendieck topos $\mathbb{E}$, the functors*

$$\textbf{\textit{Commutative C*-algebras}} \rightleftarrows \textbf{\textit{Compact, completely regular locales}}^{op}$$

determine a duality between the category of commutative C*-*algebras in $\mathbb{E}$ and the category of compact, completely regular locales in $\mathbb{E}$.*

Proof. It will be proved that the duality is in fact an adjoint duality. For any commutative C*-algebra A, the adjunction

$$A \to \mathbb{C}\,(\text{Max } A)$$

is the Gelfand representation. It may be remarked that this is natural. For, given any map

$$\varphi : A \to B$$

of commutative C*-algebras, the diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{C}\,(\text{Max } A) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\mathbb{C}(\text{Max } \varphi)} \\ B & \longrightarrow & \mathbb{C}\,(\text{Max } B) \end{array}$$

is commutative. To each $a \in A$, one now assigns firstly the continuous complex function

$$\hat{a} : \text{Max } A \to \mathbb{C},$$

then its composite with the map

$$\text{Max } \varphi : \text{Max } B \to \text{Max } A$$

of locales. The inverse image of a rational open rectangle (r, s) of the locale $\mathbb{C}$ is firstly $a \in (r, s)$, and then $\varphi(a) \in (r, s)$. This is exactly the inverse image of the rational open rectangle (r, s) under the map

$$\widehat{\varphi(a)} : \text{Max } B \to \mathbb{C}$$

of locales obtained by passing round the diagram the other way. Hence, the Gelfand representation

$$A \to \mathbb{C}(\text{Max } A)$$

is natural in the commutative C*-algebra A. Of course, the map is actually an isometric *-isomorphism for each commutative C*-algebra A, by the Stone–Weierstrass theorem applied in the corollary to the theorem of the previous section; hence, one has that the natural map is actually a natural isomorphism.

Now, for any compact, completely regular locale M, consider the map

$$M \to \text{Max } \mathbb{C}(M)$$

of locales, defined by assigning to each proposition $\alpha \in (r, s)$ determined by a continuous complex function

$$\alpha : M \to \mathbb{C},$$

the inverse image $\alpha^*(r, s)$ of the rational open rectangle (r, s) along this continuous complex function. It will be shown firstly that this indeed determines a map of locales. For each axiom of the theory of the locale Max A, it will be shown that the corresponding relation is satisfied in the locale M:

(A1): It must be proved that

$$1_M \le 1^*(A(q))$$

for all $q < 1$, in which the identity element of the commutative C*-algebra $\mathbb{C}(M)$ is given by the map

$$1 : M \to \mathbb{C}$$

of locales defined by

$$1^*(r, s) = \begin{cases} 1_M & \text{whenever } 1 \in (r, s), \\ 0_M & \text{otherwise.} \end{cases}$$

But whenever $q < 1$, one has $1 \in A(q)$, and hence there exists a rational open rectangle (r, s) for which $1 \in (r, s) \le A(q)$. Thus, $1_M \le 1^*(A(q))$, since $A(q)$ is the join of those rational open rectangles contained in it.

(A2): Conversely, it must be shown that

$$a^*(A(q)) \le 0_M$$

whenever the continuous complex function $a \in \mathbb{C}(M)$ lies in the open ball $N(q)$ of the seminormed algebra $\mathbb{C}(M)$. But $a \in N(q)$ means that $a^*(N(q)) = 1_M$. However, one has that $N(q) \wedge A(q) = 0$ in the locale $\mathbb{C}$ of complex numbers, and hence that

$$a^*(N(q)) \wedge a^*(A(q)) = 0_M$$

in the locale M. Hence, $a^*(N(q)) = 0_M$ in the locale M.

(A3): For any continuous complex function $a \in \mathbb{C}(M)$, one has that

$$a^*(A(q)) \leq \bar{a}^*(A(q))$$

for each positive rational q, in which the involution of the commutative C*-algebra $\mathbb{C}(M)$ is denoted by conjugation. But, for each rational open rectangle $(r, s) \leq A(q)$, it is also the case that $\overline{(r, s)} \leq A(q)$. Hence, because conjugation is defined by requiring that $\bar{a}^*(r, s) = a^*\overline{(r, s)}$ for each rational open rectangle (r, s) in the locale $\mathbb{C}$, the required inequality holds in the locale M.

(A4): It must next be shown that

$$(a + b)^* (A (r + s)) \leq a^* (A (r)) \wedge b^* (A (s))$$

for all $a, b \in \mathbb{C}(M)$ and any positive rationals r, s. Again, this will be deduced from an equivalent assertion concerning the map

$$\alpha : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

defining addition on the locale $\mathbb{C}$, namely that

$$\alpha^* (A (r + s)) \leq A (r) \times \mathbb{C} \vee \mathbb{C} \times A(s)$$

for all $a, b \in \mathbb{C}(M)$ and any positive rationals r, s. To prove this, we remark firstly that its dual form, that

$$N (r) \times N (s) \leq \alpha^* (N (r + s)),$$

may be proved directly by expanding in terms of rational open rectangles. Specifically, it may be shown that

$$\bigvee (p, q) \times (p', q') \leq \bigvee (u, v) \times \left(u', v'\right),$$

in which the first disjunction is taken over all $(p, q) \leq N(r)$ and $(p', q') \leq N(s)$, while in the second one consider all rectangles for which $(u, v) + \left(u', v'\right) \leq N (r + s)$.

It may be remarked that the inequalities considered are those provable algebraically concerning the lattice of subsets of the space of rational complex numbers, in this case by requiring that the vertices of the rectangles concerned have moduli less than the given positive rationals, while the operation of addition introduced is that defined algebraically on rational complex numbers. However, one then has that $(p, q) + (p', q') \leq N (r + s)$ whenever $(p, q) \leq N(r)$ and $(p', q') \leq N(s)$. Hence, each term in the disjunction on the left of the inequality appears also on the right-hand side, establishing the required condition in the locale $\mathbb{C} \times \mathbb{C}$.

Now, to deduce the inequality for $\alpha^* (A (r + s))$ which is dual to this, observe that given any $r' > r, s' > s$, one has that

$$N \left(r'\right) \vee A (r) = 1 \quad \text{and} \quad N \left(s'\right) \vee A (s) = 1$$

in the locale $\mathbb{C}$. From these, it follows that

$$N \left(r'\right) \times \mathbb{C} \vee A (r) \times \mathbb{C} = 1 \quad \text{and} \quad \mathbb{C} \times N \left(s'\right) \vee \mathbb{C} \times A (s) = 1$$

in the locale $\mathbb{C} \times \mathbb{C}$. Taking the meet of the second with the first term of the first, one obtains that

$$N \left(r'\right) \times N \left(s'\right) \vee N \left(r'\right) \times A (s) \vee A (r) \times \mathbb{C} = 1.$$

Since one has that $N \left(r'\right) \times A (s) \leq \mathbb{C} \times A (s)$, one then obtains that

$$1 = N \left(r'\right) \times A (s) \vee A (r) \times \mathbb{C} \vee \mathbb{C} \times A(s)$$

in the locale $\mathbb{C} \times \mathbb{C}$. However, one has also that $N \left(r' + s'\right) \wedge A \left(r' + s'\right) = 0$ in the locale $\mathbb{C}$, from which it follows that

$$\alpha^* \left(N \left(r' + s'\right)\right) \wedge \alpha^* \left(A \left(r' + s'\right)\right) = 0$$

in the locale $\mathbb{C} \times \mathbb{C}$, and hence that

$$N \left(r'\right) \times N \left(s'\right) \wedge \alpha^* \left(A \left(r' + s'\right)\right) = 0$$

by the inequality already established. Taking the meet of $\alpha^* \left(A \left(r' + s'\right)\right)$ with the expression already obtained for the identity of $\mathbb{C} \times \mathbb{C}$, we obtain finally that

$$\alpha^* \left(A \left(r' + s'\right)\right) \leq A (r) \times \mathbb{C} \vee \mathbb{C} \times A(s).$$

Taking the join of the expression over all $r' > r, s' > s$, and observing that

$$\bigvee_{r'>r,s'>s} A \left(r' + s'\right) = A (r + s),$$

the required condition follows on applying the inverse image of

$$\alpha : \mathbb{C} \times \mathbb{C} \to \mathbb{C},$$

together with the inequality just established. Hence,

$$\alpha^* (A (r + s)) \leq A (r) \times \mathbb{C} \vee \mathbb{C} \times A(s)$$

for all positive rationals r, s. Finally, applying the inverse image of the map

$$M \to \mathbb{C} \times \mathbb{C}$$

into the product locale, determined by the pair of continuous complex functions $a, b \in \mathbb{C}(M)$, yields the required inequality in the locale M.

(A5): It is required to prove that

$$a^* (A (r)) \times b^* (A (s)) \leq (ab)^* (A (rs))$$

for any $a, b \in \mathbb{C}(M)$ and any positive rationals r, s. Once again, it is enough to prove an equivalent fact for the locale $\mathbb{C} \times \mathbb{C}$, in this case, that

$$A(r) \times A(s) \leq \mu^* (A(rs))$$

for all positive rationals r, s, in which

$$\mu : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

denotes the multiplication of the locale of complex numbers. However, this may be proved directly, rather than by dualising first, by expanding in terms of rational open rectangles and noting that

$$(p, q) \cdot (p', q') \leq A(rs)$$

whenever $(p, q) \leq A(r)$ and $(p', q') \leq A(s)$. Again, these are inequalities considered algebraically in the lattice of subsets of the rational complex plane, proved using the properties of the modulus of a complex number. Applying the inverse image of the map from M to the product locale $\mathbb{C} \times \mathbb{C}$, induced by the continuous complex functions $a, b \in \mathbb{C}(M)$, one deduces the required inequality from that already obtained.

(A6): The fact that

$$(ab)^* (A(rs)) \leq a^* (A(r)) \vee b^* (A(s))$$

in the locale M is deduced from the equivalent assertion that

$$\mu^* (A(rs)) \leq a^* (A(r)) \vee b^* (A(s))$$

in the locale $\mathbb{C} \times \mathbb{C}$, which is proved in turn from its dual assertion, that

$$N(r) \times N(s) \leq \mu^* (N(rs)),$$

in a manner identical to that argued already in the case when addition replaced multiplication.

(A7): It must now be proved that

$$a^* (A(r)) \times b^* (A(s)) \leq (a\bar{a} + b\bar{b})^* \left(A\left(r^2 + s^2\right)\right)$$

for all continuous complex functions $a, b \in \mathbb{C}(M)$. Once again, this corresponds to an equivalent fact about the map

$$\gamma : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

of locales which, intuitively, assigns to each pair of complex numbers the sum of the squares of their moduli. In this case, it is that

$$A(r) \times A(s) \leq \gamma^* \left(A\left(r^2 + s^2\right)\right)$$

for all positive rationals r, s. The proof of this inequality again is found straightforwardly by expanding in terms of rational open rectangles and noting that

$$(p, q) \oplus (p', q') \leq A\left(r^2 + s^2\right)$$

whenever $(p, q) \leq A(r)$ and $(p', q') \leq A(s)$, in which $\oplus$ denotes the operation of summing squares of moduli. Applying the inverse image of the map

$$M \to \mathbb{C} \times \mathbb{C}$$

of locales determined by the continuous complex functions $a, b \in \mathbb{C}(M)$ then yields the required result.

(A8): Finally, it must be shown that

$$a^*(A(q)) = \bigvee_{q'>q} a^*(A(q'))$$

for each $a \in \mathbb{C}(M)$, which follows from the observation that

$$A(q) = \bigvee_{q'>q} A(q')$$

in the complex plane on applying the inverse image of the continuous complex function $a \in \mathbb{C}(M)$, which completes the proof of this part of the result.

It may be remarked that the map

$$M \to \text{Max } \mathbb{C}(M)$$

thus defined is natural in the compact, completely regular locale M. For, given any map

$$\varphi : L \to M$$

of compact, completely regular locales, the diagram

$$\begin{array}{ccc} L & \longrightarrow & \text{Max } \mathbb{C}(L) \\ \varphi \downarrow & & \downarrow \text{Max } \mathbb{C}(\varphi) \\ M & \longrightarrow & \text{Max } \mathbb{C}(M) \end{array}$$

may be seen to commute. Considering the inverse images of the maps involved, one sees that on the one hand any proposition

$$a \in (r, s)$$

of the theory of Max $\mathbb{C}(M)$ is mapped firstly to the element $a^*(r, s)$ of the locale M, and thence to the element $\varphi^*a^*(r, s)$ of the locale L, while, on the other hand, the proposition is mapped firstly to that of the theory of Max $\mathbb{C}(L)$ obtained by composing the continuous complex function with the map

$$\varphi : L \to M$$

of locales, then taking the inverse image of the rational open rectangle (r, s) in the locale $\mathbb{C}$ along the continuous complex function on L which results. One again obtains the element

$$\varphi^*a^*(r, s)$$

of the locale L. The diagram therefore commutes, since the inverse image mappings concerned agree on the propositions which generate the locale Max $\mathbb{C}(M)$.

It will now be shown that the map

$$M \to \text{Max } \mathbb{C}(M)$$

is actually an isomorphism in the category of compact, completely regular locales. To prove this, it is enough to show that the map is a dense embedding, given that the locales concerned are compact, completely regular. To show that it is an embedding, it must be proved that the inverse image mapping is surjective. Given any open subset U of the locale M, by complete regularity one has that

$$U = \bigvee_{V \lhd\!\lhd U} V.$$

However, given any open subset $V \lhd\!\lhd U$, there exists an interpolation by open subsets (V_q) indexed by the rationals $0 \le q \le 1$, for which

$$V_p \lhd V_q$$

whenever $p < q$, and for which

$$V = V_0 \quad \text{and} \quad V_1 = U.$$

By the idea central to the proof of Urysohn's Lemma, one may then find a continuous complex function

$$a_V : M \to \mathbb{C}$$

for which $a_V^*(W_q) = V_q$ for each $0 \le q \le 1$, in which W_q denotes the open left half of the complex plane determined by the rational q. In particular, $a_V^*(W) = V$, in which W denotes the open left half-plane of $\mathbb{C}$. Then, since

$$U = \bigvee_{V \lhd\!\lhd U} V,$$

it follows that

$$a \in U \;\vdash\!\dashv\; \bigvee_{V \lhd\!\lhd U} a_V \in W$$

in the theory of Max $\mathbb{C}(M)$. The map

$$M \to \text{Max } \mathbb{C}(M)$$

is therefore an embedding.

It is also dense, in the sense that an open subset of Max $\mathbb{C}(M)$ is the zero element of the locale Max $\mathbb{C}(M)$ exactly if its inverse image is zero in the locale M. Observing that it is enough to prove this for a base of open subsets of Max $\mathbb{C}(M)$, it may be remarked that we have shown that the propositions

$$a \in P$$

of the theory $\mathbb{M}$ax $\mathbb{C}(M)$ for each $a \in \mathbb{C}(M)$ together determine such a base of open subsets for the locale Max $\mathbb{C}(M)$. It suffices therefore to prove that

$$a \in P \vdash \mathit{false}$$

is provable in the theory $\mathbb{M}\text{ax}\ \mathbb{C}(M)$ whenever $a^*P \leq 0_M$ in the locale M. But, recalling that P denotes the open complement of zero in the complex plane, one has that exactly implies that

$$\alpha : M \to \mathbb{C}$$

is the zero function. Then

$$a \in P \vdash \mathit{false}$$

because $0 \in P$ in the locale $\mathbb{C}$. The map

$$M \to \text{Max}\ \mathbb{C}(M)$$

is therefore dense.

But any dense embedding of compact, completely regular locales is necessarily an isomorphism, because the image of a compact locale in a completely regular locale is closed (Banaschewski–Mulvey [3]). The map of locales

$$M \to \text{Max}\ \mathbb{C}(M)$$

is therefore an isomorphism in the category of compact, completely regular locales, which establishes that the functors

$$\textbf{Commutative C*-algebras} \rightleftarrows \textbf{Compact, completely regular locales}^{\text{op}}$$

yield the duality asserted.

Finally, for any compact, completely regular locale M, the canonical map

$$\mathbb{C}(M) \to \mathbb{C}(\text{Max}\ \mathbb{C}(M)) \to \mathbb{C}(M)$$

sends a continuous complex function

$$a : M \to \mathbb{C}$$

to the map of locales

$$M \to \text{Max}\,\mathbb{C}(M) \xrightarrow{\hat{a}} \mathbb{C}.$$

For each rational open rectangle (r, s), one has that its inverse image along this map is that of the proposition $a \in (r, s)$ along the canonical map

$$M \to \text{Max}\ \mathbb{C}(M),$$

which yields exactly $a^*(r, s)$. Hence, one obtains the continuous complex function on M given exactly by the given map

$$a : M \to \mathbb{C}.$$

The composite considered is, therefore, the identity map on the commutative C*-algebra $\mathbb{C}(M)$.

Equally, for any commutative C*-algebra A, the canonical map

$$\text{Max}\ A \to \text{Max}\,\mathbb{C}(\text{Max}\ A) \to \text{Max}\ A$$

is also the identity map on the locale Max A. For its inverse image assigns to each proposition $a \in (r, s)$ of the theory $\mathbb{M}$ax A, firstly the proposition $\hat{a} \in (r, s)$ of the theory $\mathbb{M}$ax $\mathbb{C}$(Max A), and then the inverse image of (r, s) along the map

$$\hat{a} : \text{Max } A \to \mathbb{C}.$$

However, this is exactly the element of Max A determined by the proposition $a \in (r, s)$ of the theory $\mathbb{M}$ax A, by the definition of the Gelfand transform of any $a \in A$, which shows that the canonical map of locales is indeed the identity on the locale Max A.

Together, these identities establish the adjointness of the functors involved in the Gelfand duality, which completes the proof of the theorem. □

It is interesting to note that the proofs that the adjunction and the coadjunction of this adjointness are isomorphisms each have depended on an argument involving density of one kind or another. In the first case, the Stone–Weierstrass theorem depends ultimately on the complex rationals being dense in the complex numbers, while in the second it is the denseness of the canonical map from a locale to its compactification which gave the required result, depending itself finally on the denseness of the open subset P in the complex plane.

Finally, one may remark that the Gelfand duality proved extends that known classically:

Corollary 5.2. *In any Grothendieck topos* $\mathbb{E}$ *in which the Axiom of Choice is satisfied, the duality*

$$\textbf{\textit{Commutative C*-algebras}} \rightleftarrows \textbf{\textit{Compact, completely regular locales}}^{op}$$

is exactly that between the category of commutative C*-*algebras and the category of compact Hausdorff topological spaces.*

Proof. The concept of commutative C*-algebras is then the canonical one, for in the presence of the Axiom of Choice the seminorm may again be expressed in terms of a function

$$\| \ \| : A \to \mathbb{R}_{\mathbb{E}}$$

satisfying the seminorm conditions, whilst the concept of Cauchy approximation is equivalent to that of Cauchy sequence, by applying countable dependent choice to choose a sequence from an approximation. Moreover, every compact, completely regular locale is isomorphic to the lattice of open subsets of its space of points, which is indeed compact Hausdorff, and every compact Hausdorff topological space arises in this way for a unique compact, completely regular locale. □

6. Applications

In this last section, we shall outline some of the consequences of the existence of Gelfand duality, omitting many of the details concerned, either referring to existing results in the classical situation which may be adapted to the present context, or leaving a more detailed discussion to another place. We begin with a result which establishes categorically the algebraic nature of the maximal spectrum of a commutative C*-algebra.

It was remarked earlier that it may be proved (Banaschewski–Mulvey [3]) that:

For any commutative C-algebra A in a Grothendieck topos $\mathbb{E}$ there exists a retraction*

$$\text{Spec } A \to \text{Max } A$$

from the prime spectrum of A to the maximal spectrum of A.

This will now be shown, depending on the Gelfand duality just established for commutative C*-algebras in the topos $\mathbb{E}$ which allows us to assume that A is actually the algebra $\mathbb{C}\,(\text{Max } A)$ of continuous complex functions on Max A. Before doing this, we remark that the corresponding statement for commutative C*-algebras classically is equivalent to the fact that the maximal ideal space is Hausdorff.

Indeed, this condition applied to commutative rings more generally is equivalent to the existence of a representation of a kind which generalises the Gelfand representation of a commutative C*-algebra, leading to considerable insights into the categories of modules over Gelfand rings (de Marco–Orsatti [12], Mulvey [22]).

It may also be remarked before beginning the proof that classically the condition for commutative rings generally is equivalent to requiring that any prime ideal of A be contained in a unique maximal ideal. That this is the case for commutative C*-algebras is because the closure of any prime ideal is a maximal ideal of the algebra. This motivates the description of the retraction which is now given in the context of any Grothendieck topos $\mathbb{E}$.

Firstly, we recall that the inclusion

$$\text{Max } A \to \text{Spec } A$$

of the retraction is that determined by assigning to each proposition

$$a \in P$$

of the theory of the locale Spec A the corresponding proposition of the theory $\mathbb{M}$ax A. It has already been established that the axioms of the theory $\mathbb{S}$pec A are satisfied by this interpretation within the theory $\mathbb{M}$ax A, and hence one obtains a map of locales, which identifies the locale Max A to be the sublocale of Spec A obtained by adjoining the propositions

$$a \in A(q)$$

for positive rationals q, together with the axioms of $\mathbb{M}$ax A involving these propositions. It may be recalled that the closedness of the prime P defined by the theory $\mathbb{M}$ax A is described by the axiom which requires that:

$$a \in P \;\vdash\!\dashv\; \bigvee\nolimits_q a \in A(q)$$

for each $a \in A$ and for positive rationals q. Moreover, it will be remembered that the proposition $a \in A(q)$ is provably equivalent in the theory to the proposition

$$(|a| - q.1)^+ \in P$$

in which the element $(|a| - q.1)^+ \in A$ is definable algebraically in terms of $a \in A$ and the positive rational q. In particular, the theory $\mathbb{M}$ax A therefore satisfies the condition that:

$$a \in P \dashv\vdash \bigvee_q (|a| - q.1)^+ \in P$$

for each $a \in A$. It will now be shown that forcing this axiom to hold for a prime P is equivalent to converting it to a model of the theory $\mathbb{M}$ax A.

Explicitly, define the map of locales

$$\text{Spec } A \to \text{Max } A$$

by assigning to each proposition $a \in P$ of $\mathbb{M}$ax A the proposition

$$(|a| - q.1)^+ \in P$$

of the theory of the locale Spec A. Observing that the propositions $a \in P$ generate the locale Max A, to show that this is a map of locales it suffices to prove that each axiom of the theory $\mathbb{M}$ax A is provable in the theory $\mathbb{S}$pec A under this interpretation. That this is the case follows by arguments of which the details may be found elsewhere (Banaschewski–Mulvey [3]), given that the commutative C*-algebra A is isomorphic to that of continuous complex functions on the locale Max A, yielding the required map of locales

$$\text{Spec } A \to \text{Max } A.$$

This map of locales provides a retraction of the inclusion of Max A in the locale Spec A, since the inverse image of each proposition of the theory of Max A of the form

$$a \in P$$

is given in Spec A by the proposition

$$(|a| - q.1)^+ \in P,$$

of which the inverse image in Max A is again this proposition. But it has already been remarked that this is provably equivalent to the proposition

$$a \in P$$

which establishes the required identity. The locale Max A is therefore a retract of the locale Spec A. It may be remarked further that the locale Max A is actually the complete regularisation of the locale Spec A, as in the classical situation. In particular, the commutative C*-algebra $\mathbb{C}(\text{Max } A)$ of continuous complex functions on Max A is canonically isomorphic to that on the locale Spec A.

The existence of this retraction may now be used to obtain the Gelfand–Mazur theorem for commutative C*-algebras in a more conventional form. It may be recalled (Lawvere [17]) that a commutative ring A in a topos $\mathbb{E}$ is said to be *local* provided that

$$\forall_{a \in A}\, a \in \text{Inv}(A) \vee 1 - a \in \text{Inv}(A),$$

in which Inv (A) denotes the subset

$$\text{Inv}(A) = \{a \in A \mid \exists\, b \in A\; ab = 1\}$$

consisting of the invertible elements of A. It may be verified that this is equivalent to the subset $\mathrm{Inv}\,(A)$ being a prime of the ring A, in the sense already defined. Of course, it is equivalent classically to the ring A having a unique maximal ideal, namely the complement of the subset of invertible elements.

Applying the Gelfand duality theorem, one may now obtain the following form of the Gelfand–Mazur theorem:

In any Grothendieck topos $\mathbb{E}$, let A be a commutative C-algebra which is local. Then A is isometrically *-isomorphic to the field $\mathbb{C}_{\mathbb{E}}$ of complex numbers of $\mathbb{E}$.*

It may be remarked that any commutative C*-algebra which is local is classically a field, so that this result is equivalent to the Gelfand–Mazur theorem in that situation. In the present context, the theorem is proved by showing firstly that the subset of invertible elements of A yields a model of the theory $\mathbb{M}\mathrm{ax}\ A$, and hence a point

$$\mathbf{1} \to \mathrm{Max}\ A$$

of the maximal spectrum of A. It is then straightforwardly deduced that this map of locales is actually an isomorphism, and hence that A is isometrically *-isomorphic by the Gelfand isomorphism to the algebra $\mathbb{C}(\mathrm{Max}\ A)$ of continuous complex functions on the locale $\mathbf{1}$, which is evidently the field $\mathbb{C}_{\mathbb{E}}$.

To obtain the existence of this point of the locale Max A, we consider firstly that of the locale Spec A obtained by taking the subset of A consisting of its invertible elements. Since the commutative C*-algebra A is assumed to be local, one has that this subset is indeed a prime of A, and hence determines a model of the theory $\mathbb{S}\mathrm{pec}\ A$, yielding a point

$$\mathbf{1} \to \mathrm{Spec}\ A$$

of the locale which it determines. Observe that any prime P of A yields a prime M_P which is a model of the theory $\mathbb{M}\mathrm{ax}\ A$, by composition with the retraction

$$\mathrm{Spec}\ A \to \mathrm{Max}\ A.$$

Moreover, the prime M_P obtained is contained in the prime P: for the extent to which $a \in M_P$ for each $a \in A$ is the join of the extents to which

$$(|a| - q.1)^+ \in P \ \vdash \ a \in P$$

taken over positive rationals q, by the definition of the retraction.

Within the theory of Spec A, one has that

$$(|a| - q.1)^+ \in P \ \vdash \ a \in P$$

is provable. Hence, the extent to which $a \in M_P$ is contained in that to which $a \in P$. But the prime P considered presently consists of all invertible elements of A. Hence, the prime M_P obtained consists of invertible elements of A. But any prime contains all invertible elements of A, and hence the prime P coincides with the prime M_P obtained. So the prime P of invertible elements of the commutative C*-algebra A is a model of $\mathbb{M}\mathrm{ax}\ A$, and hence determines a point

$$\mathbf{1} \to \mathrm{Max}\ A,$$

which is now asserted to be an isomorphism in the category of locales.

For the inverse image of this map of locales assigns to the proposition $a \in P$ of the theory $\mathbb{M}$ax A the extent $[\![a \in P]\!]$ to which the element $a \in A$ is invertible. To verify that this map of locales is an isomorphism it suffices to prove that this inverse image mapping is bijective. It is clearly surjective, since the identity element of A of any given extent is invertible to that extent. To see that it is also injective, observe that because the locales involved are compact, completely regular, it is enough to prove that if the inverse image of an element of the locale Max A is zero, then the element concerned is the zero of the locale. Moreover, it suffices to prove this for elements chosen from a basis of the locale Max A. But the propositions $a \in P$ form such a basis, since it has been shown that these generate the theory $\mathbb{M}$ax A, yet are closed under finite conjunctions by virtue of the axiom (P4) of the theory $\mathbb{S}$pec A. However, if $[\![a \in P]\!] = 0$, then $a = 0$: for if an element of the ring of continuous complex functions on a locale, which by Gelfand duality the commutative C*-algebra A may be taken to be, is nowhere invertible then it is zero. It follows that $a \in P$ is provably false, by the axiom (P2), as required. The canonical map

$$\mathbf{1} \to \text{Max } A$$

is therefore an isomorphism of locales. Then, by the Gelfand isomorphism of A with $\mathbb{C}(\text{Max } A)$ it follows that A is isometrically *-isomorphic to $\mathbb{C}_{\mathbb{E}}$, which completes the proof.

The Gelfand representation of commutative C*-algebras in the Grothendieck topos $\mathbb{E}$ extends to one of any C*-algebra A in $\mathbb{E}$ over the maximal spectrum of its centre $\mathrm{Z}(A)$, by constructions analogous to those considered in the case of the topos of sets [11,24]. Besides depending on the existence of non-negative partitions of unity in the commutative C*-algebra $\mathrm{Z}(A)$, the proof relies on one other important fact concerning the C*-algebra A, namely that it is locally convex over its centre. To show that the open ball $N(q)$ of radius q of A is closed under convex linear combinations determined by elements of its centre $\mathrm{Z}(A)$, one may argue by taking the inverse image of the C*-algebra A along a Barr covering

$$\gamma : \mathbb{B} \to \mathbb{E}$$

of the topos $\mathbb{E}$. The pre-C*-algebra $\gamma^*(A)$ obtained in the topos $\mathbb{B}$ admits an isometric *-homomorphism into its completion in $\mathbb{B}$, which will be a C*-algebra in the topos $\mathbb{B}$. By the density of this homomorphism into the completion, the centre of $\gamma^* A$ is mapped into the centre of the completion. By the fact that the order relation on the self-adjoint part of a commutative C*-algebra is determined algebraically, non-negative partitions of unity in the centre of $\gamma^* A$ are mapped to non-negative partitions of unity in the centre of the completion. Hence the required conclusion is reached since it is true of a C*-algebra in the topos $\mathbb{B}$ in which the Axiom of Choice is satisfied, by the classical arguments used to establish this fact.

The Gelfand representation is then obtained by observing that the centre $\mathrm{Z}(A)$ of the C*-algebra A is a commutative C*-algebra which admits a Gelfand representation

$$\mathrm{Z}(A) \to \mathbb{C}_{\text{Max } \mathrm{Z}(A)}(\text{Max } \mathrm{Z}(A))$$

into the commutative C*-algebra of continuous complex functions on the locale $\operatorname{Max} Z(A)$, considered here to be the algebra of sections of the sheaf of continuous complex functions on $\operatorname{Max} Z(A)$. This extends canonically to the existence of adjoint functors

$$\mathbf{Mod}\, Z(A) \underset{\longleftarrow}{\longrightarrow} \mathbf{Mod}_{\operatorname{Max} Z(A)}\, \mathbb{C}_{\operatorname{Max} Z(A)}$$

between the category of modules over $Z(A)$ in the topos $\mathbb{E}$ and the category of sheaves of modules over the sheaf $\mathbb{C}_{\operatorname{Max} Z(A)}$ in the topos of sheaves in $\mathbb{E}$ over the locale $\operatorname{Max} Z(A)$.

The existence of finite partitions of unity over $\operatorname{Max} Z(A)$ in the commutative C*-algebra $\mathbb{C}(\operatorname{Max} Z(A))$ implies that these adjoint functors establish an equivalence of categories (Mulvey [21]). In particular, the sheaf $A_{\operatorname{Max} Z(A)}$ assigned to the C*-algebra A, considered to be a module over its centre $Z(A)$, is canonically an involutive algebra over the sheaf $\mathbb{C}_{\operatorname{Max} Z(A)}$ of continuous complex functions on $\operatorname{Max} Z(A)$, in such a way that the canonical map

$$A \to A_{\operatorname{Max} Z(A)}(\operatorname{Max} Z(A))$$

is a *-isomorphism.

The *-algebra $A_{\operatorname{Max} Z(A)}$ obtained may be given a seminorm by assigning to each positive rational q over $\operatorname{Max} Z(A)$ the subsheaf consisting over each open set of $\operatorname{Max} Z(A)$ of those sections which are locally the restrictions of Gelfand transforms of elements of A lying in the open ball of radius q. It may be observed that the seminorm induced on the *-algebra $A_{\operatorname{Max} Z(A)}(\operatorname{Max} Z(A))$ of sections over $\operatorname{Max} Z(A)$ is given by taking its open ball of radius q to consist of those elements which are locally given by the Gelfand transforms of elements from the open ball of A of radius q. In particular, the canonical *-isomorphism

$$A \to A_{\operatorname{Max} Z(A)}(\operatorname{Max} Z(A))$$

is immediately seen to be contractive.

With these observations one may now state the following generalisation of the classical result (Dauns–Hofmann [11], Mulvey [24]) concerning the representation of C*-algebras over the maximal spectrum of their centres:

For any C-algebra A in a Grothendieck topos $\mathbb{E}$, the canonical map*

$$A \to A_{\operatorname{Max} Z(A)}(\operatorname{Max} Z(A))$$

*is an isometric *-isomorphism into the algebra of sections of a C*-algebra in the topos of sheaves in $\mathbb{E}$ over the maximal spectrum of the centre of A.*

That the canonical map is isometric may be proved by observing that whenever the Gelfand transform of $a \in A$ has seminorm less than q one may find a finite open covering $(U_i)_{i=1,\ldots,n}$ of $\operatorname{Max} Z(A)$ together with for each $i = 1, \ldots, n$ an element $a_i \in A$ lying in the open ball of A of radius q, of which the Gelfand transform coincides with that of $a \in A$ over the open set U_i. Choosing a non-negative partition of unity $(p_i)_{i=1,\ldots,n}$ in $Z(A)$ subordinate to this open covering, one may conclude that

$$a = \sum_i p_i a_i .$$

Then one observes that the convexity of the open balls of the C*-algebra A with respect to its centre $\mathrm{Z}(A)$ implies that the convex linear combination

$$a = \sum_i p_i a_i$$

of these elements of the open ball of A of radius q again lies in that open ball, establishing the isometricity of the Gelfand representation. Now, the seminormed structure obtained on the involutive algebra $A_{\mathrm{Max}\ \mathrm{Z}(A)}$ is such that its algebra of sections over $\mathrm{Max}\,\mathrm{Z}(A)$ satisfies the condition required of a C*-algebra with respect to its involution, and is necessarily complete, by virtue of this isometry with the C*-algebra A. But, again by the existence of finite non-negative partitions of unity in $\mathrm{Z}(A)$ over the locale $\mathrm{Max}\,\mathrm{Z}(A)$, it then follows that the involutive algebra $A_{\mathrm{Max}\ \mathrm{Z}(A)}$ is actually a C*-algebra in the topos of sheaves in $\mathbb{E}$ over the locale $\mathrm{Max}\,\mathrm{Z}(A)$, by arguments entirely similar to those in the classical situation.

It may be remarked that, as in the classical context, the purpose of this representation is to obtain from the C*-algebra A a topos in which it is represented isometrically *-isomorphically by a C*-algebra $A_{\mathrm{Max}\ \mathrm{Z}(A)}$ of which the centre is the commutative C*-algebra $\mathbb{C}_{\mathrm{Max}\ \mathrm{Z}(A)}$ of complex numbers in the topos. In analogous ways to those usually pursued one may equally obtain a duality between the category of C*-algebras in the topos $\mathbb{E}$ and the category consisting of compact, completely regular locales in $\mathbb{E}$ together with C*-algebras defined over their algebras of continuous complex functions.

Consider now another consequence of the Gelfand representation, namely its application to commutative C*-algebras which admit a single generator. Classically, it is this which provides the link between the spectrum of a C*-algebra and that of a normal operator on a Hilbert space, by considering the commutative C*-algebra generated by the operator within the C*-algebra of bounded operators on the space. Once again the classical situation carries over completely into the context of a Grothendieck topos, although the greater generality achieved by doing so then allows rather more to be concluded from the result obtained. To begin with we have the following:

For any commutative C-algebra A in a Grothendieck topos $\mathbb{E}$, consider an element $a \in A$. Then its Gelfand transform*

$$\hat{a} : \mathrm{Max}\ A \to \mathbb{C}$$

provides an embedding of the maximal spectrum as a bounded closed sublocale of $\mathbb{C}$ exactly to the extent that $a \in A$ exists and generates the commutative C**-algebra A.*

Of course, the spectrum of the element $a \in A$ may then be considered to be the maximal spectrum of the commutative C*-algebra which it generates, yielding that this is a closed bounded sublocale of the locale $\mathbb{C}$ of complex numbers in the Grothendieck topos $\mathbb{E}$.

To prove this it suffices to consider the case of an element $a \in A$ which is defined and generates the C*-algebra A globally. By Gelfand duality, one has that A is isomorphic to $\mathbb{C}(M)$ for M the maximal spectrum $\mathrm{Max}\ A$, and that the Gelfand transform of $a \in A$ is a map

$$\hat{a} : M \to \mathbb{C}$$

of locales which globally generates $\mathbb{C}(M)$. Now, consider the factorisation

$$\begin{array}{ccc} M & \xrightarrow{\hat{a}} & \mathbb{C} \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\iota} & \\ & F & \end{array}$$

in which F is a compact, completely regular locale. By Gelfand duality, the canonical map

$$\mathbb{C}(\pi) : \mathbb{C}(F) \to \mathbb{C}(M)$$

is an embedding, and, since the Gelfand transform is the image of the continuous complex function $\iota : F \to \mathbb{C}$ under this canonical map, one has that the image of $\mathbb{C}(\pi)$ contains the Gelfand transform $\hat{a}$, making it equal to $\mathbb{C}(M)$ by hypothesis. Hence $\mathbb{C}(\pi)$, and therefore π, is an isomorphism. The converse, that the extent to which the Gelfand transform provides an embedding of the spectrum as a closed sublocale of the locale $\mathbb{C}$ is equal to the extent to which the element $a \in A$ exists and generates the commutative C*-algebra A, is obvious.

Note that, in the above, the statement that $a \in A$ generates the commutative C*-algebra A means that each subalgebra B of A equals A to the extent that $a \in B$. The fact that $a \in A$ generates the commutative C*-algebra A in this sense does not imply that A is *singly generated*, in the sense that there exists an element which generates A, except in the case that the element $a \in A$ has global extent. This somewhat subtle point may be illuminated by the following example:

Let $\mathbb{S}$ be the Sierpinski topos, that is, the topos of sheaves on the Sierpinski space, or equivalently the topos of maps in the category of sets. Then, the natural embedding

$$\mathbb{C} \to \mathbb{C}(D)$$

of the complex number field in the category of sets into the algebra of continuous complex functions on the unit disc D is a C*-algebra A in $\mathbb{S}$. Further, if $z \in \mathbb{C}(D)$ is the identical embedding of D into D, then $z \in A$, considered as an element of non-global extent, generates the commutative C*-algebra A because it generates $\mathbb{C}(D)$. Now, if A were singly generated, that is, if it were true that there exists a generating element of A, then Sierpinski space would be covered by open sets on each of which the restriction of A has a generating element. Since Sierpinski space has no non-trivial covers, this would imply that A has a generating element of global extent. However, any global element of A already belongs to the smallest subalgebra of A, namely the identity mapping

$$\mathbb{C} \to \mathbb{C},$$

and hence cannot generate A.

It may be remarked that this example also shows that a C*-algebra may have an element generating it globally without having a generating element of global extent. Finally, note that Max A is just

$$(\text{Open}\,(D))^{\top} \to \text{Open}\,(D)\,,$$

in which $(\ )^{\top}$ signifies addition of a new top element and the map just identifies the old and the new top elements. As a space, this is just the embedding

$$D \to D^{\bullet},$$

in which $(\ )^{\bullet}$ denotes the addition of a new point whose only neighbourhood is the whole space. Then, clearly, $D \to D^{\bullet}$ cannot be embedded into the space $\mathbb{C} \to \mathbb{C}$ of complex numbers in $\mathbb{S}$ because any continuous map $D^{\bullet} \to \mathbb{C}$ is constant. Hence, Max A cannot, globally, be embedded into the locale of complex numbers.

In a similar fashion, one can give examples, for instance in the topos of sheaves on the unit circle, which show that a singly generated C*-algebra need not have any global generating element.

We now turn to a substantial generalisation of the proposition proved above. For this, we first need to remark that the concept of element which one considers in a topos is that generalised to allow consideration of any map

$$a : X \to A$$

into the object A with which we are concerned from an arbitrary object X. In particular, any element $a \in A$ of this kind of a commutative C*-algebra A has a Gelfand transform, which may be viewed explicitly as a map of locales

$$\hat{a} : \text{Max } A \to \mathbb{C}^X$$

from the maximal spectrum of A into the locale obtained by exponentiating the locale $\mathbb{C}$ by the discrete locale determined by X, in other words given by the object of subobjects of X. With this in mind it is then clear that we have also proved the following:

For any commutative C-algebra A in a Grothendieck topos $\mathbb{E}$, the Gelfand transform of any element*

$$a : X \to A$$

which generates A provides an embedding of the maximal spectrum Max A *as a bounded closed sublocale of the locale $\mathbb{C}^X$. In particular, there is a canonical embedding*

$$\text{Max } A \to \mathbb{C}^A$$

of the maximal spectrum onto a bounded closed sublocale of the power of the locale $\mathbb{C}$ indexed by the commutative C-algebra A.*

In the latter case, the embedding is that induced by the generic element of the commutative C*-algebra A, namely the identity mapping on A.

For any compact, completely regular locale M, this yields the observation that M is embedded in $\mathbb{C}^{\mathbb{C}(M)}$ by the Gelfand transform of the identity map on $\mathbb{C}(M)$, applying the Gelfand isomorphism between M and Max $\mathbb{C}(M)$. Of course, one can also show directly, as in the classical case, that the natural map from M to $\mathbb{C}^{\mathbb{C}(M)}$ is an isomorphism.

In another direction, the present proposition further implies that any element $a \in A$ defined on a subobject U of the terminal object 1 and generating the commutative C*-algebra A determines an embedding

$$\text{Max } A \to \mathbb{C}^U$$

into the locale $\mathbb{C}^U$. This locale may be considered to be the locale of complex numbers localised to the extent of U.

Finally, returning to the case of a commutative C*-algebra A with a single, globally defined generator $a \in A$, it may be remarked that the theory of the maximal spectrum Max A is then that of the locale $\mathbb{C}$ of complex numbers together with the following additional axioms:

(i) $a \in A(q) \vdash$ *false* whenever $a \in N(q)$, which ensures boundedness, and
(ii) $a \in (r, s) \vdash$ *false*,

for certain rational open rectangles (r, s) which are determined by, and determine, the particular properties of the element $a \in A$ concerned. In particular, observing that the spectrum of a bounded normal linear operator T on a Hilbert space is given by the maximal spectrum of the C*-algebra generated by T, this allows the spectrum of the operator to be described in terms of a theory given by propositions of the form

$$T \in (p, q).$$

In the context of the foundation of quantum mechanics this suggests one way in which an observable might be represented constructively by allowing a theory of the observable to determine directly its spectrum instead of the usual approach of first representing the observable as an operator on a Hilbert space.

Regarding Gelfand duality in general, we note that, although the nature of commutative C*-algebras is reasonably transparent in an arbitrary Grothendieck topos, this is rather different for compact, completely regular locales in general and for the construction of the maximal spectrum of a commutative C*-algebra in particular. There are, however, two situations in which these are well understood, namely that of the topos of sheaves on a compact Hausdorff space X and that of the topos of G-sets for a group G. We shall now discuss these in some detail.

In the case of sheaves on a compact Hausdorff space X, the category of commutative C*-algebras in **Sh** X is equivalent to the category of commutative C*-bundles $\pi : A \to X$, where the latter may be defined as follows (Hofmann–Keimel [13], Burden–Mulvey [9]): for each $x \in X$ the fibre $\pi^{-1}(x)$ of the continuous map π carries the structure of a C*-algebra such that the algebraic operations are continuous over X, the norm topology coincides with the subspace topology in the total space A, the norm is upper semi-continuous on A, and for each $x \in X$ the continuous sections on neighbourhoods of $x \in X$ meet $\pi^{-1}(x)$ densely. A map between two such bundles is a continuous fibre preserving map between the total spaces, inducing C*-algebra homomorphisms on each fibre. From commutative C*-bundles over X to commutative C*-algebras in **Sh** X, the equivalence takes each $\pi : A \to X$ to the sheaf of continuous sections of π, with the obvious definition of C*-algebra structure. On the other hand, the compact, completely regular locales M in **Sh** X result, up to isomorphism (Johnstone [16]), from the compact Hausdorff spaces over X, that is, the continuous maps $\varphi : K \to X$, K compact Hausdorff, by letting $M(U) = \mathrm{Open}(\varphi^{-1}(U))$ for each open subset U of X. Alternatively, this says that $M = \varphi_*(\Omega_K)$ for the subobject classifier Ω_K of the topos **Sh** K and the functor $\varphi_* :$ **Sh** $K \to$ **Sh** X induced by φ. Further, the locale maps $\varphi_*(\Omega_K) \to \varphi'_*(\Omega_{K'})$ for such $\varphi : K \to X$ and $\varphi' : K' \to X$ correspond exactly to the continuous maps

$f : K \to K'$ over X, the map corresponding to f taking each open $V \subseteq (\varphi')^{-1}(U)$ to $f^{-1}(V) \subseteq \varphi^{-1}(U)$. Hence the category of compact, completely regular locales in **Sh** X is equivalent to the category of compact Hausdorff spaces over X.

This description permits the following elucidation of the functor $\mathbb{C}(\ )$ of Gelfand duality in **Sh** X:

The complex number object in **Sh** X is the sheaf $\mathbb{C}_X$ of complex-valued continuous functions on X or, alternatively, the sheaf of continuous sections of the initial C*-bundle, that is, the projection $p : X \times \mathbb{C} \to X$. Further, the locale of complex numbers is the locale of open subsets of $\mathbb{C}_X$, which may be described as $p_*(\Omega_{X \times \mathbb{C}})$, the sheaf which associates Open $(U \times \mathbb{C})$ with the open subset U of X. It follows that, for any compact regular locale $M = \varphi_*(\Omega_K)$, $\varphi : K \to X$ a compact Hausdorff space over X, the object of locale maps from M to the locale of complex numbers is the sheaf assigning to each open set U in X the set of locale maps $(\varphi|\varphi^{-1}(U))_* : \Omega_{\varphi^{-1}(U)} \to \Omega_{U \times \mathbb{C}}$ in the topos of sheaves on U, the latter being essentially the same as the locale maps Open$(\varphi^{-1}(U)) \to$ Open$(U \times \mathbb{C})$, which in turn correspond exactly to the continuous maps $\varphi^{-1}(U) \to \mathbb{C}$, by the soberness of the spaces involved. This shows that $\mathbb{C}(M) = \varphi_*(\mathbb{C}_K)$. Further, it is easily checked that locale maps $M \to M'$ determine precisely the expected maps $\varphi'_*(\mathbb{C}_{K'}) \to \varphi_*(\mathbb{C}_K)$.

In all, we now have the following results:

For any compact Hausdorff space X, Gelfand duality in **Sh** *X determines an equivalence between the category of C*-bundles over X and the category of compact Hausdorff spaces over X. The C*-algebras in* **Sh** *X are exactly the $\varphi_*(\mathbb{C}_K)$, for compact Hausdorff spaces $\varphi : K \to X$ over X.*

There is an obvious alternative approach to the duality of compact Hausdorff spaces over X, based on classical Gelfand duality by which the correspondence

$$K \stackrel{\varphi}{\to} X \quad \mapsto \quad \mathbb{C}(K) \stackrel{\mathbb{C}(\varphi)}{\leftarrow} \mathbb{C}(X)$$

makes compact Hausdorff spaces over X equivalent to the category of all C*-algebra homomorphisms $\mathbb{C}(X) \to A$, for A any C*-algebra, which we shall call the C*-algebras over $\mathbb{C}(X)$, these being the C*-algebras which are equipped with an appropriate $\mathbb{C}(X)$-algebra structure. This duality is clearly different from that described in the above proposition and leads to the following observation:

The category of commutative C-bundles over a compact Hausdorff space X is equivalent to the category of commutative C*-algebras over $\mathbb{C}(X)$ by the functor taking each commutative C*-bundle $A \stackrel{\pi}{\to} X$ to the C*-algebra of its global sections.*

Note that this can also be obtained directly, by familiar arguments concerning C*-bundles and C*-sheaves over X. Also, it is the commutative C*-algebra counterpart of the equivalence between Banach bundles over X and locally convex Banach modules over $\mathbb{C}(X)$, for a compact Hausdorff space X (Burden–Mulvey [9], Hofmann–Keimel [13]).

Interpreting now our Gelfand duality in the particular case of the topos of G-sets, for an arbitrary group G, leads to the following observations:

The compact completely regular locales are exactly the locales of open sets of compact Hausdorff spaces X with G acting on X continuously (and hence as automorphisms), the G-action induced by that on X in the obvious manner. Similarly, the maps between compact completely regular locales are the maps induced by continuous G-maps (= equivariant maps), and hence there is a category equivalence between the category of compact completely regular locales and the category

$$\textbf{Compact Hausdorff spaces}/G$$

of compact Hausdorff spaces with G-action and continuous G-maps. This identifies one side of Gelfand duality. The other side is given by the category

$$\textbf{Commutative C*-algebras}/G$$

of commutative C*-algebras with G-action by automorphisms and equivariant C*-algebra homomorphisms. Gelfand duality now asserts there is a dual equivalence between these two categories. To analyse the functors involved the following facts are needed:

The complex number object in the topos of G-sets is just the usual complex number field $\mathbb{C}$ with trivial G-action, as one readily sees by tracing through the general definition of complex number (or, perhaps, more conveniently, real number) objects (Banaschewski [1]). Furthermore, the locale of complex numbers is the locale Open($\mathbb{C}$) because this is the case in the topos of sets in which the G-sets are taken. As a result, for any compact completely regular locale M, $\mathbb{C}(M)$ is, as an object, the object of locale maps $M \to \text{Open}(\mathbb{C})$, and if M equals Open(X) for some X in compact Hausdorff spaces$/G$, this is then the object of all locale maps $\text{Open}(X) \to \text{Open}(\mathbb{C})$. Now, by the soberness of the spaces involved, the latter is exactly the usual set of all continuous maps $X \to \mathbb{C}$. Taking into account the G-action and the C*-algebra structure, we see that $\mathbb{C}(M)$ is the usual C*-algebra $\mathbb{C}(X)$ of all complex-valued continuous functions on X with G-action

$$(sf)(x) = f\left(s^{-1}x\right)$$

for $s \in G$, $f \in \mathbb{C}(X)$, and $x \in X$. Furthermore, the functoriality of the correspondence $M \mapsto \mathbb{C}(M)$ is exactly analogous to that of the correspondence $X \mapsto \mathbb{C}(X)$.

On the other hand, for any C*-algebra A, the spatial locale Max A, whose points are exactly the homomorphisms $A \to \mathbb{C}$, is the locale Open(Max$_G$ A), where Max$_G$ A is the usual maximal ideal space of A with G-action induced in the obvious way by the G-action on A. Moreover, the functoriality of $A \mapsto \text{Max}\, A$ and $A \mapsto \text{Max}_G\, A$ correspond to each other. In all, this leads to the following result:

For G-sets, Gelfand duality takes the form of a dual equivalence

$$\textbf{Commutative C*-algebras}/G \rightleftarrows \textbf{Compact Hausdorff spaces}/G$$

given by $A \mapsto \text{Max}_G\, A$ *and* $X \mapsto \mathbb{C}(X)$.

We note that this is, indeed, the same equivalence obtained by taking classical Gelfand duality and considering G-sets as set-valued functors on G. The principle involved is that, for the functor category between two categories and the formation $(\ )^{op}$ of the dual category, one has that

$$\textbf{Funct}(K, L)^{\text{op}} = \textbf{Funct}(K^{\text{op}}, L^{\text{op}}),$$

together with the fact that $G^{\text{op}} \cong G$ (by $s \mapsto s^{-1}$).

The latter argument has an obvious analogue, taking into account that there may not be an isomorphism with the dual, for arbitrary monoids or, indeed, any small category. How the resulting alternative duality is related to that given by general Gelfand duality we have not yet decided.

We conclude with some conjectures concerning the true nature of Gelfand duality. The form in which that duality is presented here has an obvious asymmetry: on one side, one deals with locales, but on the other the objects involved, C^*-algebras, are spaces. It seems to us there ought to be a further duality, extending that considered here, in which the objects on either side are of the same kind, and hence locales. This presupposes that there is a notion, yet to be properly defined, of a localic C^*-algebra; we expect this to be related to C^*-algebras in somewhat the same manner in which the complex number locale is related to the complex number object in a topos. On the other hand, we speculate that compact regular locales will take the place of the compact completely regular locales in the present duality.

The two contravariant functors giving the dual equivalence of the two categories thus indicated should then be (i) an appropriate version of the present Max, and (ii) given by taking each compact regular locale M to the localic C^*-algebra $\mathbb{C}^M$ obtained by exponentiation.

Finally, the present duality should be a consequence of this new one because of the following conjectures:

(i) For any compact regular locale M, $\mathbb{C}^M$ is spatial if, and only if, M is completely regular.
(ii) For any localic C^*-algebra A, Max A is completely regular if, and only if, A is spatial.
(iii) The C^*-algebras are given exactly by taking the objects of points of localic C^*-algebras.

We hope to return to the discussion of these matters in due course.

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