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Polyhedra of Small Order and Their Hamiltonian Properties

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enumerated classes include polyhedra with up 13 vertices, simplicial polyhedra with up to 16 vertices, 4-connected polyhedra with up to 15 vertices, non-Hamiltonian polyhedra with up to 15 vertices, bipartite polyhedra with up to 24 vertices, and bipartite trivalent polyhedra with up to 44 vertices. The results of the enumeration were used to systematically search for certain smallest non-Hamiltonian polyhedral graphs. In particular, the smallest non-Hamiltonian planar graphs satisfying certain toughness-like properties are presented here, as are the smallest non-Hamiltonian, 3-connected, Delaunay tessellations and triangulations. Improved upper and lower bounds on the size of the smallest non-Hamiltonian, inscribable polyhedra are also given.

1. INTRODUCTION

This paper describes the results of a computer enumeration of several classes of polyhedra and a search for polyhedra with certain noteworthy properties. It extends work done by many other researchers; references are given in the appropriate sections. Among the classes of polyhedra enumerated are:

- Polyhedra with up to 13 vertices.
- Simplicial polyhedra with up to 16 vertices.
- Bipartite, trivalent polyhedra with up to 24 faces (44 vertices).
- 4-connected simplicial polyhedra with up to 17 vertices.
- 4-regular polyhedra with up to 22 vertices.

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- 4-connected and minimally 4-valent polyhedra with up to 15 vertices.
- Bipartite polyhedra with up to 24 vertices or 15 faces.
- Non-Hamiltonian polyhedra with up to 15 vertices, 27 edges, or 13 faces.
- Non-Hamiltonian simplicial polyhedra with up to 17 vertices.

These results are discussed in Sections 3–5. Section 3 describes the enumeration of simplicial polyhedra, including a brief sketch of several implementation details. Section 4 describes the various approaches used to enumerate different classes of polyhedra, and contains enumeration of these classes. In particular, this section contains a refinement of Tutte's inductive definition of 3-connected planar graphs that may be of independent interest. Section 5 focuses on the generation of non-Hamiltonian polyhedra.

One goal of this research was to find smallest examples of non-Hamiltonian polyhedra that satisfy certain additional graph-theoretical properties.[1] The results are summarized in Table I. The examples themselves, and the relevant definitions, appear in Section 6.

In Section 7, we turn our attention to inscribable polyhedra. We present enumerations of inscribable and circumscribable simplicial polyhedra with up to 16 vertices. We also exhibit the (unique) smallest polyhedron that is neither inscribable nor circumscribable; it has 10 vertices and is self-dual.

In Section 8, the results of the earlier sections are applied to the problem of finding smallest non-Hamiltonian Delaunay tessellations and triangulations. In particular, the smallest non-Hamiltonian, 3-connected, Delaunay tessellations have 13 vertices and 10 faces. There are four combinatorially distinct tessellations with this property; two are isomorphic as graphs, but have different outer faces in the plane embedding. The smallest non-Hamiltonian, 3-connected Delaunay triangulations have 13 vertices and 21 faces (i.e., the nontriangular face is a quadrilateral); there are two combinatorially distinct smallest examples.

Section 9 contains some results concerning smallest non-Hamiltonian inscribable polyhedra. Using the results of our enumeration, we have determined that the number of vertices in the smallest simplicial non-Hamiltonian inscribable polyhedra lies between 18 and 20, inclusive. There are at least 11 combinatorially distinct simplicial non-Hamiltonian inscribable polyhedra with 20 vertices. In the nonsimplicial case, there are exactly three combinatorially distinct non-Hamiltonian inscribable polyhedra with 19 vertices and 13 faces; we conjecture that these are the smallest non-Hamiltonian inscribable polyhedra.

[1] We say graph G is smaller than graph H if either (1) G has fewer vertices than H, or (2) they have the same number of vertices and G has fewer edges.

TABLE I

Summary of Smallest Non-Hamiltonian Planar Graphs Presented in Section 6

Properties	Count	Size		Figure(s)
		Vertices	Faces	
Planar, 1-supertough, Non-1-Hamiltonian	2	10	8	5(a)(b)
Planar, 1-supertough, Non-1-Hamiltonian, simplicial	1	10	(16)	5(c)
Planar, 3-connected, 1-tough, Non-Hamiltonian	1	13	10	6(a)
Planar, 1-tough, Non-Hamiltonian, simplicial	1	13	(22)	6(b)
Planar, 1-supertough, Non-Hamiltonian	1	15	11	7(a)
Planar, 1-supertough, Non-Hamiltonian, simplicial	1	15	(26)	7(b)

2. Preliminaries

For the relevant background in combinatorial geometry and graph theory, see [22, 5]. Throughout this paper, polyhedron means a 3-polyhedron. We make implicit use of Steinitz' theorem that a graph is realizable as a 3-polyhedron (*polyhedral*) if and only if it is planar and 3-connected. Two polyhedra are *combinatorially equivalent* if they are isomorphic. In Sections 7 and 8, we will be concerned with embeddings of graphs in the plane in which the identity of the outer (unbounded) face is important; in this context, we will say that two plane graphs are combinatorially equivalent if there is an isomorphism between them that preserves the identity of the outer face. The equivalence classes induced by the relation of combinatorially equivalence are called *combinatorial types*. A *full stellation* of a plane graph G is obtained by choosing a face f of G, inserting a new vertex inside f, and connecting the new vertex to all vertices of G on the boundary of f. If the new vertex is connected to some (but not necessarily all) of the boundary vertices of f, the resulting graph is called a *partial stellation* of G.

We use the following notation. $\mathscr{S}_n$ denotes the class of simplicial polyhedra with n vertices (i.e., polyhedra in which all faces are triangles.) A polyhedron with n vertices and k faces is called an (n, k)-polyhedron; the class of all (n, k)-polyhedra is denoted $\mathscr{P}_{n,k}$. The class of (n, k)-polyhedra is nonempty if and only if $n \leqslant 2k-4$ and $k \leqslant 2n-4$; if these inequalities are satisfied, we call (n, k) a *feasible pair*. We use $|\cdot|$ to denote cardinality, with the following conventions: if G is a graph, S is a point set, and $\mathcal{G}$ is a class of polyhedra (e.g., $\mathcal{S}_n$), then $|G|$, $|S|$, and $|\mathcal{G}|$ denote, respectively, the number of vertices in G, the number of elements in S, and the number of distinct combinatorial types in $\mathcal{G}$.

3. Generating Simplicial Polyhedra

The fundamental operation needed to generate $\mathcal{S}_n$, the simplicial polyhedra with n vertices, is the operation `augment`(G, e_1, e_2) illustrated in Fig. 1. It is defined as follows. Given two distinct oriented edges $e_1 = vw$ and $e_2 = vx$ with a common tail v, v is "stretched" into an edge uv, and edges uw and ux are added. All neighbors of v that are between w and x (moving counterclockwise about v) are then disconnected from v and attached to u. (Note that there may not be any such neighbors, in which case the new vertex u will have degree 3.) It is well known (see, for example, [6]), that for $n \geqslant 4$, $\mathcal{S}_n$ can be generated by applying the `augment` operation to every member of $\mathcal{S}_{n-1}$ in every possible way and checking for duplicates, and that $\mathcal{S}_4$ consists of a single graph (the tetrahedron).

The procedure outlined in the preceding paragraph causes many redundant candidate graphs to be generated. The total number of candidate graphs generated can be considerably reduced by applying a few simple observations.

1. Simplicial polyhedra with minimum degree 4 or 5 can be efficiently generated using an inductive procedure defined in [3], so it is only necessary to generate candidate graphs with minimum degree 3.

2. In view of Observation 1, it is only necessary to apply the operation `augment`$(e1, e2)$ in situations where e_1 and e_2 are adjacent edges.

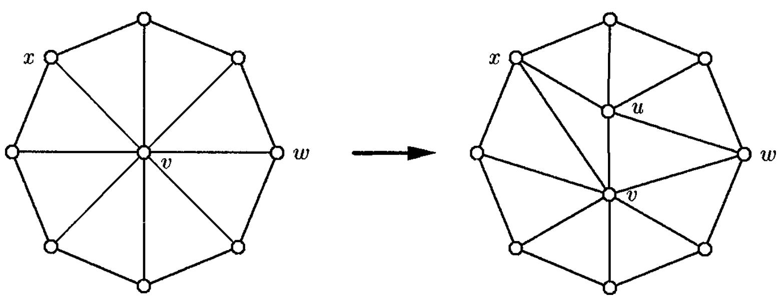

Fig. 1. The "vertex stretching" operation used to generate simplicial polyhedra.

Indeed, it is shown in [6] that any simplicial polyhedron with degree 3 can be generated by performing the `augment` operation on some pair of incident edges in some graph $\mathscr{S}_{n-1}$.

3. Since e_1 and e_2 play symmetric roles in the definition of `augment` (e_1, e_2), it is only necessary to apply the operation when e_2 is the clockwise neighbor of e_1 about their common tail.

4. Define two oriented edges e and e' in graph G to be *automorphism-equivalent* if there is an isomorphism of G onto itself mapping the tail and head of e onto the tail and head (respectively) of e'. This relation partitions the oriented edges of e into *automorphism-equivalence* classes. The preceding observations imply that for each base graph, it is adequate to choose one oriented edge e_1 from each automorphism-equivalence class, let e_2 be the clockwise neighbor about its tail, and to then apply `augment`(e_1, e_2) only to pairs of oriented edges e_1 and e_2 constructed in this fashion. An efficient and practical procedure for constructing the automorphism-equivalence classes for polyhedral graphs is described in [27].

In addition to the preceding optimizations, several implementation details are worth noting. Duplicate graph detection could be performed using the isomorphism-testing algorithm of [27], which is based on the partitioning of oriented edges into automorphism equivalence classes mentioned above. However, the simple isomorphism-testing algorithm described in [6], while asymptotically slower than the algorithm of [27], appears to be significantly faster for the small values of n relevant to this paper.

Efficient searching for possible duplicates can be done using standard chain-bucket hashing techniques [30]. The following hash function $h(\cdot)$ is invariant under isomorphism, can be computed rapidly, and seems to have nice distribution properties. Let G be a simplicial polyhedron with n vertices. For each vertex v_i, let $s(v_i)$ be the sum of the squares of the degrees of the neighbors of v_i. Let $s_1 \cdots s_n$ be the n values of $s(v_i)$, sorted in ascending order. The hash function for G is then given by

$$h(G) = \sum_{i=1}^{n} s_i p^{i-1} \mod q, \tag{3.1}$$

for suitably chosen primes p and q. In addition to its use for streamlining the search for duplicates, this hash function is useful for producing large catalogs when disk space is limited. Indeed, one can partition the range $0, 1, \ldots, q-1$ into k disjoint intervals and run the generating program k times, once for each interval, each time ignoring all candidate graphs whose hash function values fall outside the appropriate interval.

The number of simplicial polyhedra of each order up to 16 is shown in Table II. The third column shows the number of distinct degree sequences that are realized by simplicial polyhedra of the given order. The fourth column shows the number of simplicial polyhedra with minimum degree at least 4. As indicated above, these were separately generated by a program implementing Batagelj's inductive definition of this subclass [3]. The final column shows the number of 4-connected simplicial polyhedra. These were obtained by testing each minimally 4-valent simplicial polyhedron for 4-connectivity.

Counts of simplicial polyhedra with up to 11 vertices can be found in [22] (also, see [7]). Simplicial polyhedra with 12 vertices were first enumerated by Bowen and Fisk [6]. The values in the above table up to and including $n = 14$ have been independently confirmed by Warren Smith. The values in the fourth column were previously computed by Holton and McKay [26], and earlier by Hucher *et al.* for $n \leqslant 14$ [28].

Table III contains the number of simplicial polyhedra with all vertices having even degree, for $n \geqslant 24$. Equivalently, these are the counts of bipartite, trivalent polyhedra having 44 or fewer vertices. These were generated using an algorithm described in [3], modified to use efficiency techniques discussed above. The entries up through $n = 22$ were previously computed by Holton, Manvel, and McKay [25].

TABLE II

The Number of Nonisomorphic Simplicial Polyhedra and Distinct Maximal Planar Degree Sequences for $n \leqslant 15$, and the Number of Nonisomorphic 4-valent and 4-connected Simplicial Polyhedra for $n \leqslant 17$

n	Graphs	Sequences	Minimum degree $\geqslant 4$	4-connected
3	1	1		1
4	1	1		1
5	1	1	1	1
6	2	2	1	1
7	5	5	1	1
8	14	13	2	2
9	50	33	5	4
10	233	85	12	10
11	1,249	199	34	25
12	7,595	445	130	87
13	49,566	947	525	313
14	339,722	1,909	2,472	1,357
15	2,406,841	3,713	12,400	6,244
16	17,490,241	7,006	65,619	30,926
17	?	?	357,504	158,428

TABLE III

Number of n-Vertex Simplicial Polyhedra with All Vertices Having Even Degree, for $n \leqslant 24$

n	Count	n	Count
6	1	16	185
7	0	17	466
8	1	18	1,543
9	1	19	4,583
10	2	20	15,374
11	2	21	50,116
12	8	22	171,168
13	8	23	582,603
14	32	24	2,024,119
15	57		

4. Generating Polyhedra

There are two different approaches to enumerating a class of polyhedra $\mathscr{P}_{n,k}$, which we call the subtractive and additive approaches. Both have their uses.

The *subtractive approach* generates $\mathscr{P}_{n,k}$ from $\mathscr{P}_{n,k+1}$ by systematically deleting each edge from each $P \in \mathscr{P}_{n,k+1}$, verifying that the resulting graph remains 3-connected and checking for duplicates. Two simple improvements speed up the algorithm considerably: (1) deleting one edge from each automorphism equivalence class (rather than deleting each edge in the graph); and (2) generating candidate graphs only if the new face would be a maximum-valence face (since otherwise the same candidate graph will be generated from a different base graph). Since $\mathscr{P}_{n,2n-4} = \mathscr{S}_n$, the subtractive approach can in principle be used to generate all polyhedra with n vertices once $\mathscr{S}_n$ has been computed.

The *additive approach* uses the theory of 3-connected graphs developed by Tutte [48]. Tutte defined two basic operations, called *face-splitting* and *vertex-splitting*. These two operations, which are illustrated in Fig. 2, are dual to one another. The inverse operations are, respectively, called *face-merging* and *vertex-merging*. An edge in a 3-connected graph is called *removable* if deleting it (i.e., merging the two faces on either side of it) preserves 3-connectivity. An edge in a 3-connected graph is called *shrinkable* if shrinking it to a vertex (i.e., merging its two endpoints) would not create a *multiedge* (a pair of edges with the same two endpoints).

Tutte proved that if $n > 4$ and $k > 4$, any graph in $\mathscr{P}_{n,k}$, with one exception, can be obtained either by applying a face-splitting operation to a

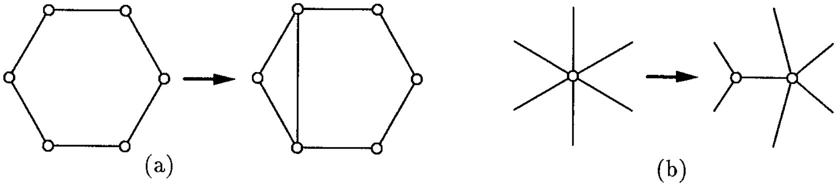

FIG. 2. The two splitting operations: (a) Face-splitting. A face is split by adding an edge. (b) Vertex-splitting. A vertex is split, the two new vertices are joined by an edge, and the edges incident on the original vertex are apportioned between the two new vertices.

graph in $\mathscr{P}_{n,k-1}$ or by applying a vertex-splitting operation to a graph in $\mathscr{P}_{n-1,k}$. The one exception is the wheel, W_n, consisting of $n-1$ vertices of degree 3 arranged in a cycle about a hub vertex of degree $n-1$. The following theorem refines Tutte's theorem by showing that, for fixed n and k, only one of these two operations need be performed.

THEOREM 4.1. *If $k \geqslant n$, every graph in $\mathscr{P}_{n,k}$ (with the single exception of the wheel W_n if $n=k$, and otherwise without exception) can be generated by applying a face-splitting operation to some graph in $\mathscr{P}_{n,k-1}$.*

Proof. Assume $n \leqslant k$, $G \in \mathscr{P}_{n,k}$, and G is not a wheel. Let m denote the number of edges in G. We show that G has a removable edge. The proof is by induction on n. For $n \leqslant 5$, the result is easily verified by inspection. Indeed, there are only three polyhedra with five or fewer vertices: the two wheels W_4 and W_5, and the triangular bypyramid, which has a removable edge.

For the induction step, we first note that since $k \geqslant n$, the average valence of a face is less than 4 (this follows easily from Euler's formula), so G has at least one triangular face. Let T be this face, and let u, v, and w be the three vertices on the boundary of T. There are three cases, depending on the number of boundary vertices that have degree 3.

Case 1. u, v, and w all have degree 3. Let x, y, and z be, respectively, the neighbors of u, v, and w that are not in the triangle uvw. Notice that x, y, and z must all be distinct. Indeed, if they were all the same, then G would be the wheel W_4. If two of them were identical (say x and y), then removing x and w would separate uv from the rest of the graph, violating 3-connectivity.

Let G' be the graph obtained by collapsing uvw to a single vertex, r (see Fig. 3). It is easy to verify that G' is 3-connected. Let k', n', and m' be, respectively, the number of faces, vertices, and edges of G'. We have $k' = k-1$, $n' = n-2$, and $m' = m-3$. Hence $k' > n'$ (so, in particular, G' is

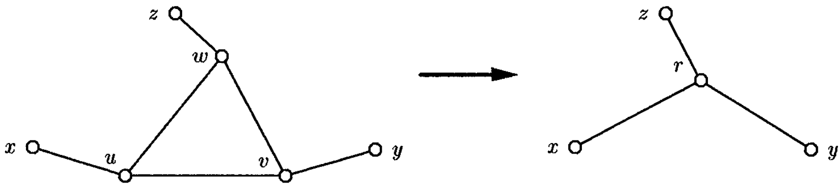

FIG. 3. Case 1 in the proof of Theorem 4.1.

not a wheel), and $m' < m$. So by the inductive assumption, G' has a removable edge, say e. The edge e cannot be incident on r, since $\mathrm{degree}(r) = 3$. Hence e is a removable edge of G as well.

Case 2. At least two of the three vertices (without loss of generality, assume u and v) have degree $\geqslant 4$. Result (3.4) in [48, p. 446] may be restated in our terminology as follows: if uvw is a triangle and neither edge uv nor uw is removable, then either u or v has degree 3.[2] It follows that either edge uv or uw is removable.

Case 3. Two vertices (say u and v) have degree 3, the remaining vertex (w) has degree > 3. Let x (respectively, y) be the neighbor of u (respectively, v) that is not a boundary vertex of T. As in Case 1, x and y must be distinct. Let G' be the graph obtained by collapsing the edge uv to a single vertex r, with neighbors w, x and y, as illustrated in Fig. 4. We claim that G' is 3-connected. Assume for the moment that the claim is true. Let n', k', and m' be, respectively, the number of vertices, faces, and edges of G'. We have $n' = n - 1$, $k' = k - 1$, and $m' = m - 2$. In particular, $k' \geqslant n'$. Also, G' is not a wheel (since, if it were, G would be a wheel with hub w). By induction, G' has a removable edge e. Since r has degree 3, r cannot be an endpoint of e. It is not hard to see that e is also a removable edge of G.

It remains to show that G' is 3-connected. We must show that given any pair of distinct vertices in G', there are 3 vertex-disjoint paths between them. If w is not in the pair, this is straightforward to verify (since G is 3-connected). So assume one vertex is w, the other some vertex a. If $a = r$, the verification is again straightforward, so assume $a \neq r$. Let F_1 (respectively, F_2) be the face opposite the edge uw (respectively, vw) from T in G, and give the corresponding faces in G' the same names. Assume that wux forms part of a clockwise walk around the boundary of F_1 (see Fig. 4). Since G is three-connected, there are three vertex-disjoint paths, Π_1, Π_2, and Π_3, from w to a in G. Assume that one of these paths (Π_1) uses the edge wu and another path (Π_2) uses the edge wv (otherwise, the paths

[2] Result (3.4) in [48] is actually stated in a weaker form, using the stronger hypothesis that neither uv nor uw is either removable or shrinkable. Nevertheless, Tutte's proof of this result uses only the assumption that neither uv nor uw is removable, so it may be stated in this stronger form.

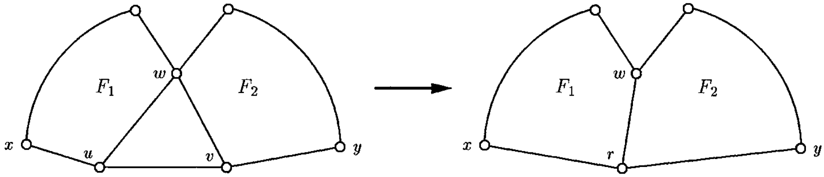

FIG. 4. Case 3 in the proof of Theorem 4.1.

correspond naturally to three disjoint paths from w to a in G' and the proof is complete).

Notice that Π_2 cannot contain a vertex on the boundary of F_1 other than w (or possibly a, if a is on the boundary of F_1). Indeed, suppose it contained a vertex $p \neq w$ on the boundary of F_1. Let H be the graph obtained from G by placing a new vertex z inside face F_1 and connecting it to p, w, and u. Then H is planar. But H would then have a set of nine vertex-disjoint paths connecting each of p, u, and w to each of z, a, and v (namely: the three edges incident on z; the portion of Π_2 from p to a, the portion of Π_1 from u to a, and Π_3; and the portion of Π_2 from v to p and the edges uv and wv). Since a planar graph cannot contain a $K_{3,3}$ minor, this is impossible. Similarly, Π_1 cannot contain a vertex on the boundary of F_2 other than w (or possibly a). Also, since w has degree at least 4 and G is 3-connected, a cannot be simultaneously on the boundary of F_1 and the boundary of F_2.

Next, modify Π_3 as follows. Let b be the last vertex on the boundary of either F_1 or F_2 encountered by Π_3 from w to a. If $b = w$, do nothing. Otherwise, b cannot belong to both F_1 and F_2. If b belongs to F_1 (respectively, F_2), replace the portion of Π_3 from w to b with the arc of the boundary of F_1 (respectively, F_2) from w to b that does not contain u (respectively, v). After this modification, there is some $i \in \{1, 2\}$ such that Π_3 does not contain any boundary vertex of F_i. Assume, without loss of generality, that $i = 1$. Let c be the last vertex on Π_1 that is also on the boundary of F_1. Since u has degree 3, $c \neq u$. Since neither Π_2 nor Π_3 contains any vertices of F_1 (other than w), we can "detour" Π_1 to go counterclockwise around the boundary of F_1 from w to c, thereby missing u. Now the modified paths $\{\Pi_i\}$ correspond exactly to three disjoint paths in G' from w to b (with r replacing v on Π_2). This proves the claim, and hence the theorem. ∎

Since vertex-splitting and face-splitting are dual to one another, we also have the following dual version of Theorem 4.1.

THEOREM 4.2. *If $n \geq k$, every graph $\mathscr{P}_{n,k}$ (with the single exception of the wheel W_n if $n = k$, and otherwise without exception) can be generated by applying a vertex-splitting operation to some graph in $\mathscr{P}_{n-1,k}$.*

Theorem 4.2 effectively halves the work required to generate $\mathscr{P}_{n,k}$ for $n > k$, since it says that we need only apply vertex-splitting to every vertex in $\mathscr{P}_{n-1,k}$. There is a natural correspondence between possible vertex-splitting operations and pairs of oriented edges (e_1, e_2) with a common tail. Hence the amount of work can be further reduced by choosing one oriented edge e_1 from each automorphism class and only applying the corresponding vertex-splitting operations. Also, it is only necessary to consider as candidates for e_2 the first half of the edges that have the same tail as e_1, moving clockwise about e_1. This is because the remaining pairs will be encountered with e_2 and e_1 playing opposite roles. Notice that it is important for efficiency reasons to modify the hash function of Section 3 to take face valences into account. We used the following modification of (3.1):

$$h(G) = \sum_{j=1}^{m} s_j p^{j-1} \mod q.$$

Here, for each edge, we compute the sum of the squares of the degrees of the two endpoints and a small multiple (we used 5) of the squares of the valences of the two faces incident on the edge. The s_j's are these m computed values, sorted into ascending order.

In the special case of generating the "diagonal" entries $\mathscr{P}_{n,n}$ from $\mathscr{P}_{n-1,n}$, further saving of work is possible. It follows from Theorem 4.2 that for any $G \in \mathscr{P}_{n,n} - \{W_n\}$, both G and its dual G^* will be generated. Hence, we introduce the notion of a *representative* of each dual pair. Whenever we generate a candidate graph G, we determine whether it is the representative. If it is the representative, we check whether it is a duplicate and proceed accordingly; if it is not the representative, we eliminate it immediately. This scheme saves disk space (since we only have to store one representative of each dual pair) and work (since we save roughly half the checks for duplicates).

To implement the representative scheme, we introduce a 2-variable selection function, $s(x, y)$, with the properties that (1) the value of $s(x, y)$ is always either x or y, and (2) $s(x, y) = s(y, x)$. Given a graph G for which $h(G) \neq h(G^*)$, we say G is the representative of the pair if and only if $s(h(G), h(G^*)) = h(G)$. (Here $h(\cdot)$ is the hash function.) A more precise description is as follows. For each candidate graph G, we compute $h(G)$ and $h(G^*)$. If $h(G) \neq h(G^*)$, we determine whether G is the representative; if so, we check whether G is a duplicate, otherwise we discard it immediately. If $h(G) = h(G^*)$, we check whether G is a duplicate; if it is not, we check whether G^* is a duplicate.

Notice that if G is self-dual, we will only do the second check the first time that G appears as a candidate graph. Otherwise, we do two duplicate checks only in the (rare) case where G is not self-dual but $h(G) = h(G^*)$.

TABLE IV

Values of $|\mathscr{P}_{n,k}|$, the Number of Nonisomorphic Polyhedral Graphs Having n Vertices and k Faces

	n											
k	4	5	6	7	8	9	10	11	12	13	14	15
4	1											
5		1	1									
6		1	2	2	2							
7			2	8	11	8	5					
8			2	11	42	74	76	38	14			
9				8	74	296	633	768	558	219	50	
10				5	76	633	2,635	6,134	8,822	7,916	4,442	1,404
11					38	768	6,134	25,626	64,439	104,213	112,082	79,773
12					14	558	8,822	64,439	268,394	709,302	1,263,032	1,556,952
13						219	7,916	104,213	709,302	2,937,495	8,085,725	15,535,572
14						50	4,442	112,082	1,263,032	8,085,725	33,310,550	?
15							1,404	79,773	1,556,952	15,535,572	?	?
16							233	36,528	1,338,853	21,395,274	?	?
17								9,714	789,749	21,317,178	?	?
18								1,249	306,470	15,287,112	?	?
19									70,454	7,706,577	?	?
20									7,595	2,599,554	?	?
21										527,235	?	?
22										49,566	?	?
23											4,037,671	?
24											339,722	?
25												?
26												2,406,841
Total	1	2	7	34	257	2,606	32,300	440,564	6,384,634	96,262,938	?	?
Self-dual	1	1	2	6	16	50	165	554	1,908	6,667	23,556	?

TABLE V

Number of Polyhedra with Up to 26 Edges

m	Total	Self-dual	Dual pairs
6	1	1	1
7	0		0
8	1	1	1
9	2		1
10	2	2	2
11	4		2
12	12	6	9
13	22		11
14	58	16	37
15	158		79
16	448	50	249
17	1,342		671
18	4,199	165	2,182
19	13,384		6,692
20	43,708	554	22,131
21	144,810		72,405
22	485,704	1,908	243,806
23	1,645,576		822,788
24	5,623,571	6,667	2,815,119
25	19,358,410		9,679,205
26	67,078,828	23,556	33,551,192

TABLE VI

Number of 4-Regular Polyhedra with Up to 22 Vertices

n	4-regular	4-connected
6	1	1
7	0	0
8	1	1
9	1	0
10	3	3
11	3	1
12	11	8
13	18	7
14	58	37
15	139	55
16	451	220
17	1,326	499
18	4,461	1,862
19	14,554	5,174
20	59,957	18,258
21	171,159	57,107
22	598,102	198,474

Notice also that the selection function should be chosen so as not to skew the uniform distribution of the hash function; for example, $s(x, y) = \max(x, y)$ would be a bad choice. In our implementation, we chose

$$s(x, y) = \begin{cases} \max(x, y) & \text{if} \quad x + y \bmod r \text{ is even} \\ \min(x, y) & \text{otherwise} \end{cases}$$

for a large prime r.

Table IV contains the values of $|\mathscr{P}_{n,k}|$ for all feasible pairs with $n \leqslant 12$ and for selected values with $n \leqslant 15$. Question marks indicate unknown values, blank entries indicate infeasible pairs.

With the exception of $|\mathscr{P}_{16, 28}| = 17{,}490{,}241$, not shown because of space, the table implicitly contains all known values (since $|\mathscr{P}_{n,k}| = |\mathscr{P}_{k,n}|$).

TABLE VII

The Number of Nonisomorphic Polyhedral Graphs Having n Vertices, k Faces, and Minimum Degree at Least 4

	n												
k	4	5	6	7	8	9	10	11	12	13	14	15	Total
4	0												0
5		0	0										0
6		0	0	0	0								0
7			0	0	0	0	0						0
8			1	0	0	0	0	0	0				1
9				0	0	0	0	0	0	0	0		0
10				1	1	0	0	0	0	0	0	0	2
11					1	1	0	0	0	0	0	0	2
12					2	4	3	0	0	0	0	0	9
13						4	10	3	0	0	0	0	17
14						5	25	36	11	0	0	0	77
15							17	107	119	18	0	0	261
16							12	159	580	456	58	0	1,265
17								89	1,095	2,815	1,714	139	5,852
18								34	1,089	7,562	14,102	6,678	
19									491	10,096	47,890	67,651	
20									130	7,485	85,805	288,534	
21										2,806	87,124	651,596	
22										525	51,844	870,969	
23											16,534	712,861	
24											2,472	355,286	
25												98,587	
26												12,400	
Total	0	0	1	1	4	14	67	428	3,515	31,763	307,543	3,064,701	

Values for $n \leqslant 9$ were first published in [21]. Values for $n \leqslant 10$ and for (11, 11), (11, 12), (11, 13), and (12, 12) first appeared in [18]. The remaining values for $n \leqslant 11$ were first published in [20]. Values for (12, 13), (12, 14), (12, 15), (13, 13), (13, 14), and (14, 14) were first computed by Duijvestijn [17]. The value for (13, 15) was independently computed by Duijvestijn [17]. All other values appearing in Table IV are new. Table V, included for completeness, contains the number of polyhedra with m edges for all $m \leqslant 26$.

We now present enumerations of certain subsets of $\mathscr{P}_{n,k}$. Table VI shows the number of 4-regular polyhedra with 22 or fewer vertices. These were generated using the inductive algorithm given in [4]. By Euler's formula, a 4-regular polyhedron with n vertices has exactly $n+2$ faces. The third column of Table VI contains the number of 4-connected 4-regular, polyhedra with n vertices.

TABLE VIII

The Number of Nonisomorphic Polyhedral Graphs Having n Vertices, k Faces, and Minimum Degree at Least 4

	n												
k	4	5	6	7	8	9	10	11	12	13	14	15	Total
4	1												1
5		0	0										0
6		0	0	0	0								0
7			0	0	0	0	0						0
8			1	0	0	0	0	0	0				
9				0	0	0	0	0	0	0	0		0
10				1	1	0	0	0	0	0	0	0	2
11					1	0	0	0	0	0	0	0	1
12					2	3	3	0	0	0	0	0	8
13						3	7	1	0	0	0	0	11
14						4	20	24	8	0	0	0	56
15							13	70	70	7	0	0	160
16							10	112	366	252	37	0	777
17								60	686	1,591	867	55	3,259
18								25	700	4,416	7,497	3,207	
19									307	5,897	25,912	33,539	
20									87	4,401	47,030	146,823	
21										1,616	47,640	335,055	
22										313	28,289	449,468	
23											8,875	366,007	
24											1,357	181,118	
25												49,504	
26												6,244	
Total	1	0	1	1	4	10	53	292	2,224	18,493	167,504	1,571,020	

TABLE IX

The Number of Nonisomorphic Bipartite Polyhedral Graphs Having n Vertices and k Faces

n	k = 6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	Total
8	1																	1
9		0																0
10		0	1															1
11			0	1														1
12			1	1	3													5
13				0	2	3												5
14				1	3	6	11											21
15				0	7	15	18											40
16					2	9	41	66	58									176
17						0	24	125	212	139								500
18						2	27	178	602	793	451							2,053
19							0	106	807	2,400	2,893	1,326						7,532
20							8	92	958	4,482	10,407	10,798	4,461					31,206
21								0	420	4,964	21,454	42,992	40,168	14,554				124,552
22								8	322	4,554	31,259	104,549	180,123	150,560	49,957			521,332
23									0	1,808	28,430	168,978	489,381	743,052	564,791	171,159		2,167,599
24									32	1,172	22,400	199,637	915,536	2,261,066	3,060,071	2,125,863	598,102	9,183,879
25										0								
26										57								
Total	1	0	2	3	10	27	126	593	3,411	20,369								

Table VII shows the number of (n, k)-polyhedra in which every vertex has degree at least 4 for $n \leqslant 15$. Each column was generated by applying the subtractive method, starting with the minimally-4-valent simplicial polyhedra with n vertices. Notice that applying the subtractive method is clearly valid (since a polyhedron obtained by adding an edge to a polyhedron with minimum degree 4 also has minimum degree 4), but the additive method may not be. Table VIII shows the number of 4-connected, (n, k)-polyhedra for $n \leqslant 15$. It was generated by testing each polyhedron with minimum degree 4 for 4-connectivity. (Notice that the tetrahedron is a special case: it is 4-connected but has minimum degree 3.)

Table IX shows the number of bipartite (n, k)-polyhedra for $n \leqslant 24$. The values for each fixed n were computed by starting with the set of 4-regular polyhedra with $n-2$ vertices (and n face), computing their duals, and then applying the subtractive method. This is valid because it is always possible to add edges to any bipartite polyhedron to obtain a quadrangulation.

Table X shows the number of *irreducible* polyhedra, which we define to be those polyhedra that do not have a removable edge. (In other words, these are wheels plus the counterexamples to the statement obtained by

TABLE X

The Number of Nonisomorphic Irreducible Polyhedral Graphs Having n Vertices and k Faces

	k											
n	4	5	6	7	8	9	10	11	12	13	14	Total
4	1											1
5		1	0									1
6		1	1	0	0							2
7			2	1	0	0	0					3
8			2	6	1	0	0	0	0			9
9				8	10	1	0	0	0	0	0	19
10				5	44	21	1	0	0	0	0	71
11					38	173	37	1	0	0	0	249
12					14	362	607	74	1	0	0	1,058
13						219	2,348	1,999	138	1	0	4,705
14						50	3,073	12,611	6,370	275	1	22,380
15							1,404	28,885	58,753	20,025	?	
16							233	26,698	209,516	253,015	?	
17								9,714	329,165	1,274,772	?	
18								1,249	232,981	3,039,562	?	
19									70,454	3,569,749	?	
20									7,595	2,038,206	?	
21										527,235	?	
22										49,566	?	
Total	1	2	5	20	107	826	7,703	81,231	914,973	10,772,406		

substituting "$k > n$" in Theorem 4.1.) The irreducible polyhedra with n vertices and k faces were generated by filtering the collection $\mathscr{P}_{n,k}$. It is an open question whether there is a more efficient way of generating them.

5. Generating Non-Hamiltonian Polyhedra

Table XI contains the number of non-Hamiltonian simplicial polyhedra for $n \geqslant 11$. Barnette and Jucocoviv̌ have shown [2] that the count is 0 for $n < 11$. The values for $n \leqslant 16$ were obtained by filtering $\mathscr{S}_n$. The value for $n = 17$ was obtained by applying the `augment` operation of Section 3 to each polyhedron in $\mathscr{S}_{16}$ (using the optimizations discussed in Section 3, but only keeping the candidate graphs that are not duplicates and are also non-Hamiltonian).

We define a non-Hamiltonian simplicial polyhedron with n vertices to be *imprimitive* if it can be obtained from some *non-Hamiltonian* simplicial polyhedron by an application of the `augment` operation. A non-Hamiltonian simplicial polyhedron is *primitive* if it cannot be so generated (i.e., if any polyhedron obtained by performing the inverse of the vertex-strechting operation shown in Fig. 1 is Hamiltonian). The primitive non-Hamiltonian simplicial polyhedra are counted in the third column of Table XI. The following conjecture is suggested by our observations for $n \leqslant 17$.

Conjecture 5.1. Every primitive non-Hamiltonian simplicial polyhedron has an odd number of vertices.

TABLE XI

Counts of Nonisomorphic, Non-Hamiltonian Simplicial Polyhedra

n	Non-Hamiltonian simplicial polyhedra	Primitive non-Hamiltonian simplicial polyhedra	Non-Hamiltonian 1-tough simplicial polyhedra	Non-Hamiltonian 1-supertough simplicial polyhedra
11	1	1	0	0
12	2	0	0	0
13	30	5	1	0
14	239	0	6	0
15	2,369	32	72	1
16	22,039	0	847	4
17	205,663	227	9,801	58
18	1,879,665[a]	0[a]	108,236[a]	698[a]

[a] Assuming Conjecture 5.1 is true.

TABLE XII

The Number of Nonisomorphic Non-Hamiltonian Polyhedra Graphs Having n Vertices and k Faces

	n														
k	11	12	13	14	15	16	17	18	19	20	21	22	23	24	Total
8	0	0													0
9	1	0	0	0	0										1
10	2	2	4	0	0	0									8
11	6	17	46	40	26	0	0	0							135
12	12	72	318	645	808	519	183	0	0	0					2,557
13	16	188	1,371	4,885	10,857	14,632	12,538	6,100	1,524	0	0	0			52,111
14	16	326	3,783	21,844	77,099								0	0	
15	12	390	7,245	64,651	344,403										
16	6	326	9,857	134,552	1,055,131										
17	2	188	9,677	203,772	2,336,015										
18	1	72	6,843	228,238	3,849,881										
19		17	3,413	189,592	4,800,886										
20		2	1,157	115,642	4,555,010										
21			240	50,449	3,274,683										
22			30	14,948	1,757,521										
23				2,711	683,283										
24				239	182,323										
25					29,925										
26					2,369										
Total	74	1,600	43,984	1,032,208	22,960,220										

We note that Observation 2, from the list in Section 3, no longer holds when computing imprimitive simplicial polyhedra. The last two columns in Table XI are discussed in the next section. The last row in Table XI was obtained by computing the imprimitive non-Hamiltonian simplicial polyhedra; hence, as indicated in the table, it is valid iff there are no primitive simplicial polyhedra with 18 vertices (which would be true if Conjecture 5.1 holds).

Table XII contains the number of non-Hamiltonian polyhedra with n vertices and k faces for all $n \leqslant 15$ and all $k \leqslant 13$. Holton and McKay have shown that there are no non-Hamiltonian trivalent polyhedra with $n < 38$ [26]; these zero values are not all reflected in the table. Also, the values for (16, 28) and (17, 30), which are not shown in Table XII, appear in Table XI.

The non-Hamiltonian (n, k)-polyhedra were computed using a combination of methods. The entries for which $\mathscr{P}_{n,k}$ had been generated were computed by filtering $\mathscr{P}_{n,k}$. The entries $(14, k)$ for $k > 14$ were computed by starting with the (14, 14) non-Hamiltonian catalog and then applying face-splitting (and, of course, filtering for non-Hamiltonian graphs, and eliminating duplicates). The (15, 14) non-Hamiltonian catalog was computed by applying vertex-splitting, filtering, and duplicate elimination to the entire catalog $\mathscr{P}_{14,14}$. The remaining $(15, k)$ entries (for $k \geqslant 15$) were then obtained in succession by starting from the (15, 14) non-Hamiltonian catalog and applying face-splitting, filtering, and elimination of duplicates.

It is not, in general, possible to compute all non-Hamiltonian (n, k)-polyhedra by starting with all non-Hamiltonian simplicial polyhedra with n vertices, applying face merging, and filtering for non-Hamiltonicity and 3-connectedness. The problem is that there exist nonsimplicial, non-Hamiltonian (n, k)-polyhedra with the property that applying any possible face-splitting operation makes the polyhedron Hamiltonian. Examples of such polyhedra with 19 vertices and 33 faces are given in Section 9. We do not know if these are the smallest such examples.

6. Some Minimum Non-Hamiltonian Polyhedra

Using the generated catalogs of polyhedra discussed above, we were able to find several smallest examples of polyhedra with interesting Hamiltonian properties. We present them here without proofs.

A graph is k-Hamiltonian if deleting any k vertices leaves a Hamiltonian graph. Thomassen has given an example of a planar graph with 105 vertices that is 1-Hamiltonian but not Hamiltonian [44]. It is shown in [16] that for $k > 1$, any k-Hamiltonian planar graph is $(k-1)$-Hamiltonian (note that $k = 2$ and $k = 3$ are the only non-vacuous cases).

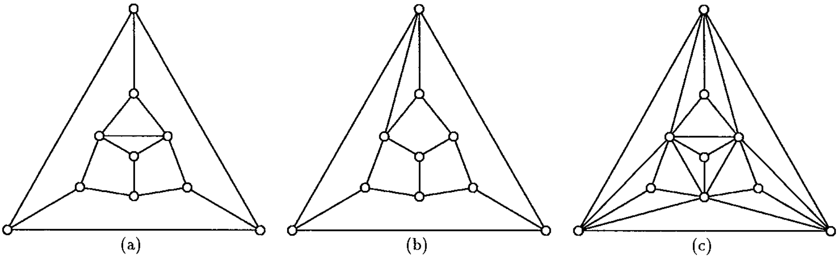

FIG. 5. Three smallest 1-supertough graphs that fail to be 1-Hamiltonian: (a), (b) are the two smallest polyhedra with this property; (c) is the smallest simplicial polyhedron with this property.

A graph is 1-*tough* if $c(G-S) \leqslant |S|$ for all nonempty $S \subseteq V(G)$. Here $G-S$ denotes the graph obtained by deleting S and all incident edges from G, and $c(\cdot)$ denotes the number of components. A graph is 1-*supertough* if deleting any vertex leaves a 1-tough graph. Any 1-tough graph is 2-connected, and any 1-supertough graph is 3-connected (so a planar, 1-supertough graph is polyhedral).

The notion of toughness of a graph was originally defined by Chvátal as a weak form of Hamiltonicity [8]. It is noted in [8] that any Hamiltonian graph is 1-tough. It follows immediately that any 1-Hamiltonian graph is 1-supertough and, hence, that any 1-Hamiltonian graph is 1-tough. The converses of these statements do not hold; here, we give smallest counterexamples for polyhedra and simplicial polyhedra.

The smallest polyhedron that is not 1-Hamiltonian is the cube, and the smallest simplicial polyhedron that is not 1-Hamiltonian is the fully stellated tetrahedron. Both these graphs are 1-tough, but not 1-supertough.

There are two nonisomorphic smallest 1-supertough planar graphs that fail to be 1-Hamiltonian. They have 10 vertices and 8 faces and are shown in Figs. 5(a) and (b). The (unique) smallest 1-supertough simplicial polyhedron that fails to be 1-Hamiltonian is shown in Fig. 5(c); it also has 10 vertices.

The smallest 1-tough, non-Hamiltonian polyhedron is the 13-vertex, 10-face example shown in Fig. 6(a). The significance of the markings on the figure will become apparent in Section 8. The smallest 1-tough, non-Hamiltonian simplicial polyhedron is the 13-vertex graph shown in Fig. 6(b). This graph has been independently discovered by Tkáč [46]. Previously, Nishizeki gave a 19-vertex example [32].

The *shortness exponent* of a class of graphs was introduced in [23] as a measure of the non-Hamiltonicity of the class. Let $h(G)$ denote the length of the longest cycle in a graph. Then for any class $\mathscr{T}$ of graphs, the shortness exponent is defined by

$$\sigma(\mathscr{T}) = \liminf_{n \to \infty} \frac{\log h(G_n)}{\log |G_n|},$$

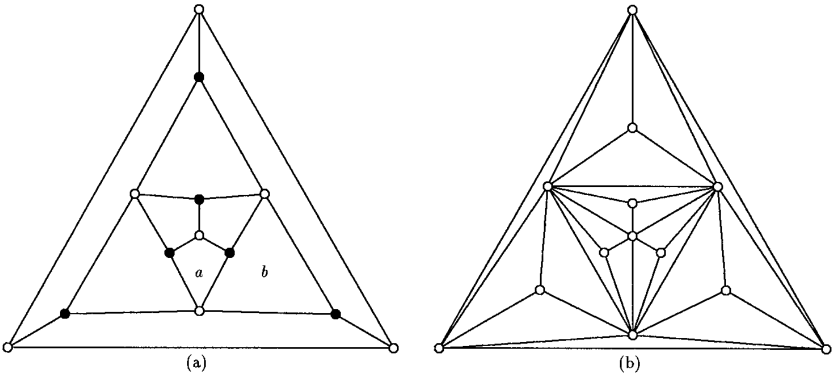

FIG. 6. (a) The smallest 1-tough, non-Hamiltonian, polyhedron. (b) The smallest 1-tough, non-Hamiltonian, simplicial polyhedron.

where the lim inf is taken over all sequences of graphs in $\mathscr{T}$ for which $|G_n| \to \infty$. By applying the construction of [12] to the graph of Fig. 6(b), it can be shown that the shortness exponent of the class of 1-tough simplicial polyhedra is at most $\log_6 5$. This observation, also made independently by Tkáč in [46], improves the bound of $\log_7 6$ given in [12].

The smallest 1-supertough, non-Hamiltonian planar graph has 15 vertices and 11 faces. It is shown in Fig. 7(a). The smallest 1-supertough, non-Hamiltonian simplicial polyhedron has 15 vertices and is shown in Fig. 7(b). The 1-tough and 1-supertough non-Hamiltonian, simplicial polyhedra with up to 17 vertices are enumerated in the last two columns of Table XI.

The structure of the simplicial examples described in this section becomes clearer if we look at the "building blocks" of Fig. 8. The smallest

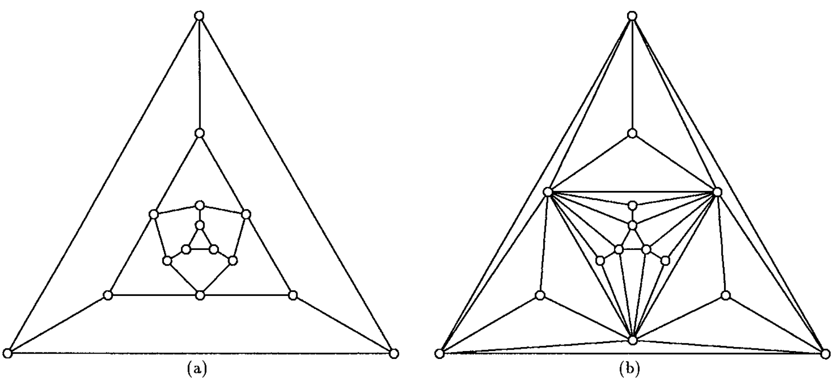

FIG. 7. (a) The smallest 1-supertough, non-Hamiltonian, planar graph. (b) The smallest 1-supertough, non-Hamiltonian, simplicial polyhedron.

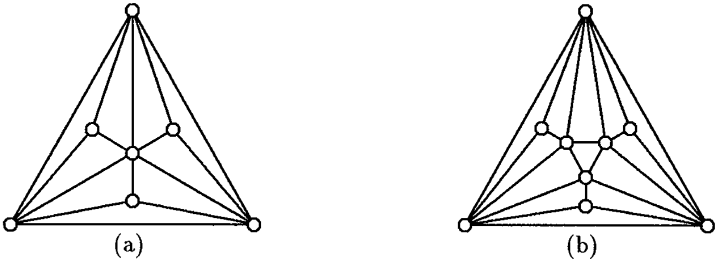

FIG. 8. Two building blocks for non-Hamiltonian simplicial polyhedra.

simplicial graph that is not 1-supertough (the stellated tetrahedron) is obtained by stellating the outer face of Fig. 8(a). The smallest simplicial graph that is 1-supertough but not 1-Hamiltonian (Fig. 5(c)) is obtained by stellating the outer face of Fig. 8(b). The 11-vertex smallest non-Hamiltonian simplicial polyhedron of [2] consists of two copies of the graph of Fig. 8(a), pasted together along a common face. The graphs of Fig. 6b and Fig. 7b consist, respectively, of one copy of Fig. 8(a) and one of Fig. 8(b), and two copies of Fig. 8(b), pasted together along a common face.

We have given the smallest examples of planar 3-connected graphs, both simplicial and non-simplicial, that remain 1-tough when j vertices are removed but fail to be k-Hamiltonian for $j, k \in \{0, 1\}$. The next logical class of graphs to consider in the progression starting with 1-tough and 1-supertough would be those graphs that remain 1-tough when two vertices are removed. However, the planar graphs with this property are exactly the 4-connected planar graphs [13, 40]. These graphs are Hamiltonian [47, 49] and 1-Hamiltonian (see [45]). Recently, Thomas and Yu have shown that all 4-connected planar graphs are 2-Hamiltonian [43], proving a long-standing conjecture of Plummer [34].

7. Inscribable Graphs and Delaunay Tessellations

A polyhedron is *inscribable* if it has a (combinatorially equivalent) realization as the edges and vertices of the convex hull of a noncoplanar set of points on the surface of a sphere in 3-space. A polyhedron is *circumscribable* if it has a (combinatorially equivalent) realization as a polyhedron each of whose faces is tangent to a common sphere. It is shown in [22] that a polyhedron is circumscribable if and only if its dual is inscribable.

A *Delaunay tessellation* is a 2-connected plane graph such that (1) the boundary vertices of the outer face are exactly the vertices of the convex hull; (2) the boundary vertices of every interior face are cocircular; and (3) no circumcircle about a face contains any vertices in its interior. A *Delaunay triangulation* is a Delaunay tessellation in which all interior faces are triangles and the boundary vertices of the outer face are exactly the extreme points of the vertex set.[3] For a more conventional definition of Delaunay triangulations and tessellations as duals of Voronoi diagrams and for a systematic exposition of their fundamental properties, see [1, 19, 35]. The word *nondegenerate* is sometimes used to distinguish Delaunay triangulations as we have defined them here. (A *degenerate* Delaunay triangulation is a triangulation obtained by adding edges to a Delaunay tessellation that is not a Delaunay triangulation.)

We state without proof several results about inscribable polyhedra, Delaunay triangulations, and the relations between them.

THEOREM 7.1 (Rivin [36]; also see [24, 38, 39]). *A polyhedron is inscribable if and only if weights w can be assigned to its edges such that*:

(W1) *For each edge e*, $0 < w(e) < 1/2$.

(W2) *For each vertex v, the total weight of all edges incident on v is equal to* 1.

(W3) *For each noncoterminous cutset* $C \subseteq E(G)$, *the total weight of all edges in C is strictly greater than* 1.

THEOREM 7.2 [15]. *A plane graph G is realizable as a Delaunay tessellation, with a given face f as the unbounded face and with a subset S of the boundary vertices of f as its extreme vertices, if and only if the graph G′ obtained by partially stellating face f, connecting the stellating vertex to the vertices of S, is inscribable. In particular, a plane graph G is realizable as a Delaunay triangulation, with a given face f as the unbounded face, if and only if the graph G′ obtained from G by fully stellating f is simplicial and inscribable.*

THEOREM 7.3. *The following properties hold*:

(a) *Every* 1-*Hamiltonian, planar graph is inscribable* [16].

(b) *Every inscribable graph is* 1-*tough* [11].

(c) *Every nonbipartite inscribable graph is* 1-*supertough* [14].

(d) *Every nonbipartite Delaunoy tessellation is* 1-*tough* [11].

[3] Here we use the definition of a triangulation commonly accepted in computational geometry [35], namely a 2-connected plane graph in which every face except possibly the unbounded face is bounded by a triangle.

(e) *If a bipartite Delaunoy tessellation is not* 1*-tough, then all the extreme vertices of the tessellation are in the same subset with respect to the bipartition* [11].

(f) *If G is inscribable and nonbipartite, any graph obtained from G by connecting two nonadjacent vertices on a common face is inscribable* [15].

Table XIII contains the number of noncircumscribable and noninscribable simplicial polyhedra for small values of n. (By duality, these numbers equal the number of inscribable and circumscribable trivalent, polyhedra with $2n-4$ vertices). Both classes of polyhedra were computed by applying filters to the collection of simplicial polyhedra. The polyhedra were tested for circumscribability using the linear-time algorithm of [14]. The simplicial noninscribable polyhedra were computed using the following "triage" procedure. By Theorem 7.3.(a), any 1-Hamiltonian polyhedron is inscribable. By Theorem 7.3(c), any simplicial polyhedron that fails to be 1-supertough is noninscribable. The remaining polyhedra (i.e., those that are 1-supertough but not 1-Hamiltonian) were then tested using an algorithm due to Rivin, based on Theorem 7.1 (See [37] for details). The counts for 1-supertough and non-1-Hamiltonian simplicial polyhedra are also included in Table XIII. All 1-supertough simplicial polyhedra with up to 14 vertices are inscribable. However, there are six noninscribable 1-supertough, simplicial polyhedra with 15 vertices. One of them is the graph of Fig. 7(b); for an explicit proof of this fact, see [14]. Smith has given

TABLE XIII

The Number of Noncircumscribable and Noninscribable Simplicial Polyhedra with n Vertices

n	$\|\mathscr{S}_n\|$	Not circ	Not 1-Ham	Not 1st	1st, not 1-Ham	Not inscr	1st, not inscr
4	1	0	0	0	0	0	0
5	1	0	0	0	0	0	0
6	2	0	0	0	0	0	0
7	5	1	0	0	0	0	0
8	14	2	1	1	0	1	0
9	50	8	1	1	0	1	0
10	233	35	10	9	1	9	0
11	1,249	168	53	48	5	48	0
12	7,595	999	383	343	40	343	0
13	49,566	6,340	2,809	2,466	343	2,466	0
14	339,722	43,133	21,884	18,905	2,979	18,905	0
15	2,406,841	305,271	172,214	146,264	25,950	146,270	6
16	17,490,241	2,231,377	1,374,908	1,150,135	224,773	1,150,197	62

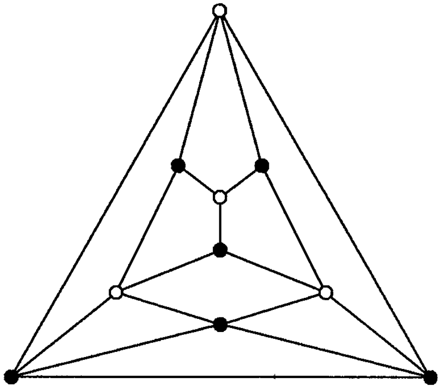

FIG. 9. A noninscribable, self-dual graph with 10 vertices.

bounds on the number of inscribable and circumscribable simplicial polyhedra [42].

The smallest noninscribable simplicial polyhedron is the fully stellated tetrahedron, and the smallest noncircumscribable simplicial polyhedron is the dual of the "clipped cube" (the polyhedron obtained by slicing off a corner of the cube with a plane, turning it into a triangle). The smallest polyhedron that is neither inscribable nor circumscribable is the self-dual polyhedron shown in Fig. 9. This polyhedron is noninscribable because it is not 1-supertough, as can be seen by deleting the four white vertices.

8. Smallest Non-Hamiltonian Delaunay Triangulations and Tessellations.

The question of whether all nondegenerate Delaunay triangulations are Hamiltonian was posed in [31, 33] and, in a closely related form, in [41]. Counterexamples are known [9, 10, 29]. Here we discuss their minimality and present (new) smallest counterexamples under the additional assumption of 3-connectivity.

The smallest non-Hamiltonian graph realizable as a Delaunay tessellation is the graph obtained by deleting a vertex from the cube; its minimality follows from the fact that the cube is the smallest non-1-Hamiltonian polyhedron. This example first appeared in [29]. The smallest non-Hamiltonian Delaunay triangulation is the example of [9]. This graph may be obtained by deleting one of the degree-7 vertices from the graph in Fig. 5(c). The minimality of the example of [9] follows from Theorem 7.2, Theorem 7.3(c), and the fact that the graph of Fig. 5(c) is the smallest 1-supertough, non-1-Hamiltonian, simplicial planar graph. Both the preceding examples fail to be 3-connected. A 3-connected, non-Hamiltonian

Delaunay triangulation with 25 vertices was constructed in [10], but this example is not minimal.

There are exactly four combinatorially distinct smallest 3-connected, non-Hamiltonian plane graphs that can be realized as Delaunay tessellations. The four graphs, each of which have with 13 vertices and 10 faces, are the two bipartite plane graphs shown in Fig. 10 and the two nonbipartite graphs shown in Fig. 11. Note that here, and throughout this section, the drawings are *not* Delaunay tessellations; rather, they are representations of plane graphs that have combinatorially equivalent realizations as a Delaunay tessellation.

Figure 11(a) and (b) are both isomorphic to the graph of Fig. 6(a). Figure 11(a) (respectively, (b)) is a reembedding of the graph of Fig. 6(a) such that the face marked *a* (respectively, *b*), becomes the outer face. In both Fig. 11(a) and Fig. 11(b), the outer faces can be stellated to obtain an inscribable polyhedron either by a full stellation or by a partial stellation in which the stellating vertex is connected to the two dark vertices and one of the two white vertices. The partial stellation can be done in two different ways in Fig. 11(a), but only one in Fig. 11(b).

The minimality of the graphs of Figs. 10 and 11 can be argued as follows. We saw in Section 6 that the smallest 1-tough non-Hamiltonian polyhedron is the graph of Fig. 6(a). Hence, by Theorem 7.3(d) and (e), if there is a 3-connected, non-Hamiltonian, Delaunay tessellation G smaller than those represented in Figs. 10 and 11, then G is bipartite, and all the extreme vertices are in the same half of the bipartition. Let G' be the graph obtained by adding a new vertex in the outer face of G and connecting it to the extreme vertices of G. Then G' is bipartite, inscribable (by Theorem 7.2), and hence 1-tough (by Theorem 7.3(b)). Also, G' must have the property that deleting some vertex yields a 3-connected,

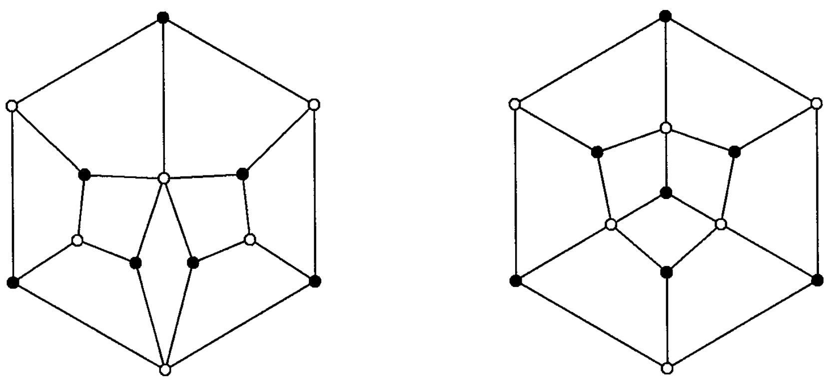

FIG. 10. The two combinatorially distinct 3-connected, non-Hamiltonian, bipartite plane graphs with 13 vertices and 10 faces that are realizable as Delaunay tessellations.

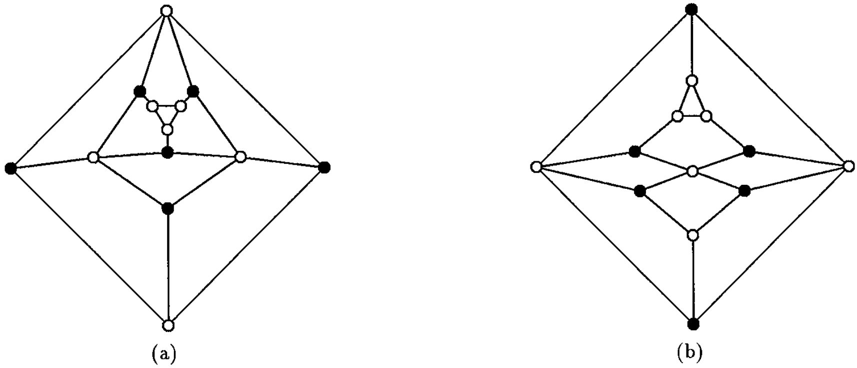

FIG. 11. The two combinatorially distinct 3-connected, non-Hamiltonian, nonbipartite plane graphs with 13 vertices and 10 faces that are realizable as Delaunay tessellations.

Hamiltonian graph. A computer scan of the bipartite polyhedra shows that there are exactly two bipartite polyhedra with these properties and 14 or fewer vertices; these are the two polyhedra obtained by partially stellating the outer faces of the graphs in Fig. 10, connecting the stellating vertices to the dark vertices on the outer faces.

There are exactly two combinatorially distinct smallest 3-connected, non-Hamiltonian plane graphs that can be realized as Delaunay triangulations. These two graphs, which have 13 vertices and 21 faces, are shown in Fig. 12. Their minimality can be argued as follows. Let G' be the graph obtained by stellating the outer face of a 3-connected, non-Hamiltonian Delaunay triangulation. By Theorem 7.2 and Theorem 7.3(c), G' must be simplicial, 1-supertough, and have the property that removing some vertex leaves a 3-connected, non-Hamiltonian graph (so, in particular, G' cannot be 1-Hamiltonian). The only two simplicial polyhedra with at most 14

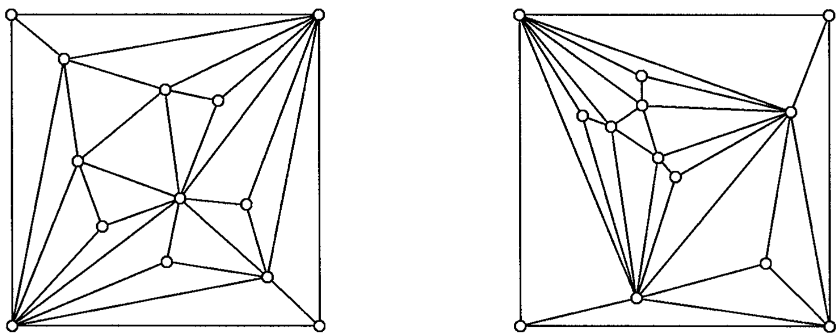

FIG. 12. The two minimal 3-connected, non-Hamiltonian triangulation that are realizable as Delaunay triangulations (13 vertices, 21 faces).

vertices that have these properties are the two 14-vertex graphs obtained by fully stellating the outer faces of the graphs of Fig. 12.

9. Small Non-Hamiltonian Inscribable Polyhedra

A 25-vertex, non-Hamiltonian, inscribable simplicial polyhedron was constructed in [10]. It follows from Theorem 7.3(a) that Thomassen's example of a 105-vertex planar graph that is 1-Hamiltonian but not Hamiltonian [44] represents an earlier discovery of a non-Hamiltonian inscribable polyhedron, although it was not so identified at the time. Here we present improved lower and upper bounds for the size of the smallest non-Hamiltonian inscribable polyhedra, both in general and in the simplicial cases. We deal with the unrestricted case first.

There are three non-Hamiltonian, inscribable polyhedra with 19 vertices and 13 polyhedra (Fig. 13). We have verified that there are no non-Hamiltonian inscribable polyhedra for any other value of n and k that has a nonempty entry in Table XII. We have also verified that there are no non-Hamiltonian bipartite inscribable polyhedra with 24 or fewer vertices; indeed, there are no 1-tough, non-Hamiltonian, bipartite polyhedra with 24 or fewer vertices. We conjecture that the three examples of Fig. 13 are the smallest examples.

For the remainder of this section, let N denote the number of vertices in the smallest non-Hamiltonian, inscribable, simplicial polyhedron. We have determined that $18 \leqslant N \leqslant 20$, and we conjecture that the true answer is 20.

The bound $N \leqslant 20$ holds because there are at least 11 nonisomorphic non-Hamiltonian, inscribable, simplicial polyhedra with 20 vertices. They are shown in Fig. 14. They were constructed by looking for "partner graphs" that could be "pasted" together with the 9-vertex "building block"

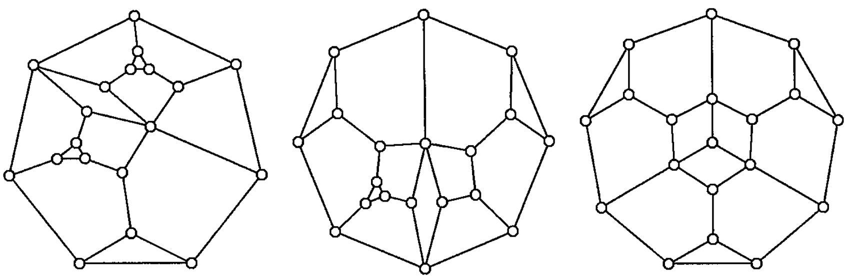

FIG. 13. Schlegel diagrams of the three non-Hamiltonian, inscribable polyhedra with 19 vertices and 13 faces.

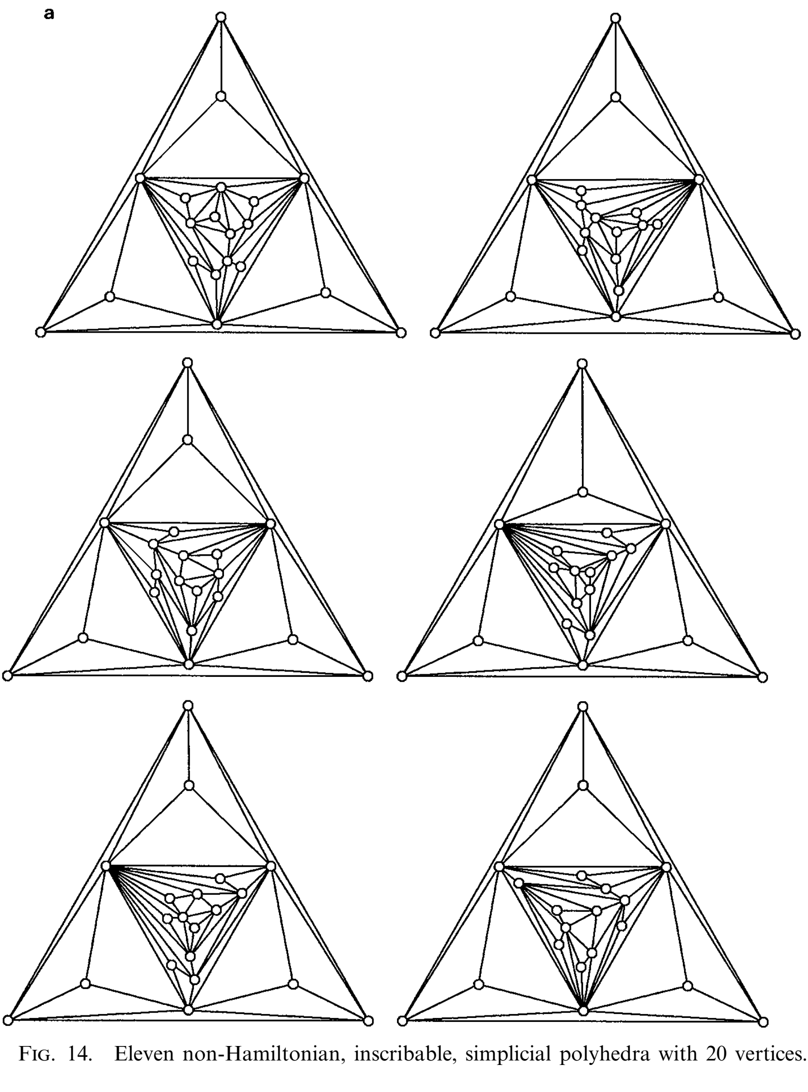

FIG. 14. Eleven non-Hamiltonian, inscribable, simplicial polyhedra with 20 vertices.

of Fig. 8(b) (which we temporarily call T_9) to obtain a non-Hamiltonian inscribable simplicial polyhedron, in the same way that pasting T_9, together with itself, creates the non-Hamiltonian, 1-supertough, simplicial planar graph of Fig. 7(b). Let K be such a "partner graph," and let f be the boundary of the face along which K is to be pasted. Assume that the orientation is such that f is the outer face of K. Clearly K must be simplicial. It can be shown that K must also have the following properties (i.e., they are necessary, but perhaps not sufficient):

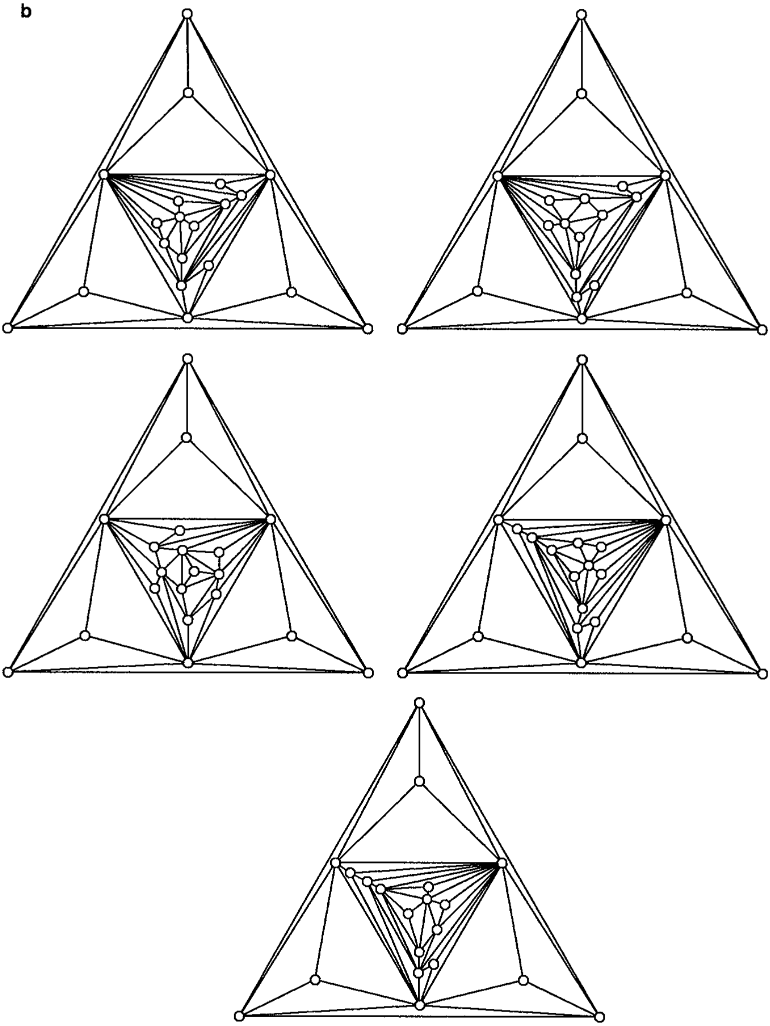

FIG. 14—*Continued*

1. *K* must have the property that for any two vertices of *f*, any path between the two vertices that visits all vertices inside *f* must also visit the third vertex of *f*. (Otherwise, the pasted graph would be Hamiltonian.)

2. *K* must have the property that when *f* is stellated, the resulting graph is 1-supertough. (Otherwise, the pasted graph would not be 1-supertough.)

3. Define a Delaunay labeling of a simplicial polyhedron to be a labeling of the interior angles so that (1) the angles about each interior

vertex sum to 360, (2) the angles about each triangle sum to 180, (3) all angles are positive, and (4) the sum of two angles facing a common edges is less than 180. K must have a Delaunay labeling in which the three angles facing the three outer edges have a total value less than 450.

We filtered $\mathscr{S}_n$ for polyhedra with these properties. We found none with 13 or fewer vertices, but 11 with 14 vertices. Each of the 20-vertex polyhedra obtained by pasting one of these 14-vertex "partner graphs" with T_9 is, indeed, non-Hamiltonian, inscribable, and simplicial. These 11 simplicial polyhedra are shown in Fig. 14.

The bound $N > 17$ was determined by examining the non-Hamiltonian, 1-supertough, simplicial polyhedra with $n \leqslant 17$ (see column 5 of Table XI) and verifying that none were inscribable.

As indicated in Table XI, there are 698 imprimitive, non-Hamiltonian, 1-supertough, simplicial polyhedra with 18 vertices. None of these are inscribable. So if Conjecture 5.1 is true, then $N > 18$.

The conjecture that $N > 19$ is based on the following experiment. We applied the `augment` operation in every possible way to the imprimitive non-Hamiltonian, 1-tough simplicial polyhedra with 18 vertices. In this fashion, we generated 9232 19-vertex, non-Hamiltonian, 1-supertough, simplicial polyhedra. None of these were inscribable. So we could conclude that $N > 19$ (and, hence, that there are no examples smaller than those of Fig. 14) if we could establish Conjecture 5.1 and the following conjectures.

Conjecture 9.1. Every imprimitive, non-Hamiltonian simplicial polyhedron with an odd number of vertices fails to be 1-tough.

Conjecture 9.2. Every non-Hamiltonian, 1-supertough simplicial polyhedron with n vertices can be generated by applying the `augment` operation in some way to some non-Hamiltonian, 1-tough simplicial polyhedron with $n-1$ vertices.

Both these conjecture are true for up to $n \leqslant 17$ vertices. In fact, a somewhat stronger statement than Conjecture 9.1 holds for these values of n: every imprimitive, non-Hamiltonian simplicial polyhedra with $n = 2j + 1$ vertices has an independent set of size $j + 1$. Also, if Conjecture 5.1 holds for $n = 18$, then so does Conjecture 9.2, as every one of the 698 imprimitive non-Hamiltonian, 1-supertough, 18-vertex simplicial polyhedron can be generated by applying the `augment` operation to some non-Hamiltonian, 1-tough, 17-vertex simplicial polyhedron.

We conclude with one more collection of relevant counterexamples. We generated, for each $k \geqslant 13$, all non-Hamiltonian $(19, k)$-polyhedra that could be obtained by starting with the three graphs of Fig. 13 and applying sequences of face-splitting operations. By Theorem 7.3(f), all polyhedra

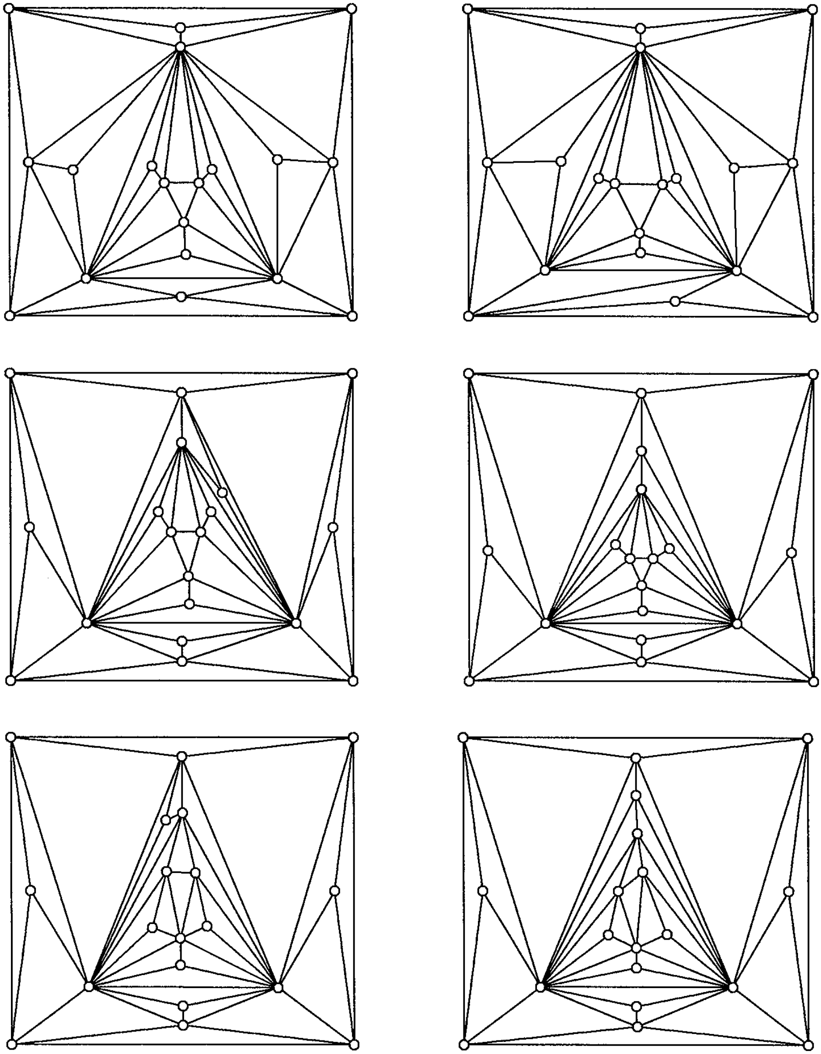

FIG. 15. Six non-Hamiltonian, inscribable polyhedra with 19 vertices and 33 faces.

obtained in this way are inscribable. This process ultimately produced the six non-Hamiltonian inscribable (19, 33)-polyhedra shown in Fig. 15. These polyhedra are inscribable triangulations, but they have one quadrangular face so they are not simplicial. In each case, adding a diagonal to the outer face (to make them simplicial) also makes them Hamiltonian. These examples justify the remarks made at the end of Section 5.

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