**Note**

Disjoint Covering Systems of Rational Beatty Sequences

**Marc A. Berger, Alexander Felzenbaum, and Aviezri S. Fraenkel**

*Faculty of Mathematical Sciences, The Weizmann Institute of Science, Rehovot 76100, Israel*

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We prove that for a disjoint covering system with rational moduli, the two largest numerators of the moduli are identical. Furthermore, if the two moduli corresponding to these two identical numerators are distinct, then actually the three largest numerators of the moduli are identical for a system with at least three moduli.

The purpose of this note is to generalize the result of Davenport, Mirsky, D. Newman, and Radó from disjoint covering systems of residue classes to disjoint covering systems of rational Beatty sequences. Recall that their result establishes equality of the two largest moduli. (See Newman [3].) If all the moduli are irrational, then also two moduli are identical (see Graham [2]).

A rational Beatty sequence, $B$, is a set of integers

$$B = \{\lfloor \alpha + nr \rfloor \mid n \in \mathbb{Z}^+\},$$

where $\alpha \in \mathbb{R}$, $r \in \mathbb{Q}$ is the modulus, $r \geq 1$, $\lfloor \cdot \rfloor$ denotes the greatest integer function and $\mathbb{Z}^+$ denotes the set of nonnegative integers (cf. Stolarsky [4]). In the theorem below we deal with rational Beatty sequences

$$B_k = \{\lfloor \alpha_k + nr_k \rfloor \mid n \in \mathbb{Z}^+ \} \quad (1 \leq k \leq t)$$

which together partition $\mathbb{Z}^+$. Surprisingly, unlike the integer and irrational cases, a partition of $\mathbb{Z}^+$ with rational Beatty sequences can exist in which all the moduli are distinct:

$$\left\{ \left\lfloor \frac{7}{4^n} \right\rfloor + 1, \left\lfloor \frac{7}{2^n} \right\rfloor, 7n + 5 : n \in \mathbb{Z}^+ \right\}.$$
We write \( r_k = p_k/q_k \) with \( p_k, q_k \in \mathbb{N} \) and \((p_k, q_k) = 1\) \((1 \leq k \leq t)\), and assume \( p_1 < p_2 < \cdots < p_t \). We also assume \( t > 1 \).

**Theorem.** If \((B_1, \ldots, B_t)\) is a partition of \( \mathbb{Z}^+ \), then

(a) \( p_{t-1} = p_t \).

(b) If also \( t \geq 3 \) and \( r_{t-1} \neq r_t \), then \( p_{t-2} = p_{t-1} - p_t \).

**Proof.**

(a) Set \( m_k = \lceil x_k/q_k \rceil \) and let

\[
a_{kn} = \lceil x_k + nr_k \rceil = \left\lfloor \frac{m_k + np_k}{q_k} \right\rfloor
\]

(1)

(where we used the fact that for any positive integer \( m \), \( \lfloor x/m \rfloor = \lfloor \lfloor x \rfloor/m \rfloor \)). Then

\[
\frac{1}{1-z} = \sum_{k=1}^{t} \sum_{n=0}^{q_k-1} \frac{z^{akn}}{1-z^{pk}},
\]

where \( z \) represents a formal complex variable. It clearly suffices (as in Newman [3, p. 281]) to establish that

\[
\sum_{n=0}^{q_t-1} \omega^{a_{kn}} \neq 0 \quad \text{for} \quad \omega = \exp \left( \frac{2\pi i}{p_t} \right).
\]

(2)

Choose \( u, v \in \mathbb{N} \) such that

\[
up_t - vq_t = 1.
\]

(3)

Then for \(-m_t \leq n < q_t - m_t\), we have

\[
\left\lfloor \frac{m_t + np_t}{q_t} \right\rfloor = nv.
\]

(4)

Let \( u_n \) denote the least nonnegative residue of \( nu \mod q_t \), that is, \( u_n \equiv nu \mod q_t \), \( 0 \leq u_n < q_t \). Then clearly

\[
\left\lfloor \frac{m_t + np_t}{q_t} \right\rfloor = \left\lfloor \frac{m_t + u_n p_t}{q_t} \right\rfloor \quad \text{(mod } p_t).\]

(5)

Since \((u, q_t) = 1\) we have

\[
\{ u_n : -m_t \leq n < q_t - m_t \} = \{0, 1, \ldots, q_t - 1\}.
\]

(6)
Let $a_n$ and $v_n$ denote the least nonnegative residues of $a_n$ and $m_n$ respectively, mod $p_t$. By (1), (4), (5), and (6),

$$\{a_n : 0 \leq n < q_t\} = \{v_n : -m_t \leq n < q_t - m_t\}.$$ 

Since for any integers, $a, k$ we have $\omega^a = \omega^{a + kp_t}$, we get from (3),

$$\sum_{n=0}^{q_t-1} \omega^{a_n} = \sum_{n=0}^{q_t-1} \omega^{a_n} = \sum_{n=-m_t}^{q_t-m_t-1} \omega^{m_n} = \frac{1-\omega^{-1}}{1-\omega^p} \omega^{-m_v},$$

and (2) follows, since $t > 1$ implies $p_t > 1$, so $\omega^{-1} \neq 1$.

(b) Using part (a) of this theorem we have

$$\frac{1}{1-z} = \sum_{k=1}^{t-2} \sum_{n=0}^{q_t-1} \frac{z^{a_{kn}}}{1-z^{p_t}} + \sum_{n=0}^{q_t-1} \frac{z^{a_{kn}}}{1-z^{p_t}}.$$ 

We use proof by contradiction. Assume that $p_{t-2} < p_{t-1}$ but the result is false. Then, as we reasoned in (2) above,

$$\sum_{n=0}^{q_{t-1}-1} \omega^{a_{t-1,n}} + \sum_{n=0}^{q_{t-1}-1} \omega^{a_{m_n}} = 0 \quad \text{for} \quad \omega = \exp\left(\frac{2\pi i}{p_t}\right).$$

As in (3) choose $u', v'$ such that

$$u'p_t - v'q_{t-1} = 1.$$ 

Applying (7) to each of the sums in (8), we obtain

$$\frac{1-\omega}{1-\omega^p} = -\omega^{m_{t-1}v' - m_v}.$$ 

In particular,

$$|1 - \omega^p| = |1 - \omega^v|$$

and thus $v \equiv \pm v'(\text{mod } p_t)$. From (3), (9) it follows that $q_{t-1} \equiv \pm q_t$ (mod $p_t$), and since $q_{t-1}, q_t \leq p_t$ and $r_{t-1} \neq r_t$ it must be that $q_{t-1} = p_t - q_t$. But this implies that $B_{t-1} \cup B_t = \mathbb{Z}^+$ by Beatty's $\sum r_k^{-1} = 1$ test for complementary sequences—which is impossible since $t \geq 3$.

**Remark.** Fraenkel [1] conjectured that if $t \geq 3$ then there must be two moduli, one of which is an integral multiple of the other. It is an immediate consequence of (b) that this conjecture is valid if at most two of the moduli are nonintegral. In fact if this is the case then either two of the moduli are equal, or else one of the integral moduli is an integral multiple of both non-integral moduli.
REFERENCES

2. R. L. GRAHAM, Covering the positive integers by disjoint sets of the form \([na+b]: n = 1, 2, \ldots\), *J. Combin. Theory* 15 (1973), 354–358.