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Factorizable quasi-Hopf algebras—applications

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Abstract

We define the notion of factorizable quasi-Hopf algebra by using a categorical point of view. We show that the Drinfeld double $D(H)$ of any finite dimensional quasi-Hopf algebra H is factorizable, and we characterize $D(H)$ when H itself is factorizable. Finally, we prove that any finite dimensional factorizable quasi-Hopf algebra is unimodular. In particular, we obtain that the Drinfeld double $D(H)$ is a unimodular quasi-Hopf algebra.

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0. Introduction

The concept of a quasi-triangular (or braided) bialgebra is due to Drinfeld [9]. Roughly speaking, a bialgebra H is quasi-triangular if the monoidal category of left H -modules is braided in the sense of Joyal and Street [16]. In other words, H is quasi-triangular if there exists an invertible element $R \in H \otimes H$ satisfying some additional relations (see the complete definition below). In the Hopf algebra case a reformulation of this definition was given by Radford [24]. Ribbon and factorizable Hopf algebras are special classes of quasi-triangular Hopf algebras, and in this theory a particular interest is produced by the Drinfeld double $D(H)$. By the Drinfeld double

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construction [9], every finite dimensional Hopf algebra H can be embedded into a finite dimensional quasi-triangular Hopf algebra $D(H)$.

As we pointed out, factorizable Hopf algebras belong to the class of quasi-triangular Hopf algebras. Suppose that (H, R) is a quasi-triangular Hopf algebra and denote by $R^1 \otimes R^2$ and $r^1 \otimes r^2$ two copies of the R -matrix R of H . Then (H, R) is called factorizable if

$$\mathcal{Q} : H^* \rightarrow H, \quad \mathcal{Q}(\chi) = \chi(R^2 r^1) R^1 r^2 \quad \forall \chi \in H^*$$

is a linear isomorphism or, equivalently, if the map

$$\overline{\mathcal{Q}} : H^* \rightarrow H, \quad \overline{\mathcal{Q}}(\chi) = \chi(R^1 r^2) R^2 r^1 \quad \forall \chi \in H^*$$

is a linear isomorphism. Factorizable Hopf algebras were introduced and studied by Reshetikhin and Semenov-Tian-Shansky [26]. They are important in the Hennings investigation of 3-manifold invariants [15]. Hennings shows how we can construct 3-manifold invariants using some finite dimensional ribbon Hopf algebras which are, in particular, factorizable. Afterwards, Kauffman reworked the Hennings construction, see [19] or [25] for more details. We note that factorizable Hopf algebras are also important in the representation theory [27], notably with applications to the classification of a certain classes of Hopf algebras, see [11].

Now, quasi-bialgebras and quasi-Hopf algebras were introduced by Drinfeld [10]. They come out from categorical considerations: putting some additional structure on the category of modules over an algebra H , the definition of a quasi-bialgebra H ensures that the category of left H -modules ${}_H\mathcal{M}$ is a monoidal category. So H is a unital associative algebra together with a comultiplication $\Delta : H \rightarrow H \otimes H$ and a usual counit $\varepsilon : H \rightarrow k$ such that Δ and ε are algebra maps, and Δ is quasi-coassociative, in the sense that it is coassociative up to conjugation by an invertible element $\Phi \in H \otimes H \otimes H$. Consequently the definition of a quasi-bialgebra is not self dual. For a quasi-Hopf algebra H the definition ensures that the category of finite dimensional left H -modules is a monoidal category with left duality. In a similar manner one can define quasi-triangular (ribbon, at least in the finite dimensional case) quasi-bialgebras: a quasi-bialgebra is called quasi-triangular (ribbon) if the monoidal category ${}_H\mathcal{M}$ is braided (ribbon, respectively). In the quasi-triangular case, this means that there exists an invertible element $R \in H \otimes H$ satisfying some additional conditions. If H is a quasi-Hopf algebra then the definition of a quasi-triangular quasi-bialgebra can be reformulated, see [7]. As we have already explained the study of quasi-Hopf algebras, or quasi-triangular (ribbon) quasi-Hopf algebras is strictly connected to the study of monoidal, or braided (ribbon) categories. Thus, in general, when we want to define some classes of quasi-Hopf algebras we should look at the classical Hopf case in the sense that we should try to reformulate their basic properties at a categorical level, and then we must come back to the quasi-Hopf case; if this is not possible then we have to be lucky in order to define (and then study) them.

As far as we are concerned, in the classical case, the definition of the map \mathcal{Q} given above has a categorical interpretation due to Majid [22]. Hence, if in the quasi-Hopf case a suitable map satisfies the same categorical interpretation then it makes sense

to define the factorizable notion. This is way we propose in Section 2 the following definition for the map \mathcal{Q} :

$$\mathcal{Q}(\chi) = \langle \chi, S(X_2^2 \tilde{p}^2) f^1 R^2 r^1 U^1 X^3 \rangle X^1 S(X_1^2 \tilde{p}^1) f^2 R^1 r^2 U^2,$$

for all $\chi \in H^*$, where $r^1 \otimes r^2$ is another copy of R , $\Phi = X^1 \otimes X^2 \otimes X^3 = Y^1 \otimes Y^2 \otimes Y^3$, and $f^1 \otimes f^2$, $U^1 \otimes U^2$, $\tilde{p}^1 \otimes \tilde{p}^2 \in H \otimes H$ are some special elements which we will define below. Moreover, we will see that in the quasi-Hopf case the analogues of the map $\overline{\mathcal{Q}}$ is

$$\overline{\mathcal{Q}}(\chi) = \langle \chi, S^{-1}(X^3) q^2 R^1 r^2 X_2^2 \tilde{p}^2 \rangle q^1 R^2 r^1 X_1^2 \tilde{p}^1 S^{-1}(X^1),$$

for all $\chi \in H^*$, where $q^1 \otimes q^2 \in H \otimes H$ is another special element which will be defined.

Following [22], in Section 4 we will give the categorical interpretation of the map \mathcal{Q} in the quasi-Hopf case. For this, we develop first in Section 3 the transmutation theory for dual quasi-Hopf algebras. Using the dual reconstruction theorem (also due to Majid) we will show that to any co-quasi-triangular dual quasi-Hopf algebra A we can associate a braided commutative Hopf algebra \underline{A} in the category of right A -comodules. Keeping the same terminology as in the Hopf case we will call \underline{A} the function algebra braided group associated to A . This procedure is the formal dual of the one performed in [6] where to any quasi-triangular quasi-Hopf algebra H is associated a braided cocommutative group \underline{H} in the braided category of left H -modules. We call \underline{H} the associated enveloping algebra braided group of H . We notice that, in the finite dimensional case, \underline{A} cannot be obtained from \underline{H} by (usual) dualisation. In fact, if H is finite dimensional then the map \mathcal{Q} provides a braided Hopf algebra morphism between the function algebra braided group \underline{H}^* associated to H^* and \underline{H} (Proposition 4.1). Moreover, \underline{H}^* is always isomorphic to the categorical dual of \underline{H} as braided Hopf algebra (Proposition 4.2). So the true meaning of the map \mathcal{Q} is that \underline{H} and \underline{H}^* are self dual (in a categorical sense) provided \mathcal{Q} is bijective, i.e. H is factorizable.

Let H be a finite dimensional quasi-Hopf algebra and $D(H)$ the Drinfeld double of H . Similarly to the Hopf case we will show in Section 2 that $D(H)$ is always factorizable. The description of $D(H)$ when H is quasi-triangular was given in [4]. In this case $D(H)$ is a biproduct (in the sense of [6]) of a braided Hopf algebra B^i and H , and, as a vector space, B^i is isomorphic to H^* . Furthermore, in Section 5 we will see that the Drinfeld double $D(H)$ has a very simple description when H itself is factorizable. In fact, we will give a quasi-Hopf version of a result claimed in [26] and proved in [27]. We will show that if H is factorizable then $D(H)$ is isomorphic as a quasi-Hopf algebra to a twist of a usual (componentwise) tensor product quasi-Hopf algebra $H \otimes H$. To this end we need the alternative definition for the space of coinvariants of a right quasi-Hopf H -bimodule and the second structure theorem for right quasi-Hopf H -bimodules proved in [3]. Finally, in Section 6 we will prove that any finite dimensional factorizable quasi-Hopf algebra is unimodular. In particular, we deduce that the Drinfeld double $D(H)$ is a unimodular quasi-Hopf algebra.

As we will see, the theory of quasi-Hopf algebras is technically more complicated than the classical Hopf algebra theory. This happens because of the nature of the problems which occur in the quasi-Hopf algebra theory. When we pass from Hopf algebras

to quasi-Hopf algebras the appearance of the reassociator Φ and of the elements α and β in the definition of the antipode increases the complexity of formulas, and therefore of computations and proofs.

1. Preliminaries

1.1. Quasi-Hopf algebras

We work over a commutative field k . All algebras, linear spaces etc. will be over k ; unadorned \otimes means \otimes_k . Following Drinfeld [10], a quasi-bialgebra is a four-tuple $(H, \Delta, \varepsilon, \Phi)$ where H is an associative algebra with unit, Φ is an invertible element in $H \otimes H \otimes H$, and $\Delta: H \rightarrow H \otimes H$ and $\varepsilon: H \rightarrow k$ are algebra homomorphisms satisfying the identities

$$(id \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes id)(\Delta(h))\Phi^{-1}, \quad (1.1)$$

$$(id \otimes \varepsilon)(\Delta(h)) = h \otimes 1, \quad (\varepsilon \otimes id)(\Delta(h)) = 1 \otimes h, \quad (1.2)$$

for all $h \in H$, and Φ has to be a 3-cocycle, in the sense that

$$(1 \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi)(\Phi \otimes 1) = (id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi), \quad (1.3)$$

$$(id \otimes \varepsilon \otimes id)(\Phi) = 1 \otimes 1 \otimes 1. \quad (1.4)$$

The map Δ is called the coproduct or the comultiplication, ε the counit and Φ the reassociator. As for Hopf algebras we denote $\Delta(h) = h_1 \otimes h_2$, but since Δ is only quasi-coassociative we adopt the further convention (summation understood):

$$(\Delta \otimes id)(\Delta(h)) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_2, \quad (id \otimes \Delta)(\Delta(h)) = h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},$$

for all $h \in H$. We will denote the tensor components of Φ by capital letters, and the ones of Φ^{-1} by small letters, namely

$$\Phi = X^1 \otimes X^2 \otimes X^3 = T^1 \otimes T^2 \otimes T^3 = V^1 \otimes V^2 \otimes V^3 = \dots$$

$$\Phi^{-1} = x^1 \otimes x^2 \otimes x^3 = t^1 \otimes t^2 \otimes t^3 = v^1 \otimes v^2 \otimes v^3 = \dots$$

H is called a quasi-Hopf algebra if, moreover, there exists an anti-morphism S of the algebra H and elements $\alpha, \beta \in H$ such that, for all $h \in H$, we have:

$$S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad h_1\beta S(h_2) = \varepsilon(h)\beta, \quad (1.5)$$

$$X^1\beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad S(x^1)\alpha x^2\beta S(x^3) = 1. \quad (1.6)$$

Our definition of a quasi-Hopf algebra is different to the one given by Drinfeld [10] in the sense that we do not require the antipode to be bijective. Nevertheless, in the finite dimensional or quasi-triangular case the antipode is automatically bijective, cf. [3] and [7]. In this way we omit the bijectivity of S in the definition of a quasi-Hopf algebra.

For a quasi-Hopf algebra the antipode is determined uniquely up to a transformation $\alpha \mapsto \alpha_{\cup} := \cup\alpha$, $\beta \mapsto \beta_{\cup} := \beta\cup^{-1}$, $S(h) \mapsto S_{\cup}(h) := \cup S(h)\cup^{-1}$, where $\cup \in H$ is invertible. In this case we will denote by H^{\cup} the new quasi-Hopf algebra $(H, \Delta, \varepsilon, \Phi, S_{\cup}, \alpha_{\cup}, \beta_{\cup})$.

The axioms for a quasi-Hopf algebra imply that $\varepsilon \circ S = \varepsilon$ and $\varepsilon(\alpha)\varepsilon(\beta) = 1$, so, by rescaling α and β , we may assume without loss of generality that $\varepsilon(\alpha) = \varepsilon(\beta) = 1$. Identities (1.2)–(1.4) also imply that

$$(\varepsilon \otimes id \otimes id)(\Phi) = (id \otimes id \otimes \varepsilon)(\Phi) = 1 \otimes 1 \otimes 1. \tag{1.7}$$

Next we recall that the definition of a quasi-Hopf algebra is “twist coinvariant” in the following sense. An invertible element $F \in H \otimes H$ is called a gauge transformation or twist if $(\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1$. If H is a quasi-Hopf algebra and $F = F^1 \otimes F^2 \in H \otimes H$ is a gauge transformation with inverse $F^{-1} = G^1 \otimes G^2$, then we can define a new quasi-Hopf algebra H_F by keeping the multiplication, unit, counit and antipode of H and replacing the comultiplication, reassociator and the elements α and β by

$$\Delta_F(h) = F\Delta(h)F^{-1}, \tag{1.8}$$

$$\Phi_F = (1 \otimes F)(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1), \tag{1.9}$$

$$\alpha_F = S(G^1)\alpha G^2, \quad \beta_F = F^1\beta S(F^2). \tag{1.10}$$

It is well-known that the antipode of a Hopf algebra is an anti-coalgebra morphism. For a quasi-Hopf algebra, we have the following statement: there exists a gauge transformation $f \in H \otimes H$ such that

$$f\Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{op}(h)), \quad \text{for all } h \in H, \tag{1.11}$$

where $\Delta^{op}(h) = h_2 \otimes h_1$. f can be computed explicitly. First set

$$A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (\Phi \otimes 1)(\Delta \otimes id \otimes id)(\Phi^{-1}),$$

$$B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes id \otimes id)(\Phi)(\Phi^{-1} \otimes 1)$$

and then define $\gamma, \delta \in H \otimes H$ by

$$\gamma = S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4 \quad \text{and} \quad \delta = B^1\beta S(B^4) \otimes B^2\beta S(B^3). \tag{1.12}$$

f and f^{-1} are then given by the formulas

$$f = (S \otimes S)(\Delta^{op}(x^1))\gamma\Delta(x^2\beta S(x^3)), \tag{1.13}$$

$$f^{-1} = \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{op}(x^3)). \tag{1.14}$$

Moreover, f satisfies the following relations:

$$f\Delta(\alpha) = \gamma, \quad \Delta(\beta)f^{-1} = \delta. \tag{1.15}$$

Furthermore the corresponding twisted reassociator (see (1.9)) is given by

$$\Phi_f = (S \otimes S \otimes S)(X^3 \otimes X^2 \otimes X^1). \tag{1.16}$$

In a Hopf algebra H , we obviously have the identity

$$h_1 \otimes h_2 S(h_3) = h \otimes 1, \quad \text{for all } h \in H.$$

We will need the generalization of this formula to quasi-Hopf algebras. Following [12,13], we define

$$p_R = p^1 \otimes p^2 = x^1 \otimes x^2 \beta S(x^3), \quad q_R = q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3) X^2. \quad (1.17)$$

$$p_L = p_L^1 \otimes p_L^2 = X^2 S^{-1}(X^1 \beta) \otimes X^3, \quad q_L = q_L^1 \otimes q_L^2 = S(x^1) \alpha x^2 \otimes x^3. \quad (1.18)$$

For all $h \in H$, we then have

$$\Delta(h_1) p_R [1 \otimes S(h_2)] = p_R [h \otimes 1], \quad [1 \otimes S^{-1}(h_2)] q_R \Delta(h_1) = (h \otimes 1) q_R, \quad (1.19)$$

$$\Delta(h_2) p_L [S^{-1}(h_1) \otimes 1] = p_L (1 \otimes h), \quad [S(h_1) \otimes 1] q_L \Delta(h_2) = (1 \otimes h) q_L, \quad (1.20)$$

and

$$\Delta(q^1) p_R [1 \otimes S(q^2)] = 1 \otimes 1, \quad [1 \otimes S^{-1}(p^2)] q_R \Delta(p^1) = 1 \otimes 1, \quad (1.21)$$

$$[S(p_L^1) \otimes 1] q_L \Delta(p_L^2) = 1 \otimes 1, \quad \Delta(q_L^2) p_L [S^{-1}(q_L^1) \otimes 1] = 1 \otimes 1, \quad (1.22)$$

$$\begin{aligned} & \Phi(\Delta \otimes id)(p_R)(p_R \otimes id) \\ &= (id \otimes \Delta)(\Delta(x^1) p_R)(1 \otimes f^{-1})(1 \otimes S(x^3) \otimes S(x^2)), \end{aligned} \quad (1.23)$$

$$(1 \otimes q_L)(id \otimes \Delta)(q_L) \Phi = (S(x^2) \otimes S(x^1) \otimes 1)(f \otimes 1)(\Delta \otimes id)(q_L \Delta(x^3)), \quad (1.24)$$

where $f = f^1 \otimes f^2$ is the twist defined in (1.13).

Note that some of the above formulas use the bijectivity of the antipode S . Nevertheless, we will restrict ourselves to apply them only in this case.

1.2. Quasi-triangular quasi-Hopf algebras

A quasi-Hopf algebra H is quasi-triangular (QT for short) if there exists an element $R \in H \otimes H$ such that

$$(\Delta \otimes id)(R) = \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi, \quad (1.25)$$

$$(id \otimes \Delta)(R) = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi^{-1}, \quad (1.26)$$

$$\Delta^{\circ p}(h) R = R \Delta(h), \quad \text{for all } h \in H, \quad (1.27)$$

$$(\varepsilon \otimes id)(R) = (id \otimes \varepsilon)(R) = 1, \quad (1.28)$$

where, if σ denotes a permutation of $\{1, 2, 3\}$, we set $\Phi_{\sigma(1)\sigma(2)\sigma(3)} = X^{\sigma^{-1}(1)} \otimes X^{\sigma^{-1}(2)} \otimes X^{\sigma^{-1}(3)}$, and R_{ij} means R acting non-trivially in the i th and j th positions of $H \otimes H \otimes H$.

In [7] it is shown that, consequently, R is invertible. Furthermore, the element

$$u = S(R^2 p^2) \alpha R^1 p^1 \tag{1.29}$$

(with $p_R = p^1 \otimes p^2$ defined as in (1.17)) is invertible in H , and

$$u^{-1} = X^1 R^2 p^2 S(S(X^2 R^1 p^1) \alpha X^3), \tag{1.30}$$

$$\varepsilon(u) = 1 \quad \text{and} \quad S^2(h) = uhu^{-1} \tag{1.31}$$

for all $h \in H$. Consequently the antipode S is bijective, so, as in the Hopf algebra case, the assumptions about invertibility of R and bijectivity of S can be dropped. Moreover, the R -matrix $R = R^1 \otimes R^2$ satisfy the identities (see [1,13,7]):

$$f_{21} R f^{-1} = (S \otimes S)(R), \tag{1.32}$$

$$S(R^2) \alpha R^1 = S(\alpha) u. \tag{1.33}$$

1.3. Hopf algebras in braided categories

For further use we briefly recall some concepts concerning braided categories and braided Hopf algebras. For more details the reader is invited to consult [18] or [22].

A monoidal category means a category \mathcal{C} with objects U, V, W etc., a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ equipped with an associativity natural transformation consisting of functorial isomorphisms $a_{U,V,W}: (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ satisfying a pentagon identity, and a compatible unit object $\underline{1}$ and associated functorial isomorphisms (the left and right unit constraints, $l_V: V \cong V \otimes \underline{1}$ and $r_V: V \cong \underline{1} \otimes V$, respectively).

Let \mathcal{C} be a monoidal category. An object $V \in \mathcal{C}$ has a left dual or is left rigid if there is an object V^* and morphisms $ev_V: V^* \otimes V \rightarrow \underline{1}$, $coev_V: \underline{1} \rightarrow V \otimes V^*$ such that

$$l_V^{-1} \circ (id_V \otimes ev_V) \circ a_{V,V^*,V} \circ (coev_V \otimes id_V) \circ r_V = id_V, \tag{1.34}$$

$$r_{V^*}^{-1} \circ (ev_V \otimes id_{V^*}) \circ a_{V^*,V,V^*}^{-1} \circ (id_{V^*} \otimes coev_V) \circ l_{V^*} = id_{V^*}. \tag{1.35}$$

If every object in the category has a left dual, then we say that \mathcal{C} is a left rigid monoidal category or a monoidal category with left duality.

A braided category is a monoidal category equipped with a commutativity natural transformation consisting of functorial isomorphisms $c_{U,V}: U \otimes V \rightarrow V \otimes U$ compatible with the unit and the associativity structures in a natural way (for a complete definition see [18,22]).

Suppose that $(H, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra. If U, V, W are left H -modules, define $a_{U,V,W}: (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ by

$$a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)). \tag{1.36}$$

Then the category ${}_H\mathcal{M}$ of left H -modules becomes a monoidal category with tensor product \otimes given via Δ , associativity constraints $a_{U,V,W}$, unit k as a trivial H -module and the usual left and right unit constraints. If H is a quasi-Hopf algebra then the category of finite dimensional left H -modules is left rigid; the left dual of V is V^*

with the H -module structure given by $(h \cdot v^*)(v) = v^*(S(h) \cdot v)$, for all $v \in V$, $v^* \in V^*$, $h \in H$ and with

$$ev_V(v^* \otimes v) = v^*(\alpha \cdot v), \quad coev_V(1) = \beta \cdot {}_i v \otimes {}^i v, \tag{1.37}$$

where $\{{}_i v\}$ is a basis in V with dual basis $\{{}^i v\}$. Now, if H is a QT quasi-Hopf algebra with R -matrix $R = R^1 \otimes R^2$, then ${}_H \mathcal{M}$ is a braided category with braiding

$$c_{U,V}(u \otimes v) = R^2 \cdot v \otimes R^1 \cdot u, \quad \text{for all } u \in U, v \in V. \tag{1.38}$$

Finally, the definition of a Hopf algebra B in a braided category \mathcal{C} is obtained in the obvious way in analogy with the standard axioms [28]. Thus, a bialgebra in \mathcal{C} is $(B, m_B, \eta_B, \Delta_B, \varepsilon_B)$ where B is an object in \mathcal{C} and the morphism $m_B: B \otimes B \rightarrow B$ forms a multiplication that is associative up to the isomorphism a . Similarly for the coassociativity of the comultiplication $\Delta_B: B \rightarrow B \otimes B$. The identity in the algebra B is expressed as usual by $\eta_B: \underline{1} \rightarrow B$ such that $m_B \circ (\eta_B \otimes id) = m_B \circ (id \otimes \eta_B) = id$. The counit axiom is $(\varepsilon_B \otimes id) \circ \Delta_B = (id \otimes \varepsilon_B) \circ \Delta_B = id$. In addition, Δ_B is required to be an algebra morphism where $B \otimes B$ has the multiplication

$$\begin{aligned} m_{B \otimes B} : (B \otimes B) \otimes (B \otimes B) &\xrightarrow{a} B \otimes (B \otimes (B \otimes B)) \xrightarrow{id \otimes a^{-1}} B \otimes ((B \otimes B) \otimes B) \\ &\xrightarrow{id \otimes (c \otimes id)} B \otimes ((B \otimes B) \otimes B) \xrightarrow{id \otimes a} B \otimes (B \otimes (B \otimes B)) \xrightarrow{a^{-1}} (B \otimes B) \otimes (B \otimes B) \\ &\xrightarrow{m_B \otimes m_B} B \otimes B. \end{aligned}$$

A Hopf algebra B is a bialgebra with a morphism $S_B : B \rightarrow B$ in \mathcal{C} (the antipode) satisfying the usual axioms $m_B \circ (S_B \otimes id) \circ \Delta_B = \eta_B \circ \varepsilon_B = m_B \circ (id \otimes S_B) \circ \Delta_B$.

2. Factorizable QT quasi-Hopf algebras

In this Section we will introduce the notion of factorizable quasi-Hopf algebra and we will show that the quantum double is an example of this type.

If (H, R) is a QT quasi-Hopf algebra then we will see that the k -linear map $\mathcal{Q}: H^* \rightarrow H$, given for all $\chi \in H^*$ by

$$\mathcal{Q}(\chi) = \langle \chi, S(X_2^2 \tilde{p}^2) f^1 R^2 r^1 U^1 X^3 \rangle X^1 S(X_1^2 \tilde{p}^1) f^2 R^1 r^2 U^2, \tag{2.1}$$

where $r^1 \otimes r^2$ is another copy of R and

$$U = g^1 S(q^2) \otimes g^2 S(q^1) \tag{2.2}$$

(here $f^{-1} = g^1 \otimes g^2$ and $q_R = q^1 \otimes q^2$ are the elements defined by (1.14) and (1.17), respectively) has most of the properties satisfied by the map $H^* \ni \chi \mapsto \langle \chi, R^2 r^1 \rangle R^1 r^2 \in H$ defined in the Hopf case. For this reason we will propose the following.

Definition 2.1. A QT quasi-Hopf algebra (H, R) is called factorizable if the map \mathcal{Q} defined by (2.1) is bijective.

We will see in the next Section that the above definition has a categorical explanation. In fact in this way we were able to find a suitable definition for the map \mathcal{Q} .

In the sequel we will need a second formula for the map \mathcal{Q} . Also, another k -linear map $\bar{\mathcal{Q}}: H^* \rightarrow H$ is required.

Proposition 2.2. *Let (H, R) be a QT quasi-Hopf algebra.*

(i) *The map \mathcal{Q} defined by (2.1) has a second formula given for all $\chi \in H^*$ by*

$$\mathcal{Q}(\chi) = \langle \chi, \tilde{q}^1 X^1 R^2 r^1 p^1 \rangle \tilde{q}_1^2 X^2 R^1 r^2 p^2 S(\tilde{q}_2^2 X^3), \tag{2.3}$$

where $q_L = \tilde{q}^1 \otimes \tilde{q}^2$ and $p_R = p^1 \otimes p^2$ are the elements defined by (1.18) and (1.17), respectively.

(ii) *Let $\bar{\mathcal{Q}}: H^* \rightarrow H$ be the k -linear map defined for all $\chi \in H^*$ by*

$$\bar{\mathcal{Q}}(\chi) = \langle \chi, S^{-1}(X^3) q^2 R^1 r^2 X_2^2 \tilde{p}^2 \rangle q^1 R^2 r^1 X_1^2 \tilde{p}^1 S^{-1}(X^1), \tag{2.4}$$

where $q_R = q^1 \otimes q^2$ and $p_L = \tilde{p}^1 \otimes \tilde{p}^2$ are the elements defined by (1.17) and (1.18), respectively. Then \mathcal{Q} is bijective if and only if $\bar{\mathcal{Q}}$ is bijective.

Proof. (i) It is not hard to see that (1.3) and (1.7) imply

$$X^1 S(X_1^2 \tilde{p}^1) \otimes X_2^2 \tilde{p}^2 \otimes X^3 = S(x^1 \tilde{p}^1) \otimes x^2 \tilde{p}_1^2 \otimes x^3 \tilde{p}_2^2. \tag{2.5}$$

We need also the formulas

$$p_R = \Delta(S(\tilde{p}^1))U(\tilde{p}^2 \otimes 1), \tag{2.6}$$

$$U^1 \otimes U^2 S(h) = S(h_1)_1 U^1 h_2 \otimes S(h_1)_2 U^2, \tag{2.7}$$

which can be found in [14]. Now, we claim that

$$R^1 U^1 \otimes R^2 U^2 = \tilde{q}_2^1 R^1 p^1 \otimes \tilde{q}_1^1 R^2 p^2 S(\tilde{q}^2). \tag{2.8}$$

Indeed, we calculate:

$$\begin{aligned} \tilde{q}_2^1 R^1 p^1 \otimes \tilde{q}_1^1 R^2 p^2 S(\tilde{q}^2) &\stackrel{(1,27)}{=} R^1 \tilde{q}_1^1 p^1 \otimes R^2 \tilde{q}_2^1 p^2 S(\tilde{q}^2), \\ (2.6) &= R^1 (\tilde{q}^1 S(\tilde{p}^1))_1 U^1 \tilde{p}^2 \otimes R^2 (\tilde{q}^1 S(\tilde{p}^1))_2 U^2 S(\tilde{q}^2), \\ (2.7) &= R^1 (\tilde{q}^1 S(\tilde{q}_1^2 \tilde{p}^1))_1 U^1 \tilde{q}_2^2 \tilde{p}^2 \otimes R^2 (\tilde{q}^1 S(\tilde{q}_1^2 \tilde{p}^1))_2 U^2, \\ (1.22) &= R^1 U^1 \otimes R^2 U^2, \end{aligned}$$

as needed. Now, if we denote by $\tilde{Q}^1 \otimes \tilde{Q}^2$ another copy of q_L then for all $\chi \in H^*$ we have

$$\begin{aligned} \mathcal{Q}(\chi) &\stackrel{(2,1,2,5)}{=} \langle \chi, S(x^2 \tilde{p}_1^2) f^1 R^2 r^1 U^1 x^3 \tilde{p}_2^2 \rangle S(x^1 \tilde{p}^1) f^2 R^1 r^2 U^2, \\ (2.8, 1.19, 1.27) &= \langle \chi, S(x^2 \tilde{p}_1^2) f^1 \tilde{q}_1^1 (x^3 \tilde{p}_2^2)_{(1,1)} R^2 r^1 p^1 \rangle \\ &\quad \times S(x^1 \tilde{p}^1) f^2 \tilde{q}_2^1 (x^3 \tilde{p}_2^2)_{(1,2)} R^1 r^2 p^2 S(\tilde{q}^2 (x^3 \tilde{p}_2^2)_2), \end{aligned}$$

$$\begin{aligned}
(1.24) &= \langle \chi, S(\tilde{p}_1^2) \tilde{Q}^1 X^1 (\tilde{p}_2^2)_{(1,1)} R^2 r^1 p^1 \rangle \\
&\quad \times S(\tilde{p}^1) \tilde{q}^1 \tilde{Q}_1^2 X^2 (\tilde{p}_2^2)_{(1,2)} R^1 r^2 p^2 S(\tilde{q}^2 \tilde{Q}_2^2 X^3 (\tilde{p}_2^2)_2), \\
(1.1, 1.20) &= \langle \chi, \tilde{Q}^1 X^1 R^2 r^1 p^1 \rangle S(\tilde{p}^1) \tilde{q}^1 \tilde{p}_1^2 \tilde{Q}_1^2 X^2 R^1 r^2 p^2 S(\tilde{q}^2 \tilde{p}_2^2 \tilde{Q}_2^2 X^3), \\
(1.22) &= \langle \chi, \tilde{Q}^1 X^1 R^2 r^1 p^1 \rangle \tilde{Q}_1^2 X^2 R^1 r^2 p^2 S(\tilde{Q}_2^2 X^3),
\end{aligned}$$

so we have proved relation (2.3).

(ii) For all $\chi \in H^*$ we have

$$\begin{aligned}
\mathcal{Q}(\chi) &\stackrel{(2.1,2.2)}{=} \langle \chi, S(X_2^2 \tilde{p}^2) f^1 R^2 r^1 g^1 S(q^2) X^3 \rangle X^1 S(X_1^2 \tilde{p}^1) f^2 R^1 r^2 g^2 S(q^1), \\
(\text{twice 1.32}) &= \langle \chi, S(q^2 r^1 R^2 X_2^2 \tilde{p}^2) X^3 \rangle X^1 S(q^1 r^2 R^1 X_1^2 \tilde{p}^1), \\
(2.3) &= S(\overline{\mathcal{Q}}(\chi \circ S)).
\end{aligned}$$

Since the antipode S is bijective we conclude that \mathcal{Q} is bijective if and only if $\overline{\mathcal{Q}}$ is bijective, so our proof is complete. \square

In the Hopf case perhaps the most important example of factorizable Hopf algebra is the Drinfeld double. We will see that this is also true in the quasi-Hopf case. Firstly, from [12,13,4], we recall the definition of the Drinfeld double $D(H)$ of a finite dimensional quasi-Hopf algebra H . We point out that the quasi-Hopf algebra $D(H)$ was first described by Majid [23] in the form of an implicit Tannaka–Krein reconstruction theorem.

Let $\{e_i\}_{i=1,n}$ be a basis of H and $\{e^i\}_{i=1,n}$ the corresponding dual basis of H^* . It is well known that H^* is a coassociative coalgebra with comultiplication

$$\Delta(\chi) = \chi_1 \otimes \chi_2 := \chi(e_i e_j) e^i \otimes e^j$$

and counit ε . Moreover, H^* is a H -bimodule, by

$$\langle h' \rightharpoonup \chi \leftarrow h'', h \rangle = \langle \chi, h'' h h' \rangle$$

for all $h, h', h'' \in H$ and $\chi \in H^*$. The convolution is a multiplication on H . It is not associative but ensures us that H^* is an algebra in the monoidal category of H -bimodules. We also introduce $\overline{S} : H^* \rightarrow H^*$ as the coalgebra antimorphism dual to S , i.e. $\langle \overline{S}(\chi), h \rangle = \langle \chi, S(h) \rangle$. Now, consider $\Omega \in H^{\otimes 5}$ given by

$$\begin{aligned}
\Omega &= \Omega^1 \otimes \Omega^2 \otimes \Omega^3 \otimes \Omega^4 \otimes \Omega^5 \\
&= X_{(1,1)}^1 y^1 x^1 \otimes X_{(1,2)}^1 y^2 x_1^2 \otimes X_2^1 y^3 x_2^2 \otimes S^{-1}(f^1 X^2 x^3) \otimes S^{-1}(f^2 X^3),
\end{aligned} \tag{2.9}$$

where $f = f^1 \otimes f^2$ is the Drinfeld twist defined in (1.13). The quantum double $D(H)$ is defined as follows. As k vector spaces $D(H) = H^* \otimes H$ and the multiplication is given by

$$\begin{aligned}
(\chi \bowtie h)(\psi \bowtie h') &= [(\Omega^1 \rightharpoonup \chi \leftarrow \Omega^5)(\Omega^2 h_{(1,1)} \rightharpoonup \psi \leftarrow S^{-1}(h_2) \Omega^4)] \\
&\quad \bowtie \Omega^3 h_{(1,2)} h'
\end{aligned} \tag{2.10}$$

for all $\chi, \psi \in H^*$ and $h, h' \in H$. The unit is $\varepsilon \bowtie 1$. By the above, it is easy to see that

$$\begin{aligned} (\varepsilon \bowtie h)(\chi \bowtie h') &= h_{(1,1)} \rightarrow \chi \leftarrow S^{-1}(h_2) \bowtie h_{(1,2)}h' \quad \text{and} \\ (\chi \bowtie h)(\varepsilon \bowtie h') &= \chi \bowtie hh' \end{aligned} \tag{2.11}$$

for any $h, h' \in H$, $\chi \in H^*$. The explicit formulas for the comultiplication, the counit and the antipode are

$$\begin{aligned} \Delta_D(\chi \bowtie h) &= (\varepsilon \bowtie X^1 Y^1)(p_1^1 x^1 \rightarrow \chi_2 \leftarrow Y^2 S^{-1}(p^2) \bowtie p_2^1 x^2 h_1) \\ &\quad \otimes (X_1^2 \rightarrow \chi_1 \leftarrow S^{-1}(X^3) \bowtie X_2^2 Y^3 x^3 h_2), \end{aligned} \tag{2.12}$$

$$\varepsilon_D(\chi \bowtie h) = \varepsilon(h)\chi(S^{-1}(\alpha)), \tag{2.13}$$

$$S_D(\chi \bowtie h) = (\varepsilon \bowtie S(h)f^1)(p_1^1 U^1 \rightarrow \bar{S}^{-1}(\chi) \leftarrow f^2 S^{-1}(p^2) \bowtie p_2^1 U^2), \tag{2.14}$$

$$\alpha_D = \varepsilon \bowtie \alpha, \tag{2.15}$$

$$\beta_D = \varepsilon \bowtie \beta. \tag{2.16}$$

Here $p_R = p^1 \otimes p^2$, $f = f^1 \otimes f^2$ and $U = U^1 \otimes U^2$ are the elements defined by (1.17), (1.13) and (2.2), respectively. Thus, $D(H)$ is a quasi-Hopf algebra and H is a quasi-Hopf subalgebra via the canonical morphism $i_D : H \rightarrow D(H)$, $i_D(h) = \varepsilon \bowtie h$. Moreover, $D(H)$ is QT, the R -matrix being defined by

$$\mathcal{R} = (\varepsilon \bowtie S^{-1}(p^2)_i e p_1^1) \otimes ({}^i e \bowtie p_2^1). \tag{2.17}$$

We are now able to prove the following result.

Proposition 2.3. *Let H be a finite dimensional quasi-Hopf algebra and $D(H)$ its Drinfeld double. Then $D(H)$ is a factorizable quasi-Hopf algebra.*

Proof. We will show that in the Drinfeld double case the map $\bar{\mathcal{Q}}$ defined by (2.3) is bijective, so by Proposition 2.2 it follows that $D(H)$ is factorizable. For this we will compute first the element $\mathcal{R}^2 \mathbf{R}^1 \otimes \mathcal{R}^1 \mathbf{R}^2$, where we denote by $\mathbf{R}^1 \otimes \mathbf{R}^2$ another copy of the R -matrix \mathcal{R} of $D(H)$. In fact, if we denote by $P^1 \otimes P^2$ another copy of the element p_R then we compute

$$\begin{aligned} \mathcal{R}^2 \mathbf{R}^1 \otimes \mathcal{R}^1 \mathbf{R}^2 &= ({}^i e \bowtie p_2^1)(\varepsilon \bowtie S^{-1}(P^2)_j e P_1^1) \otimes (\varepsilon \bowtie S^{-1}(p^2)_i e p_1^1)({}^j e \bowtie P_2^1), \\ (2.11) &= {}^i e \bowtie p_2^1 S^{-1}(P^2)_j e P_1^1 \otimes (S^{-1}(p^2)_i e p_1^1)_{(1,1)} \rightarrow {}^j e \\ &\quad \leftarrow S^{-1}((S^{-1}(p^2)_i e p_1^1)_2) \bowtie (S^{-1}(p^2)_i e p_1^1)_{(1,2)} P_2^1, \\ &= {}^i e \bowtie p_2^1 S^{-1}((S^{-1}(p^2)_i e p_1^1)_2 P^2)_j e (S^{-1}(p^2)_i e p_1^1)_{(1,1)} P_1^1 \otimes {}^j e \\ &\quad \bowtie (S^{-1}(p^2)_i e p_1^1)_{(1,2)} P_2^1, \\ (1.19) &= {}^i e \bowtie S^{-1}((S^{-1}(p^2)_i e)_2 P^2)_j e (S^{-1}(p^2)_i e)_{(1,1)} P_1^1 P_1^1 \otimes {}^j e \\ &\quad \bowtie (S^{-1}(p^2)_i e)_{(1,2)} P_2^1 P_2^1. \end{aligned}$$

Now, H is a quasi-Hopf subalgebra of $D(H)$, so we have to calculate the element

$$b^1 \otimes b^2 := (\varepsilon \bowtie S^{-1}(X^3)q^2) \mathcal{R}^1 \mathbf{R}^2 (\varepsilon \bowtie X_2^2 \tilde{p}^2) \otimes (\varepsilon \bowtie q^1) \mathcal{R}^2 \mathbf{R}^1 (\varepsilon \bowtie X_1^2 \tilde{p}^1 S^{-1}(X^1)).$$

By dual basis and (2.11) we have

$$\begin{aligned} b^1 \otimes b^2 &= (\varepsilon \bowtie S^{-1}(X^3)) ({}^j e \bowtie (q^2 S^{-1}(q_2^1 p^2))_{i e})_{(1,2)} ((q_1^1)_{(1,1)} P^1)_2 p_2^1 (\varepsilon \bowtie X_2^2 \tilde{p}^2) \\ &\quad \otimes {}^i e \bowtie S^{-1}((q^2 S^{-1}(q_2^1 p^2))_{i e})_2 (q_1^1)_{(1,2)} P^2 S((q_1^1)_2))_j e \\ &\quad \times (q^2 S^{-1}(q_2^1 p^2))_{i e} ({}_{(1,1)} (q_1^1)_{(1,1)} P^1)_1 p_1^1 (\varepsilon \bowtie X_1^2 \tilde{p}^1 S^{-1}(X^1)), \\ (1.19, 1.21) &= (\varepsilon \bowtie S^{-1}(X^3)) ({}^j e \bowtie ({}_{i e})_{(1,2)} P_2^1) (\varepsilon \bowtie X_2^2 \tilde{p}^2) \\ &\quad \otimes ({}^i e \bowtie S^{-1}(({}_{i e})_2 P^2))_j e ({}_{i e})_{(1,1)} P_1^1 (\varepsilon \bowtie X_1^2 \tilde{p}^1 S^{-1}(X^1)), \\ (2.11, 2.5) &= (\varepsilon \bowtie S^{-1}(x^3 \tilde{p}_2^2))_{i e} ({}^j e \bowtie P_2^1 x^2 \tilde{p}_1^2) \otimes ({}^i e \bowtie S^{-1}(P^2))_j e P_1^1 x_1 \tilde{p}^1. \end{aligned}$$

Now we want an explicit formula for the element $S_D(b^1) \otimes b^2$. To this end we need the following relations:

$$\begin{aligned} &S(P_2^1 x^2 \tilde{p}_1^2)_{i e} f_1^1 p^1 \otimes S(P_2^1 x^2 \tilde{p}_1^2)_{i e} f_2^1 p^2 S(f^2) S^2(P_1^1 x^1 \tilde{p}^1) \\ &= g^1 S(P^1 y^3 x_2^2 \tilde{p}_{(1,2)}) \otimes g^2 S(S(y^1 x^1 \tilde{p}^1) \alpha y^2 x_1^2 \tilde{p}_{(1,1)}^2), \end{aligned} \quad (2.18)$$

$$S(P^1)_2 U^2 \otimes S(P^1)_1 U^1 P^2 = g^2 \otimes g^1. \quad (2.19)$$

The first one follows applying (1.11, 1.9, 1.16, 1.1, 1.5) and then $f^1 \beta S(f^2) = S(\alpha)$ and (1.1, 1.5). The second one can be proved more easily by using (2.2, 1.11) and (1.21), we leave the details to the reader. Therefore, if we denote by $G^1 \otimes G^2$ another copy of f^{-1} then from the definition (2.14) of S_D , (2.18, 2.19) and the axioms of a quasi-Hopf algebra we obtain

$$\begin{aligned} S_D(b^1) \otimes b^2 &= (\overline{S}^{-1}({}^j e) \bowtie S({}_{i e})) \otimes (X_1^2 S^{-1}(g_2^1 G^2) \rightarrow {}^i e \leftarrow S^{-1}(X^3) \\ &\quad \bowtie X_2^2 S^{-1}(g_1^1 G^1))_j e S^{-2}(g^2 S(X^1)). \end{aligned}$$

We are able now to prove that $\overline{\mathcal{Q}}$ is injective. To this end, let $\mathbf{D} \in (D(H))^*$ such that $\overline{\mathcal{Q}}(\mathbf{D} \circ S_D) = 0$. That means $\mathbf{D}(S_D(b^1)) b^2 = 0$ and it is equivalent to

$$\begin{aligned} &\mathbf{D}(\overline{S}^{-1}({}^j e) \bowtie S({}_{i e})) ({}^i e, S^{-1}(X^3) h X_1^2 S^{-1}(g_2^1 G^2)) \\ &\quad \langle \chi, X_2^2 S^{-1}(g_1^1 G^1) \rangle_j e S^{-2}(g^2 S(X^1)) = 0, \end{aligned}$$

for all $h \in H$ and $\chi \in H^*$. In particular,

$$\begin{aligned} &\mathbf{D}(\overline{S}^{-1}({}^j e) \bowtie S({}_{i e})) ({}^i e, S^{-1}(X^3) (S^{-1}(x^3) h S^{-1}(F^2 f_2^1) x_1^2) X_1^2 S^{-1}(g_2^1 G^2)) \\ &\quad \langle S^{-2}(S(x^1) f^2) \rightarrow \chi \leftarrow S^{-1}(F^1 f_1^1) x_2^2, X_2^2 S^{-1}(g_1^1 G^1) \rangle_j e S^{-2}(g^2 S(X^1)) = 0, \end{aligned}$$

for all $h \in H$ and $\chi \in H^*$, and therefore

$$\mathbf{D}(\overline{S}^{-1}(\chi) \bowtie S(h)) = 0 \quad \forall \chi \in H^* \quad \text{and} \quad h \in H.$$

Since the antipode S is bijective (H is finite dimensional) we conclude that $\mathbf{D} = 0$ and using the bijectivity of S_D it follows that $\overline{\mathcal{Q}}$ is injective. Finally, $\overline{\mathcal{Q}}$ is bijective because of finite dimensionality of $D(H)$, so the proof is finished. \square

3. Transmutation theory for dual quasi-Hopf algebras

As we pointed out, our definition of a factorizable quasi-Hopf algebra has a categorical interpretation. Toward this end we first need to associate to any co-quasi-triangular dual quasi-Hopf algebra A a braided commutative Hopf algebra \underline{A} in the category of right A -comodules, \mathcal{M}^A . For this we will use the dual reconstruction theorem due to Majid [20]. We notice that this reconstruction theorem is the formal dual case of the reconstruction theorem used in [6] but, even in the finite dimensional case, the resulting object \underline{A} cannot be viewed as the formal dual of the object obtained in [6]. Thus, we will present all the details concerning how we can get the structure of \underline{A} as a Hopf algebra in \mathcal{M}^A .

Throughout this section, A will be a dual quasi-bialgebra or a dual quasi-Hopf algebra. Following [22], a dual quasi-bialgebra A is a coassociative coalgebra A with comultiplication Δ and counit ε together with coalgebra morphisms $m_A : A \otimes A \rightarrow A$ (the multiplication; we write $m_A(a \otimes b) = ab$) and $\eta_A : k \rightarrow A$ (the unit; we write $\eta_A(1) = 1$), and an invertible element $\varphi \in (A \otimes A \otimes A)^*$ (the reassociator), such that for all $a, b, c, d \in A$ the following relations hold (summation understood):

$$a_1(b_1c_1)\varphi(a_2, b_2, c_2) = \varphi(a_1, b_1, c_1)(a_2b_2)c_2, \tag{3.1}$$

$$1a = a1 = a, \tag{3.2}$$

$$\varphi(a_1, b_1, c_1d_1)\varphi(a_2b_2, c_2, d_2) = \varphi(b_1, c_1, d_1)\varphi(a_1, b_2c_2, d_2)\varphi(a_2, b_3, c_3), \tag{3.3}$$

$$\varphi(a, 1, b) = \varepsilon(a)\varepsilon(b). \tag{3.4}$$

A is called a dual quasi-Hopf algebra if, moreover, there exist an anti-morphism S of the coalgebra A and elements $\alpha, \beta \in A^*$ such that, for all $a \in A$:

$$S(a_1)\alpha(a_2)a_3 = \alpha(a)1, \quad a_1\beta(a_2)S(a_3) = \beta(a)1, \tag{3.5}$$

$$\varphi(a_1\beta(a_2), S(a_3), \alpha(a_4)a_5) = \varphi^{-1}(S(a_1), \alpha(a_2)a_3, \beta(a_4)S(a_5)) = \varepsilon(a). \tag{3.6}$$

It follows from the axioms that $S(1) = 1$ and $\alpha(1)\beta(1) = 1$, so we can assume that $\alpha(1) = \beta(1) = 1$. Moreover (3.3) and (3.4) imply

$$\varphi(1, a, b) = \varphi(a, b, 1) = \varepsilon(a)\varepsilon(b), \quad \forall a, b \in A. \tag{3.7}$$

For dual quasi-Hopf algebras the antipode is an anti-algebra morphism up to a conjugation by a twist. Let $\gamma, \delta \in (A \otimes A)^*$ be defined by

$$\gamma(a, b) = \varphi(S(b_2), S(a_2), a_4)\alpha(a_3)\varphi^{-1}(S(b_1)S(a_1), a_5, b_4)\alpha(b_3), \tag{3.8}$$

$$\delta(a, b) = \varphi(a_1b_1, S(b_5), S(a_4))\beta(a_3)\varphi^{-1}(a_2, b_2, S(b_4))\beta(b_3), \tag{3.9}$$

for all $a, b \in A$. If we define $f, f^{-1} \in (A \otimes A)^*$,

$$f(a, b) = \varphi^{-1}(S(b_1)S(a_1), a_3b_3, S(a_5b_5))\beta(a_4b_4)\gamma(a_2, b_2), \tag{3.10}$$

$$f^{-1}(a, b) = \varphi^{-1}(S(a_1b_1), a_3b_3, S(b_5)S(a_5))\alpha(a_2b_2)\delta(a_4, b_4), \tag{3.11}$$

then f and f^{-1} are inverses in the convolution algebra and

$$f(a_1, b_1)S(a_2b_2)f^{-1}(a_3, b_3) = S(b)S(a), \quad (3.12)$$

for all $a, b \in A$. Moreover, the following relations hold:

$$\gamma(a, b) = f(a_1, b_1)\alpha(a_2b_2) \quad \text{and} \quad \delta(a, b) = \beta(a_1b_1)f^{-1}(a_2, b_2). \quad (3.13)$$

Suppose that A is a dual quasi-bialgebra or a dual quasi-Hopf algebra. A right A -comodule M is a k -vector space together with a linear map $\rho_M: M \rightarrow M \otimes A$ required to satisfy

$$(\rho_M \otimes id_A) \circ \rho_M = (id_M \otimes \Delta) \circ \rho_M \quad \text{and} \quad (id_M \otimes \varepsilon) \circ \rho_M = id_M.$$

As usual, we denote $\rho_M(m) = m_{(0)} \otimes m_{(1)}$. The category of right A -comodules is denoted by \mathcal{M}^A and it is a monoidal category. The tensor product is given via m_A , i.e. for any $M, N \in \mathcal{M}^A$, $M \otimes N \in \mathcal{M}^A$ via the structure map

$$\rho_{M \otimes N}(m \otimes n) = m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)}. \quad (3.14)$$

The associativity constraints $\mathbf{a}_{M,N,P}: (M \otimes N) \otimes P \rightarrow M \otimes (N \otimes P)$ are defined by

$$\mathbf{a}_{M,N,P}(m, n, p) = \varphi(m_{(1)}, n_{(1)}, p_{(1)})m_{(0)} \otimes (n_{(0)} \otimes p_{(0)}) \quad (3.15)$$

for all $M, N, P \in \mathcal{M}^A$. The unit is k as a trivial right A -comodule, and the left and right unit constraints are the usual ones.

If A is a dual quasi-Hopf algebra then any finite dimensional object M of \mathcal{M}^A has a left dual, i.e. the category of finite dimensional right A -comodules is left rigid. Indeed, the left dual of M is M^* with the right A -comodule structure

$$\rho_{M^*}(m^*) = \langle m^*, \cdot \rangle m_{(0)} \otimes S(\cdot m_{(1)}),$$

for all $m^* \in M^*$, where $(\cdot m)_i$ is a basis in M with dual basis $({}^i m)_i$. The evaluation and coevaluation maps are defined by

$$ev_M: M^* \otimes M \rightarrow k, \quad ev_M(m^* \otimes m) = \alpha(m_{(1)})m^*(m_{(0)}), \quad (3.16)$$

$$coev_M: k \rightarrow M \otimes M^*, \quad coev_M(1) = \beta(\cdot m_{(1)})_i m_{(0)} \otimes {}^i m. \quad (3.17)$$

A dual quasi-bialgebra or dual quasi-Hopf algebra is called co-quasi-triangular (CQT for short) if there exists a k -bilinear form $\sigma: A \otimes A \rightarrow k$ such that the following relations hold:

$$\sigma(ab, c) = \varphi(c_1, a_1, b_1)\sigma(a_2, c_2)\varphi^{-1}(a_3, c_3, b_2)\sigma(b_3, c_4)\varphi(a_4, b_4, c_5), \quad (3.18)$$

$$\sigma(a, bc) = \varphi^{-1}(b_1, c_1, a_1)\sigma(a_2, c_2)\varphi(b_2, a_3, c_3)\sigma(a_4, b_3)\varphi^{-1}(a_5, b_4, c_4), \quad (3.19)$$

$$\sigma(a_1, b_1)a_2b_2 = b_1a_1\sigma(a_2, b_2), \quad (3.20)$$

$$\sigma(a, 1) = \sigma(1, a) = \varepsilon(a), \quad (3.21)$$

for all $a, b, c \in A$.

As in the Hopf case, if A is a CQT dual quasi-Hopf algebra then we can prove that the bilinear form σ is convolution invertible, and that the antipode S is bijective.

Proposition 3.1. *Let (A, σ) be a CQT dual quasi-Hopf algebra. Then:*

- (i) σ is convolution invertible. More exactly, its inverse (denoted by σ^{-1}) is given by

$$\begin{aligned} \sigma^{-1}(a, b) &= \varphi(a_1, S(a_3), b_4 a_{10}) \beta(a_2) \varphi(b_1, S(a_6), a_8) \sigma(S(a_5), b_2) \\ &\quad \times \varphi^{-1}(S(a_4), b_3, a_9) \alpha(a_7), \end{aligned} \tag{3.22}$$

for all $a, b \in A$.

- (ii) The element $u \in A^*$, given by

$$u(a) = \varphi^{-1}(a_7, S(a_3), S^2(a_1)) \sigma(a_6, S(a_4)) \alpha(a_5) \beta(S(a_2)) \tag{3.23}$$

for all $a \in A$, is invertible. Its inverse is given for all $a \in A$, by

$$\begin{aligned} u^{-1}(a) &= \varphi(a_1, S^2(a_8), S(a_6)) \beta(a_4) \sigma(S^2(a_9), a_2) \alpha(S(a_7)) \\ &\quad \times \varphi^{-1}(S^2(a_{10}), a_3, S(a_5)). \end{aligned} \tag{3.24}$$

- (iii) For all $a \in A$,

$$S^2(a) = u(a_1) a_2 u^{-1}(a_3). \tag{3.25}$$

In particular, the antipode S is bijective.

Proof. If A is finite dimensional then the proof follows from [7] by duality. This is why we restrict to give a sketch of the proof, leaving other details to the reader.

- (i) Follows by [7, Lemma 2.2], by duality.
- (ii) Firstly, one can prove that

$$\sigma(S(a_1), S(b_1)) \gamma(a_2, b_2) = \gamma(b_1, a_1) \sigma(a_2, b_2) \tag{3.26}$$

for all $a, b \in A^*$ and then that

$$f(b_1, a_1) \sigma(a_2, b_2) f^{-1}(a_3, b_3) = \sigma(S(a), S(b)). \tag{3.27}$$

Note that these formulas are the formal duals of [7, Lemma 2.3]. Secondly, using (3.27) and the equalities

$$u(a_2) S^2(a_1) = u(a_1) a_2 \quad \text{and} \quad \alpha(S(a_1)) u(a_2) = \sigma(a_3, S(a_1)) \alpha(a_2) \tag{3.28}$$

one can show that $u \circ S^2 = u$ (see [7, Lemmas 2.4 and 2.5] for the dual case). Now, using (3.28), (3.19) and (3.21) it can be proved that u^{-1} defined by (3.24) is a left inverse of u . It is also a right inverse since

$$u(a_1) u^{-1}(a_2) = u^{-1}(S^2(a_1)) u(a_2) = u^{-1}(S^2(a_1)) u(S^2(a_2)) = \varepsilon(S^2(a)) = \varepsilon(a),$$

because of (3.28) and $u \circ S^2 = u$ (for the dual case see [7, Theorem 2.6]). \square

By the above Proposition, if (A, σ) is a CQT dual quasi-Hopf algebra then it follows that \mathcal{M}^A is a braided category. For any $M, N \in \mathcal{M}^A$, the braiding is given by

$$c_{M,N}(m \otimes n) = \sigma(m_{(1)}, n_{(1)})n_{(0)} \otimes m_{(0)}.$$

We will use now the dual braided reconstruction theorem in order to obtain the structure of \underline{A} as a braided Hopf algebra in \mathcal{M}^A . Let \mathcal{C} and \mathcal{D} be two monoidal categories with \mathcal{D} braided. If $\mathbb{F}, \mathbb{G}: \mathcal{C} \rightarrow \mathcal{D}$ are two functors then we denote by $\text{Nat}(\mathbb{F}, \mathbb{G})$ the set of natural transformations $\xi: \mathbb{F} \rightarrow \mathbb{G}$, by $\mathbb{F} \otimes M: \mathcal{C} \rightarrow \mathcal{D}$ the functor $(\mathbb{F} \otimes M)(N) = \mathbb{F}(N) \otimes M$, where $N \in \mathcal{C}, M \in \mathcal{D}$, and by $\text{Hom}(M, M')$ the set of morphism between M and M' in \mathcal{D} . Suppose that there is an object $B \in \mathcal{D}$ such that for all $M \in \mathcal{D}$

$$\text{Hom}(B, M) \cong \text{Nat}(\mathbb{F}, \mathbb{F} \otimes M)$$

by functorial bisections θ_M and let $\mu = \{\mu_N: \mathbb{F}(N) \rightarrow \mathbb{F}(N) \otimes B \mid N \in \mathcal{C}\}$ be the natural transformation corresponding to the identity morphism id_B . Then, using μ and the braiding in \mathcal{D} we have induced maps

$$\Theta_M^s: \text{Hom}(B^{\otimes s}, M) \cong \text{Nat}(\mathbb{F}^s, \mathbb{F}^s \otimes M)$$

and we assume that these are bijections. This is the representability assumption for comodules and is always satisfied if \mathcal{D} is co-complete and if the image of \mathbb{F} is rigid, cf. [20]. Then, using $(\Theta_B^2)^{-1}, \mu_{\perp}, \theta_{B \otimes B}^{-1}$ and θ_{\perp}^{-1} we can define a multiplication, a unit, a comultiplication and a counit for B .

Theorem 3.2 (Majid [20]). *Let \mathcal{C} and \mathcal{D} be monoidal categories with \mathcal{D} braided and $\mathbb{F}: \mathcal{C} \rightarrow \mathcal{D}$ a monoidal functor satisfying the representability assumption for comodules. Then B as above is a bialgebra in \mathcal{D} . If \mathcal{C} is rigid then B is a Hopf algebra in \mathcal{D} .*

Let now (A, σ) be a CQT dual quasi-Hopf algebra, A_R the k -vector space A viewed as a right A -comodule via Δ and \underline{A} the same k -vector space A but viewed now as an object of \mathcal{M}^A via the right adjoint coaction:

$$\rho_{\underline{A}}(a) = a_2 \otimes S(a_1)a_3, \quad (3.29)$$

for all $a \in A$. We apply now the Theorem 3.2 in the case that $\mathcal{C} = \mathcal{D} = \mathcal{M}^A$ and $\mathbb{F} = id$. The first step is to show that \underline{A} is the representability object which we need.

Dual to the quasi-Hopf case, since the antipode S is bijective, we define the elements $p_L, q_L \in (A \otimes A)^*$ given by

$$\begin{aligned} p_L(a, b) &= \varphi(S^{-1}(a_3), a_1, b)\beta(S^{-1}(a_2)) \quad \text{and} \\ q_L(a, b) &= \varphi^{-1}(S(a_1), a_3, b)\alpha(a_2) \end{aligned} \quad (3.30)$$

for all $a, b \in A$. Then, for all $a, b \in A$, the following relations hold:

$$\begin{aligned} p_L(a_2, b_2)S^{-1}(a_3)(a_1b_1) &= p_L(a, b_1)b_2, \\ q_L(a_2, b_1)S(a_1)(a_3b_2) &= q_L(a, b_2)b_1, \end{aligned} \quad (3.31)$$

$$\begin{aligned} p_L(S(a_1), a_3b_2)q_L(a_2, b_1) &= \varepsilon(a)\varepsilon(b), \\ q_L(S^{-1}(a_3), a_1b_1)p_L(a_2, b_2) &= \varepsilon(a)\varepsilon(b). \end{aligned} \quad (3.32)$$

Lemma 3.3. *Let A be a dual quasi-Hopf algebra and $M \in \mathcal{M}^A$. If we define*

$$\begin{aligned} \theta_M: \text{Hom}(\underline{A}, M) &\rightarrow \text{Nat}(id, id \otimes M), \\ \theta_M(\chi)_N(n) &= p_L(S(n_{(1)}), n_{(3)})n_{(0)} \otimes \chi(n_{(2)}), \end{aligned} \tag{3.33}$$

for all $\chi \in \text{Hom}(\underline{A}, M)$, $N \in \mathcal{M}^A$ and $n \in N$, then θ_M is well defined and a bijection. Its inverse, $\theta_M^{-1}: \text{Nat}(id, id \otimes M) \rightarrow \text{Hom}(\underline{A}, M)$, is given for all $\xi \in \text{Nat}(id, id \otimes M)$ by

$$\theta_M^{-1}(\xi)(a) = q_L(a_1, (a_2)_{(1)(1)})\varepsilon((a_2)_{(0)})(a_2)_{(1)(0)}, \tag{3.34}$$

for all $a \in A$, where we denote $\xi_{A_R}(a) = a_{(0)} \otimes a_{(1)}$.

Proof. We have to prove first that θ_M is well defined, that means $\theta_M(\chi)_N$ is a right A -colinear map and $\theta_M(\chi)$ is a natural transformation. Since $\chi: \underline{A} \rightarrow M$ is a morphism in \mathcal{M}^A we have

$$\chi(a)_{(0)} \otimes \chi(a)_{(1)} = \chi(a_2) \otimes S(a_1)a_3, \tag{3.35}$$

for all $a \in A$. Now, if $n \in N$ then:

$$\begin{aligned} \rho_{N \otimes M}(\theta_M(\chi)_N(n)) &= p_L(S(n_{(1)}), n_{(3)})\rho_{N \otimes M}(n_{(0)} \otimes \chi(n_{(2)})), \\ (3.14) &= p_L(S(n_{(2)}), n_{(4)})n_{(0)} \otimes \chi(n_{(3)})_{(0)} \otimes n_{(1)}\chi(n_{(3)})_{(1)}, \\ (3.35) &= p_L(S(n_{(2)}), n_{(6)})n_{(0)} \otimes \chi(n_{(4)}) \otimes n_{(1)}(S(n_{(3)})n_{(5)}), \\ (3.31) &= p_L(S(n_{(1)}), n_{(3)})n_{(0)} \otimes \chi(n_{(2)}) \otimes n_{(4)}, \\ (3.33) &= \theta_M(\chi)_N(n_{(0)}) \otimes n_{(1)} = (\theta_M(\chi)_N \otimes id_A)(\rho_N(n)), \end{aligned}$$

as needed. It is not hard to see that $\theta_M(\chi)$ is a natural transformation, so we are left to show that θ_M^{-1} is also well defined, and that θ_M and θ_M^{-1} are inverses. The first assertion follows from the following. Since ξ_{A_R} is a right A -comodule map we have

$$(a_1)_{(0)} \otimes (a_1)_{(1)} \otimes a_2 = a_{(0)1} \otimes a_{(1)(0)} \otimes a_{(0)2}a_{(1)(1)}, \tag{3.36}$$

for all $a \in A$. On the other hand, for all $a^* \in A^*$ the map $\lambda_{a^*}: A_R \rightarrow A_R$, $\lambda_{a^*}(a) := a^*(a_1)a_2$, is right A -colinear. Since ξ is functorial under morphism λ_{a^*} we obtain that

$$a^*(a_{(0)1})a_{(0)2} \otimes a_{(1)} = a^*(a_1)(a_2)_{(0)} \otimes (a_2)_{(1)},$$

for all $a^* \in A^*$ and $a \in A$, and this is equivalent to

$$a_{(0)1} \otimes a_{(0)2} \otimes a_{(1)} = a_1 \otimes (a_2)_{(0)} \otimes (a_2)_{(1)}, \tag{3.37}$$

for all $a \in A$. Then for $a \in A$ we have

$$\begin{aligned} (\theta_M^{-1}(\xi) \otimes id_A)(\rho_A(a)) &= \theta_M^{-1}(\xi)(a_2) \otimes S(a_1)a_3 \\ &= q_L(a_2, (a_3)_{(1)(1)})\varepsilon((a_3)_{(0)})(a_3)_{(1)(0)} \otimes S(a_1)a_4, \\ (3.36) &= q_L(a_2, (a_3)_{(1)(1)})\varepsilon((a_3)_{(0)1})(a_3)_{(1)(0)} \otimes S(a_1)((a_3)_{(0)2}(a_3)_{(1)(2)}) \\ &= q_L(a_2, (a_3)_{(1)(1)})\varepsilon((a_3)_{(0)2})(a_3)_{(1)(0)} \otimes S(a_1)((a_3)_{(0)1}(a_3)_{(1)(2)}), \end{aligned}$$

$$(3.37) = q_L(a_2, (a_4)_{\langle 1 \rangle_{(1)}}) \varepsilon((a_4)_{\langle 0 \rangle}) (a_4)_{\langle 1 \rangle_{(0)}} \otimes S(a_1)(a_3(a_4)_{\langle 1 \rangle_{(2)}}),$$

$$(3.31) = q_L(a_1, (a_2)_{\langle 1 \rangle_{(2)}}) \varepsilon((a_2)_{\langle 0 \rangle}) (a_2)_{\langle 1 \rangle_{(0)}} \otimes (a_2)_{\langle 1 \rangle_{(1)}} \\ = (\rho_M \circ \theta_M^{-1}(\xi))(a),$$

so $\theta_M^{-1}(\xi)$ is a right A -comodule map. We show now that θ_M^{-1} is a left inverse for θ_M . Indeed, from definitions we have

$$\theta_M(\xi)_{A_R}(a) = p_L(S(a_2), a_4)a_1 \otimes \xi(a_3) = a_{\langle 0 \rangle} \otimes a_{\langle 1 \rangle},$$

for all $a \in A$, and therefore

$$(\theta_M^{-1} \circ \theta_M)(\chi)(a) = q_L(a_1, \chi(a_4)_{\langle 1 \rangle}) \varepsilon(a_2) p_L(S(a_3), a_5) \chi(a_4)_{\langle 0 \rangle} \\ = q_L(a_1, S(a_3)a_4) p_L(S(a_2), a_5) \chi(a_2), \\ (3.32) = \varepsilon(a_1) \varepsilon(a_3) \chi(a_2) = \chi(a),$$

for all $\chi \in \text{Hom}(A, M)$ and $a \in A$. In order to prove that θ_M^{-1} is a right inverse for θ_M observe first that for any $N \in \mathcal{M}^A$ and $n^* \in N^*$, the map $\lambda_{n^*}: N \rightarrow A_R, \lambda_{n^*}(n) = n^*(n_{\langle 0 \rangle})n_{\langle 1 \rangle}$, is right A -colinear. The fact that ξ is functorial under the morphism λ_{n^*} means

$$n^*(n_{[0]_{\langle 0 \rangle}})n_{[0]_{\langle 1 \rangle}} \otimes n_{[1]} = n^*(n_{\langle 0 \rangle})n_{\langle 1 \rangle_{\langle 0 \rangle}} \otimes n_{\langle 1 \rangle_{\langle 1 \rangle}},$$

where we denote $\xi_N(n) = n_{[0]} \otimes n_{[1]}$. Since it is true for any $n^* \in N^*$ we obtain

$$n_{[0]_{\langle 0 \rangle}} \otimes n_{[0]_{\langle 1 \rangle}} \otimes n_{[1]} = n_{\langle 0 \rangle} \otimes n_{\langle 1 \rangle_{\langle 0 \rangle}} \otimes n_{\langle 1 \rangle_{\langle 1 \rangle}} \quad (3.38)$$

for all $n \in N$. Now, for all $n \in N$ we compute:

$$(\theta_M \circ \theta_M^{-1})(\xi)_N(n) = \theta_M(\theta_M^{-1}(\xi))_N(n) = p_L(S(n_{\langle 1 \rangle}), n_{\langle 3 \rangle})n_{\langle 0 \rangle} \otimes \theta_M^{-1}(\xi)(n_{\langle 2 \rangle}) \\ = p_L(S(n_{\langle 1 \rangle}), n_{\langle 4 \rangle})q_L(n_{\langle 2 \rangle}, (n_{\langle 3 \rangle})_{\langle 1 \rangle_{(1)}}) \varepsilon((n_{\langle 3 \rangle})_{\langle 0 \rangle})n_{\langle 0 \rangle} \otimes (n_{\langle 3 \rangle})_{\langle 1 \rangle_{(0)}}, \\ (3.36) = p_L(S(n_{\langle 1 \rangle}), (n_{\langle 3 \rangle})_{\langle 0 \rangle_{(2)}}(n_{\langle 3 \rangle})_{\langle 1 \rangle_{(2)}})q_L(n_{\langle 2 \rangle}, (n_{\langle 3 \rangle})_{\langle 1 \rangle_{(1)}}) \\ \times \varepsilon((n_{\langle 3 \rangle})_{\langle 0 \rangle_{(1)}})n_{\langle 0 \rangle} \otimes (n_{\langle 3 \rangle})_{\langle 1 \rangle_{(0)}}, \\ (3.38) = p_L(S(n_{[0]_{\langle 1 \rangle}}, n_{[0]_{\langle 3 \rangle}}n_{[1]_{\langle 2 \rangle}})q_L(n_{[0]_{\langle 2 \rangle}}, n_{[1]_{\langle 1 \rangle}})n_{[0]_{\langle 0 \rangle}} \otimes n_{[1]_{\langle 0 \rangle}}, \\ (3.32) = \varepsilon(n_{[0]_{\langle 1 \rangle}}) \varepsilon(n_{[1]_{\langle 1 \rangle}})n_{[0]_{\langle 0 \rangle}} \otimes n_{[1]_{\langle 0 \rangle}} = n_{[0]} \otimes n_{[1]} = \xi_N(n),$$

as needed, and this finishes the proof. \square

We are now able to begin our reconstruction. The natural transformation $\mu \in \text{Nat}(id, id \otimes A)$ corresponding to the identity morphism id_A is given by

$$\mu_N(n) = \theta_A(id_A)_N(n) = p_L(S(n_{\langle 1 \rangle}), n_{\langle 3 \rangle})n_{\langle 0 \rangle} \otimes n_{\langle 2 \rangle}$$

for all $N \in \mathcal{M}^A$ and $n \in N$. By [20, Lemma 2.4] the multiplication of \underline{A} is characterized as being the unique morphism $\underline{m}: \underline{A} \otimes \underline{A}$ in \mathcal{M}^A such that

$$\begin{aligned} \mu_{M \otimes N} &= (id_{M \otimes N} \otimes \underline{m}) \circ \mathbf{a}_{M,N,\underline{A} \otimes \underline{A}}^{-1} \circ (id_M \otimes \mathbf{a}_{N,\underline{A},\underline{A}}) \circ (id_M \otimes (c_{\underline{A},N} \otimes id_{\underline{A}})) \\ &\circ (id_M \otimes \mathbf{a}_{\underline{A},N,\underline{A}}^{-1}) \circ \mathbf{a}_{M,\underline{A},N \otimes \underline{A}} \circ (\mu_M \otimes \mu_N), \end{aligned}$$

for any $M, N \in \mathcal{M}^A$. Using the braided categorical structure of \mathcal{M}^A and the definition of μ it is not hard to see that \underline{m} is the unique morphism in \mathcal{M}^A which satisfies

$$\begin{aligned} &p_L(S(m_{(1)}n_{(1)}), m_{(3)}n_{(3)})(m_{(0)} \otimes n_{(0)}) \otimes m_{(2)}n_{(2)} \\ &= p_L(S(m_{(3)}), m_{(15)}) p_L(S(n_{(5)}), n_{(13)}) \\ &\times \varphi(m_{(2)}, S(m_{(4)})m_{(14)}, n_{(14)}) \varphi^{-1}(S(m_{(5)})m_{(13)}, n_{(4)}, S(n_{(6)})n_{(12)}) \sigma(S(m_{(6)})m_{(12)}, n_{(3)}) \\ &\times \varphi(n_{(2)}, S(m_{(7)})m_{(11)}, S(n_{(7)})n_{(11)}) \varphi^{-1}(m_{(1)}, n_{(1)}, m_{(9)}n_{(9)}) \\ &\times (m_{(0)} \otimes n_{(0)}) \otimes (S(m_{(8)})m_{(10)}) \dot{-} (S(n_{(8)})n_{(10)}) \end{aligned}$$

for all $M, N \in \mathcal{M}^A$ and $m \in M, n \in N$, where we denote by $a \dot{-} b := \underline{m}(a \otimes b)$. We can easily check that the above equality is equivalent to

$$\begin{aligned} &p_L(S(a_1b_1), a_3b_3)a_2b_2 \\ &= p_L(S(a_3), a_{15}) p_L(S(b_5), b_{13}) \varphi(a_2, S(a_4)a_{14}, b_{14}) \\ &\times \varphi^{-1}(S(a_5)a_{13}, b_4, S(b_6)b_{12}) \sigma(S(a_6)a_{12}, b_3) \varphi(b_2, S(a_7)a_{11}, S(b_7)b_{11}) \\ &\times \varphi^{-1}(a_1, b_1, a_9b_9) (S(a_8)a_{10}) \dot{-} (S(b_8)b_{10}) \end{aligned} \tag{3.39}$$

for all $a, b \in A$. Now, the explicit formula for the multiplication $\dot{-}$ is the following:

$$\begin{aligned} a \dot{-} b &= \varphi(S(a_1), a_{10}, S(b_1)b_{12}) f(b_6, a_3) \sigma(a_8, S(b_3)) \varphi^{-1}(S(a_2), S(b_5), a_6b_9) \sigma(a_4, b_7) \\ &\times \varphi^{-1}(a_9, S(b_2), b_{11}) \varphi(S(b_4), a_7, b_{10}) a_5b_8, \end{aligned} \tag{3.40}$$

for all $a, b \in A$. Indeed, it is easy to see that the multiplication $\dot{-}$ defined by (3.40) is a right A -colinear map. A straightforward but tedious computation ensures that $\dot{-}$ satisfies the relation (3.39), we leave all these details to the reader. It is not hard to see that the unit of \underline{A} is 1, the unit of A .

Following [20], the comultiplication of \underline{A} is obtained as $\underline{\Delta} = \theta_{\underline{A} \otimes \underline{A}}^{-1}(\xi)$, where ξ is defined by the following composition

$$\xi_N: N \xrightarrow{\mu_N} N \otimes \underline{A} \xrightarrow{\mu_N} (N \otimes \underline{A}) \otimes \underline{A} \xrightarrow{\mathbf{a}_{N,\underline{A},\underline{A}}} N \otimes (\underline{A} \otimes \underline{A}),$$

for all $N \in \mathcal{M}^A$. Explicitly, for all $n \in N$,

$$\begin{aligned} \xi_N(n) &= \varphi(n_{(1)}, S(n_{(3)})n_{(5)}, S(n_{(8)})n_{(10)}) p_L(S(n_{(7)}), n_{(11)}) \\ &p_L(S(n_{(2)}), n_{(6)})n_{(0)} \otimes (n_{(4)} \otimes n_{(9)}). \end{aligned} \tag{3.41}$$

The counit $\underline{\varepsilon}$ is obtained as $\underline{\varepsilon}(a) = \theta_k^{-1}(l)(a)$, where l is the left unit constraint.

Proposition 3.4. *Let A be a dual quasi-Hopf algebra. Then the comultiplication of \underline{A} is given for all $a \in A$ by*

$$\underline{\Delta}(a) = \varphi^{-1}(S(a_1), a_5, S(a_7))\beta(a_6)\varphi(S(a_2)a_4, S(a_8), a_{10})a_3 \otimes a_9. \quad (3.42)$$

The counit of \underline{A} is $\underline{\varepsilon} = \alpha$.

Proof. Let us start by noting that (3.3) and the definitions (3.30) of p_L and q_L imply:

$$q_L(a_1, b_1c_1)\varphi(a_2, b_2, c_2) = \alpha(a_3)\varphi^{-1}(S(a_2), a_4, b_1)\varphi^{-1}(S(a_1), a_5b_2, c), \quad (3.43)$$

$$\varphi^{-1}(a, b_1, S(b_3)c_1)p_L(S(b_2), c_2) = \varphi(a_1b_1, S(b_5), c)\varphi^{-1}(a_2, b_2, S(b_4))\beta(b_3), \quad (3.44)$$

for all $a, b, c \in A$. On the other hand, from (3.41) we can easily see that

$$\begin{aligned} \check{\xi}_{A_R}(a) = a_{\langle 0 \rangle} \otimes a_{\langle 1 \rangle} &= p_L(S(a_3), a_7)p_L(S(a_8), a_{12})\varphi(a_2, S(a_4)a_6, S(a_9)a_{11}) \\ & a_1 \otimes (a_5 \otimes a_{10}). \end{aligned} \quad (3.45)$$

Now, for all $a \in A$ we compute:

$$\begin{aligned} \underline{\Delta}_{\underline{A}}(a) &= \theta_{\underline{A} \otimes \underline{A}}^{-1}(\check{\xi}) = q_L(a_1, (a_2)_{\langle 1 \rangle \langle 1 \rangle})\varepsilon((a_2)_{\langle 0 \rangle})(a_2)_{\langle 1 \rangle \langle 0 \rangle}, \\ (3.45) &= q_L(a_1, (a_5 \otimes a_{10})_{\langle 1 \rangle})p_L(S(a_8), a_{12})p_L(S(a_3), a_7) \\ & \quad \times \varphi(a_2, S(a_4)a_6, S(a_9)a_{11})(a_5 \otimes a_{10})_{\langle 0 \rangle}, \\ (3.14) &= q_L(a_1, (S(a_5)a_7)(S(a_{12})a_{14}))p_L(S(a_{10}), a_{16})p_L(S(a_3), a_9) \\ & \quad \times \varphi(a_2, S(a_4)a_8, S(a_{11})a_{15})a_6 \otimes a_{13}, \\ (3.43) &= \alpha(a_3)\varphi^{-1}(S(a_2), a_4, S(a_8)a_{10})\varphi^{-1}(S(a_1), a_5(S(a_7)a_{11}), S(a_{14})a_{16}) \\ & \quad \times p_L(S(a_{13}), a_{17})p_L(S(a_6), a_{12})a_9 \otimes a_{15}, \\ (3.31) &= \alpha(a_3)\varphi^{-1}(S(a_2), a_4, S(a_6)a_8)\varphi^{-1}(S(a_1), a_{10}, S(a_{12})a_{14}) \\ & \quad \times p_L(S(a_{11}), a_{15})p_L(S(a_5), a_9)a_7 \otimes a_{13}, \\ (3.44) &= \alpha(a_4)\varphi^{-1}(S(a_3), a_5, S(a_7)a_9)\varphi(S(a_2)a_{11}, S(a_{15}), a_{17}) \\ & \quad \times \varphi^{-1}(S(a_1), a_{12}, S(a_{14}))\beta(a_{13})p_L(S(a_6), a_{10})a_8 \otimes a_{16}, \\ (3.30) &= q_L(a_3, S(a_5)a_7)\varphi(S(a_2)a_9, S(a_{13}), a_{15})\varphi^{-1}(S(a_1), a_{10}, S(a_{12})) \\ & \quad \times \beta(a_{11})p_L(S(a_4), a_8)a_6 \otimes a_{14}, \\ (3.32) &= \varphi^{-1}(S(a_1), a_5, S(a_7))\beta(a_6)\varphi(S(a_2)a_4, S(a_8), a_{10})a_3 \otimes a_9, \end{aligned}$$

for all $a \in A$. The counit of \underline{A} is $\underline{\varepsilon}(a) = \theta_k^{-1}(l)(a) = q_L(a, 1) = \alpha(a)$ for all $a \in A$, so $\underline{\varepsilon} = \alpha$. \square

Let now M be a finite dimensional right A -comodule and M^* its left dual. According to [20, Proposition 2.9], the reconstructed antipode \underline{S} of \underline{A} is characterized as being the unique morphism in \mathcal{C} satisfying

$$\begin{aligned} (id_M \circ \underline{S}) \circ \mu_M &= l_{M \otimes \underline{A}}^{-1} \circ (id_{M \otimes \underline{A}} \otimes ev_M) \circ (a_{M, \underline{A}, M^*} \otimes id_M) \circ (a_{M, \underline{A}, M^*}^{-1} \otimes id_M) \\ &\circ ((id_M \otimes c_{\underline{A}, M^*}^{-1}) \otimes id_M) \circ ((id_M \otimes \mu_{M^*}) \otimes id_M) \\ &\circ (coev_M \otimes id_M) \circ r_M, \end{aligned}$$

for any finite dimensional object M of \mathcal{M}^A , where l, r, a, c, ev and $coev$ are the left unit constraints, the right unit constraints, the associativity constraints, the braiding of \mathcal{M}^A , and the evaluation and coevaluation map, respectively. This comes out explicitly as

$$\begin{aligned} &p_L(S(m_{(1)}), m_{(3)})m_{(0)} \otimes \underline{S}(m_{(2)}) \\ &= \beta(m_{(3)})p_L(S^2(m_{(12)}), S(m_{(4)})) \\ &\times \sigma^{-1}(S^2(m_{(11)})S(m_{(5)}), S(m_{(13)}))\varphi^{-1}(m_{(2)}, S^2(m_{(10)})S(m_{(6)}), S(m_{(14)})) \\ &\times \varphi(m_{(1)}[S^2(m_{(9)})S(m_{(7)})], S(m_{(15)}), m_{(17)})\alpha(m_{(16)})m_{(0)} \otimes S(m_{(8)}), \end{aligned}$$

for all finite dimensional right A -comodule M and $m \in M$. It follows that the above relation is equivalent to

$$\begin{aligned} p_L(S(a_1), a_3)\underline{S}(a_2) &= \beta(a_3)p_L(S^2(a_{12}), S(a_4))\sigma^{-1}(S^2(a_{11})S(a_5), S(a_{13})) \\ &\times \varphi^{-1}(a_2, S^2(a_{10})S(a_6), S(a_{14})) \\ &\times \varphi(a_1[S^2(a_9)S(a_7)], S(a_{15}), a_{17})\alpha(a_{16})S(a_8), \\ (3.3, 3.5) &= \beta(a_2)p_L(S^2(a_{11}), S(a_3))\sigma^{-1}(S^2(a_{10})S(a_4), S(a_{12})) \\ &\times \varphi(S^2(a_8)S(a_6), S(a_{14}), a_{16})\varphi(a_1, [S^2(a_9)S(a_5)]S(a_{13}), a_{17}) \\ &\alpha(a_{15})S(a_7), \\ (3.31, 3.30) &= p_L(S(a_1), a_{13})p_L(S^2(a_8), S(a_2))\sigma^{-1}(S^2(a_7)S(a_3), S(a_9)) \\ &\times \varphi(S^2(a_6)S(a_4), S(a_{10}), a_{12})\alpha(a_{11})S(a_5) \end{aligned}$$

for all $a \in A$, and therefore

$$\begin{aligned} \underline{S}(a) &= p_L(S^2(a_7), S(a_1))\sigma^{-1}(S^2(a_6)S(a_2), S(a_8)) \\ &\varphi(S^2(a_5)S(a_3), S(a_9), a_{11})\alpha(a_{10})S(a_4) \end{aligned} \tag{3.46}$$

for all $a \in A$ (it is not hard to see that \underline{S} defined above is right A -colinear). We summarize all this in the following.

Theorem 3.5. *Let (A, σ) be a CQT dual quasi-Hopf algebra. Then there is a braided Hopf algebra \underline{A} in the category \mathcal{M}^A . \underline{A} coincides with A as k -linear space, and it is*

an object in \mathcal{M}^A by the right coadjoint action

$$\rho_{\underline{A}}(a) = a_2 \otimes S(a_1)a_3.$$

The algebra structure, the coalgebra structure and the antipode are transmuted to

$$\begin{aligned} a \cdot b &= \varphi(S(a_1), a_{10}, S(b_1)b_{12})f(b_6, a_3)\sigma(a_8, S(b_3)) \\ &\quad \times \varphi^{-1}(S(a_2), S(b_5), a_6b_9)\sigma(a_4, b_7) \\ &\quad \times \varphi^{-1}(a_9, S(b_2), b_{11})\varphi(S(b_4), a_7, b_{10})a_5b_8, \end{aligned}$$

$$\underline{\Delta}(a) = \varphi^{-1}(S(a_1), a_5, S(a_7))\beta(a_6)\varphi(S(a_2)a_4, S(a_8), a_{10})a_3 \otimes a_9,$$

$$\underline{S}(a) = p_L(S^2(a_7), S(a_1))\sigma^{-1}(S^2(a_6)S(a_2), S(a_8))$$

$$\varphi(S^2(a_5)S(a_3), S(a_9), a_{11})\alpha(a_{10})S(a_4),$$

for all $a, b \in \underline{A}$. The unit element is 1 of A , and the counit is $\underline{\varepsilon} = \alpha$. As in the Hopf case, we will call \underline{A} the associated function algebra braided group of A .

Remark 3.6. The braided group \underline{A} is braided commutative in the sense of [20]. More precisely, \underline{A} has a second multiplication (denoted by $\underline{m}^{\text{op}}$) also making \underline{A} into a braided bialgebra, and there exists a convolution invertible morphism $\mathfrak{R}: \underline{A} \otimes \underline{A} \rightarrow k$ relating $\underline{m}^{\text{op}}$ by conjugation to \underline{m} . Moreover, \mathfrak{R} makes \underline{A} with its products into a CQT Hopf algebra in \mathcal{M}^A in some sense, analogues to the definition of an ordinary CQT Hopf algebra, see [21,22]. Now, \underline{A} is braided commutative mean that $\underline{m}^{\text{op}} = \underline{m}$ and $\mathfrak{R} = \underline{\varepsilon} \otimes \underline{\varepsilon}$.

We would like to stress that the opposite multiplication $\underline{m}^{\text{op}}$ is characterized by

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\mu_M \otimes \mu_N} & (M \otimes \underline{A}) \otimes (N \otimes \underline{A}) & M \otimes N & \xrightarrow{\mu_M \otimes id_N} & (M \otimes \underline{A}) \otimes N \\ & \xrightarrow{\mathbf{a}_{M, \underline{A}, N \otimes \underline{A}}} & M \otimes (\underline{A} \otimes (N \otimes \underline{A})) & & \xrightarrow{\mathbf{a}_{M, \underline{A}, N}} & M \otimes (\underline{A} \otimes N) \\ & \xrightarrow{id_M \otimes \mathbf{a}_{\underline{A}, N, \underline{A}}^{-1}} & M \otimes ((\underline{A} \otimes N) \otimes \underline{A}) & & \xrightarrow{id_M \otimes c_{N, \underline{A}}^{-1}} & M \otimes (N \otimes \underline{A}) \\ & \xrightarrow{id_M \otimes (c_{\underline{A}, N} \otimes id_{\underline{A}})} & M \otimes ((N \otimes \underline{A}) \otimes \underline{A}) & = & \xrightarrow{id_M \otimes (\mu_N \otimes id_{\underline{A}})} & M \otimes ((N \otimes \underline{A}) \otimes \underline{A}) \\ & \xrightarrow{id_M \otimes \mathbf{a}_{N, \underline{A}, \underline{A}}} & M \otimes (N \otimes (\underline{A} \otimes \underline{A})) & & \xrightarrow{id_M \otimes \mathbf{a}_{N, \underline{A}, \underline{A}}} & M \otimes (N \otimes (\underline{A} \otimes \underline{A})) \\ & \xrightarrow{\mathbf{a}_{M, N, \underline{A} \otimes \underline{A}}^{-1}} & (M \otimes N) \otimes (\underline{A} \otimes \underline{A}) & & \xrightarrow{\mathbf{a}_{M, N, \underline{A} \otimes \underline{A}}^{-1}} & (M \otimes N) \otimes (\underline{A} \otimes \underline{A}) \\ & \xrightarrow{id_{M \otimes N} \otimes \underline{m}^{\text{op}}} & (M \otimes N) \otimes \underline{A} & & \xrightarrow{id_{M \otimes N} \otimes \underline{m}} & (M \otimes N) \otimes \underline{A}. \end{array}$$

The above equality and $\underline{m}^{\text{op}} = \underline{m}$ reduce to an intrinsic form of braided commutativity of \underline{A} , we leave the details to the reader.

4. The categorical interpretation

In this Section our goal is to give a categorical interpretation for our definition of a factorizable quasi-Hopf algebra. In the Hopf case it was given by Majid [22]. If (H, R)

is a Hopf algebra then we can associate to H a braided cocommutative Hopf algebra \underline{H} in the braided category ${}_H\mathcal{M}$. As we have already seen in the previous Section, we can associate to any CQT (dual quasi-) Hopf algebra (A, σ) a braided commutative Hopf algebra \underline{A} in the category of right A -comodules \mathcal{M}^A . Now, let (H, R) be a finite dimensional factorizable Hopf algebra and (A, σ) the CQT Hopf algebra dual to H . If \underline{A} is viewed as a braided Hopf algebra in ${}_H\mathcal{M}$ then \underline{H} and \underline{A} are isomorphic as braided Hopf algebras. Moreover, the isomorphism is given by the canonical map \mathcal{Q} considered in Section 2. Also, \underline{A} is always isomorphic to the categorical left dual of \underline{H} .

We will generalize the above results to the quasi-Hopf case. In [8] there was introduced another multiplication on H , denoted by \bullet , given by the formula

$$h \bullet h' = X^1 h S(x^1 X^2) \alpha x^2 X_1^3 h' S(x^3 X_2^3), \tag{4.1}$$

$$(1.3, 1.5) = X^1 x_1^1 h S(X^2 x_2^1) \alpha X^3 x^2 h' S(x^3) \tag{4.2}$$

for all $h, h' \in H$ and it was proved that, if we denote by H_0 this structure, then H_0 becomes an algebra within the monoidal category of left H -modules, with unit β and left H -action given by

$$h \triangleright h' = h_1 h' S(h_2), \tag{4.3}$$

for all $h, h' \in H$. If (H, R) is quasi-triangular then H_0 is a Hopf algebra with bijective antipode in ${}_H\mathcal{M}$, with the additional structures (see [6]):

$$\underline{\Delta}(h) = h_{\underline{1}} \otimes h_{\underline{2}} := x^1 X^1 h_1 g^1 S(x^2 R^2 y^3 X_2^3) \otimes x^3 R^1 \triangleright y^1 X^2 h_2 g^2 S(y^2 X_1^3), \tag{4.4}$$

$$\underline{\varepsilon}(h) = \varepsilon(h), \tag{4.5}$$

$$\underline{S}(h) = X^1 R^2 p^2 S(q^1 (X^2 R^1 p^1 \triangleright h) S(q^2) X^3), \tag{4.6}$$

for all $h \in H$, where $R = R^1 \otimes R^2$ is the R -matrix R of H , and $f^{-1} = g^1 \otimes g^2$, $p_R = p^1 \otimes p^2$ and $q_R = q^1 \otimes q^2$ are the elements defined by (1.14) and (1.17), respectively. Thus, in the quasi-Hopf case, $\underline{H} = H_0$ as an algebra with the additional structures (4.4), (4.5) and (4.6). As in the Hopf case, we will call \underline{H} the associated enveloping algebra braided group of H . Note that, all the above structures were obtained by using the braided reconstruction theorem also due to Majid [22] (see [6] for full details).

Suppose now that (H, R) is a finite dimensional QT quasi-Hopf algebra. Then H^* , the linear dual of H , it is in an obvious way a CQT dual quasi-Hopf algebra, so it makes sense to consider \underline{H}^* , the function algebra braided group associated to H^* . It is a braided Hopf algebra in the category of right H^* -comodules, hence it is a braided Hopf algebra in the category of left H -modules. By Theorem 3.5, \underline{H}^* is a braided Hopf algebra in ${}_H\mathcal{M}$. From (3.29), \underline{H}^* is a left H -module via

$$h \blacktriangleright \chi = h_2 \dashv \chi \leftarrow S(h_1) \tag{4.7}$$

for all $h \in H$ and $\chi \in H^*$. By Theorem 3.5, the structure of \underline{H}^* as a Hopf algebra in ${}_H\mathcal{M}$ is given by:

$$\begin{aligned} \chi \cdot \psi &= [x_1^3 Y^2 r^1 y^1 X^2 \dashv \chi \leftarrow S(x^1 X^1) f^2 R^1] \\ &\quad \times [x_2^3 Y^3 y^3 X_2^3 \dashv \psi \leftarrow S(x^2 Y^1 r^2 y^2 X_1^3) f^1 R^2], \end{aligned} \tag{4.8}$$

$$1_{\underline{H}^*} = \varepsilon, \quad (4.9)$$

$$\underline{\Delta}_{H^*}(\chi) = \chi_1 \leftarrow S(x^1) \otimes x_2^3 X^3 \rightarrow \chi_2 \leftarrow x^2 X^1 \beta S(x_1^3 X^2), \quad (4.10)$$

$$\underline{\varepsilon}_{H^*}(\chi) = \chi(\alpha), \quad (4.11)$$

$$\underline{S}(\chi) = q_2^1 \bar{R}_2^1 \tilde{p}^2 \rightarrow \chi S \leftarrow q^2 \bar{R}^2 S(q_1^1 \bar{R}_1^1 \tilde{p}^1), \quad (4.12)$$

for all $\chi, \psi \in H^*$. Here $p_R = p^1 \otimes p^2$ and $q_R = q^1 \otimes q^2$ are the elements defined by (1.17), $f = f^1 \otimes f^2$ is the Drinfeld's twist defined by (1.13), $R^{-1} = \bar{R}^1 \otimes \bar{R}^2$, and $q_L = \tilde{q}^1 \otimes \tilde{q}^2$ is the element given by (1.18), respectively.

We would like to stress that formula (2.1) was chosen in such a way that it provides a left H -module morphism from \underline{H}^* to \underline{H} . Indeed, for all $\chi \in H^*$ and $h \in H$ we have:

$$h \triangleright \mathcal{Q}(\chi) \stackrel{(2.1)}{=} \langle \chi, S(X_2^2 \tilde{p}^2) f^1 R^2 r^1 U^1 X^3 \rangle h_1 X^1 S(X_1^2 \tilde{p}^1) f^2 R^1 r^2 U^2 S(h_2)$$

$$(2.7, 1.27, 1.11) = \langle \chi, S((h_{(2,1)} X^2)_2 \tilde{p}^2) f^1 R^2 r^1 U^1 h_{(2,2)} X^3 \rangle$$

$$\times h_1 X^1 S((h_{(2,1)} X^2)_1 \tilde{p}^1) f^2 R^1 r^2 U^2$$

$$(1.1, 1.20) = \langle \chi, S(X_2^2 \tilde{p}^2 h_1) f^1 R^2 r^1 U^1 X^3 h_2 \rangle X^1 S(X_1^2 \tilde{p}^1) f^2 R^1 r^2 U^2$$

$$(2.1, 4.7) = \mathcal{Q}(h_2 \rightarrow \chi \leftarrow S(h_1)) = \mathcal{Q}(h \blacktriangleright \chi).$$

It is quite remarkable that (2.1) is a braided Hopf algebra morphism, too.

Proposition 4.1. *Let (H, R) be a finite dimensional QT quasi-Hopf algebra, \underline{H} the associated enveloping algebra braided group of H and \underline{H}^* the function algebra braided group associated to H^* . Then the map \mathcal{Q} defined by (2.1) is a braided Hopf algebra morphism from \underline{H}^* to \underline{H} .*

Proof. We have already seen that \mathcal{Q} is a morphism in ${}_H\mathcal{M}$. Hence, it remains to show that \mathcal{Q} is an algebra and a coalgebra morphism. To this end, we will use the second formula (2.3) for the map \mathcal{Q} . From (1.3), (1.5) it follows that

$$\tilde{q}^1 X^1 \otimes \tilde{q}_1^2 X^2 \otimes \tilde{q}_2^3 X^3 = S(x^1) \tilde{q}^1 x_1^2 \otimes \tilde{q}^2 x_2^3 \otimes x^3, \quad (4.13)$$

$$x^1 \otimes x_1^2 p^1 \otimes x_2^3 p^2 S(x^3) = X^1 p_1^1 \otimes X^2 p_2^1 \otimes X^3 p^2. \quad (4.14)$$

We set $R = R^1 \otimes R^2 = r^1 \otimes r^2 = \mathbf{R}^1 \otimes \mathbf{R}^2 = \mathfrak{R}^1 \otimes \mathfrak{R}^2 = \mathfrak{r}^1 \otimes \mathfrak{r}^2 = \mathcal{R}^1 \otimes \mathcal{R}^2$, $q_L = \tilde{q}^1 \otimes \tilde{q}^2 = \tilde{Q}^1 \otimes \tilde{Q}^2$ and $p_R = p^1 \otimes p^2 = P^1 \otimes P^2$. Now, for all $\chi, \psi \in H^*$ we compute:

$$\begin{aligned} \mathcal{Q}(\chi \cdot \psi) &= \langle \chi, S(x^1 X^1) f^2 R^1 \tilde{q}_1^1 Z_1^1 \mathfrak{R}_1^2 \mathfrak{r}_1^1 p_1^1 x_1^3 Y^2 r^1 y^1 X^2 \rangle \\ &\quad \times \langle \psi, S(x^2 Y^1 r^2 y^2 X_1^3) f^1 R^2 \tilde{q}_2^1 Z_2^1 \mathfrak{R}_2^2 \mathfrak{r}_2^1 p_2^1 x_2^3 Y^3 y^3 X_2^3 \rangle \tilde{q}_1^2 Z^2 \mathfrak{R}^1 \mathfrak{r}^2 p^2 S(\tilde{q}_2^2 Z^3), \end{aligned}$$

$$(1.27, 4.14) = \langle \chi, S(x^1 X^1) f^2 \tilde{q}_2^1 [Z^1 \mathfrak{R}^2 \mathfrak{r}^1 x_{(1,1)}^3]_2 R^1 Y^2 r^1 y^1 X^2 \rangle$$

$$\times \langle \psi, S(x^2 Y^1 r^2 y^2 X_1^3) f^1 \tilde{q}_1^1 [Z^1 \mathfrak{R}^2 \mathfrak{r}^1 x_{(1,1)}^3]_1 R^2 Y^3 y^3 X_2^3 \rangle$$

$$\times \tilde{q}_1^2 Z^2 \mathfrak{R}^1 \mathfrak{r}^2 x_{(1,2)}^3 p^2 S(\tilde{q}_2^2 Z^3 x_2^3),$$

$$\begin{aligned}
 (1.27, 1.1, 1.24) &= \langle \chi, S(X^1) \tilde{q}^1 \tilde{Q}_1^2 T^2 Z_2^1 \mathfrak{R}_2^2 \tau_2^1 p_2^1 R^1 Y^2 r^1 y^1 X^2 \rangle \\
 &\quad \times \langle \psi, S(Y^1 r^2 y^2 X_1^3) \tilde{Q}^1 T^1 Z_1^1 \mathfrak{R}_1^2 \tau_1^1 p_1^1 R^2 Y^3 y^3 X_2^3 \rangle \\
 &\quad \times \tilde{q}_1^2 \tilde{Q}_{(2,1)}^2 T_1^3 Z^2 \mathfrak{R}^1 \tau^2 p^2 S(\tilde{q}_2^2 \tilde{Q}_{(2,2)} T_2^3 Z^3), \\
 (1.3, 1.1, 4.13, 1.27) &= \langle \chi, S(X^1) \tilde{q}^1 V^1 \tilde{Q}_1^2 x_{(2,1)}^2 Z^2 \mathfrak{R}_2^2 R^1 \tau_1^1 p_1^1 Y^2 r^1 y^1 X^2 \rangle \\
 &\quad \times \langle \psi, S(x^1 Y^1 r^2 y^2 X_1^3) \tilde{Q}^1 x_1^2 Z^1 \mathfrak{R}_1^2 R^2 \tau_2^1 p_2^1 Y^3 y^3 X_2^3 \rangle \\
 &\quad \times \tilde{q}_1^2 V^2 \tilde{Q}_2^2 x_{(2,2)}^2 Z^3 \mathfrak{R}^1 \tau^2 p^2 S(\tilde{q}_2^2 V^3 x^3), \\
 (1.1, 1.27, 4.14) &= \langle \chi, S(X^1) \tilde{q}^1 V^1 \tilde{Q}_1^2 Z^2 \mathfrak{R}_2^2 R^1 \tau_1^1 T_1^2 Y^2 r^1 y^1 p_1^1 X^2 \rangle \\
 &\quad \times \langle \psi, S(T^1 Y^1 r^2 y^2 (p_2^1 X^3)_1) \tilde{Q}^1 Z^1 \mathfrak{R}_1^2 R^2 \tau_2^1 T_2^2 Y^3 y^3 (p_2^1 X^3)_2 \rangle \\
 &\quad \times \tilde{q}_1^2 V^2 \tilde{Q}_2^2 Z^3 \mathfrak{R}^1 \tau^2 T^3 p^2 S(\tilde{q}_2^2 V^3), \\
 (1.3, 1.27, 1.25, 1.26) &= \langle \chi, S(X^1) \tilde{q}^1 V^1 \tilde{Q}_1^2 \mathfrak{R}^2 Z^3 x^3 R^1 W^2 \tau^1 z^1 T^2 r^1 Y_1^1 y^1 p_1^1 X^2 \rangle \\
 &\quad \times \langle \psi, S(T^1 r^2 Y_2^1 y^2 (p_2^1 X^3)_1) \tilde{Q}^1 Z^1 \mathbf{R}^2 x^2 R^2 W^3 z^3 \mathfrak{R}^1 T_1^3 Y^2 y^3 (p_2^1 X^3)_2 \rangle \\
 &\quad \times \tilde{q}_1^2 V^2 \tilde{Q}_2^2 \mathfrak{R}^1 Z^2 \mathbf{R}^1 x^1 W^1 \tau^2 z^2 \mathfrak{R}^2 T_2^3 Y^3 p^2 S(\tilde{q}_2^2 V^3), \\
 (1.3, 1.27, 1.26) &= \langle \chi, S(X^1) \tilde{q}^1 V^1 \tilde{Q}_1^2 \mathfrak{R}^2 Z^3 x^3 W_2^3 R^1 T^2 r^1 D^1 z_1^1 Y_1^1 y^1 p_1^1 X^2 \rangle \\
 &\quad \times \langle \psi, S(W^1 T_1^1 r_1^2 D^2 z_2^1 Y_2^1 y^2 (p_2^1 X^3)_1) \tilde{Q}^1 Z^1 \mathbf{R}^2 x^2 W_1^3 R^2 T^3 z^3 \mathfrak{R}^1 Y^2 y^3 (p_2^1 X^3)_2 \rangle \\
 &\quad \times \tilde{q}_1^2 V^2 \tilde{Q}_2^2 \mathfrak{R}^1 Z^2 \mathbf{R}^1 x^1 W^2 T_2^1 r_2^2 D^3 z^2 \mathfrak{R}^2 Y^3 p^2 S(\tilde{q}_2^2 V^3), \\
 (1.26, 1.3, 1.1) &= \langle \chi, S(X^1) \tilde{q}^1 V^1 \tilde{Q}_1^2 \mathfrak{R}^2 Z^3 T^3 r^1 Y^1 p_1^1 X^2 \rangle \\
 &\quad \times \langle \psi, S(T^1 r_1^2 C^1 Y_1^2 (p_2^1 X^3)_1) \tilde{Q}^1 Z^1 \mathbf{R}^2 T_2^2 r_{(2,2)}^2 \mathfrak{R}^1 C^2 Y_2^2 (p_2^1 X^3)_2 \rangle \\
 &\quad \times \tilde{q}_1^2 V^2 \tilde{Q}_2^2 \mathfrak{R}^1 Z^2 \mathbf{R}^1 T_1^2 r_{(2,1)}^2 \mathfrak{R}^2 C^3 Y^3 p^2 S(\tilde{q}_2^2 V^3), \\
 (4.13, 1.27, 1.1, 1.5) &= \langle \chi, S(y^1 X^1) \tilde{q}^1 \mathfrak{R}^2 r^1 y_1^2 Y^1 p_1^1 X^2 \rangle \\
 &\quad \times \langle \psi, S(x^1 C^1 (Y^2 p_2^1)_1 X_1^3) \alpha x^2 \mathbf{R}^2 \mathfrak{R}^1 C^2 (Y^2 p_2^1)_2 X_2^3 \rangle \\
 &\quad \times \tilde{q}^2 \mathfrak{R}^1 r^2 y_2^2 x^3 \mathbf{R}^1 \mathfrak{R}^2 C^3 Y^3 p^2 S(y^3), \\
 (4.14, 1.27, 1.1, 1.5) &= \langle \chi, S(y^1 X^1) \tilde{q}^1 \mathfrak{R}^2 r^1 y_1^2 z^1 X^2 \rangle \\
 &\quad \times \langle \psi, S(C^1 p_1^1 X_1^3) \tilde{Q}^1 \mathbf{R}^2 \mathfrak{R}^1 C^2 p_2^1 X_2^3 \rangle \\
 &\quad \times \tilde{q}^2 \mathfrak{R}^1 r^2 y_2^2 z^2 \tilde{Q}^2 \mathbf{R}^1 \mathfrak{R}^2 C^3 p^2 S(y^3 z^3).
 \end{aligned}$$

On the other hand, if we denote by $P^1 \otimes P^2$ another copy of p_R then by (4.2, 1.17, 2.3) we have:

$$\begin{aligned}
\mathcal{Q}(\chi) \bullet \mathcal{Q}(\psi) &= \langle \chi, \tilde{q}^1 Y^1 R^2 r^1 P^1 \rangle \langle \psi, \tilde{Q}^1 Z^1 \mathfrak{R}^2 \mathfrak{t}^1 p^1 \rangle \\
&\quad q^1 y_1^1 \tilde{q}_1^2 Y^2 R^1 r^2 P^2 S(q^2 y_2^1 \tilde{q}_2^2 Y^3) y^2 \tilde{Q}_1^2 Z^2 \mathfrak{R}^1 \mathfrak{t}^2 p^2 S(y^3 \tilde{Q}_2^2 Z^3), \\
(4.13, 1.27, 4.14) &= \langle \chi, S(X^1 P_1^1) \tilde{q}^1 R^2 r^1 X^2 P_2^1 \rangle \langle \psi, \tilde{Q}^1 Z^1 \mathfrak{R}^2 \mathfrak{t}^1 p^1 \rangle \\
&\quad \times q^1 y_1^1 \tilde{q}^2 R^1 r^2 X^3 P^2 S(q^2 y_2^1) y^2 \tilde{Q}_1^2 Z^2 \mathfrak{R}^1 \mathfrak{t}^2 p^2 S(y^3 \tilde{Q}_2^2 Z^3), \\
(1.20, 1.27, 1.1, 1.19) &= \langle \chi, S(X^1(q_1^1 P^1)_1 y_1^1) \tilde{q}^1 R^2 r^1 X^2(q_1^1 P^1)_2 y_2^1 \rangle \\
&\quad \times \langle \psi, \tilde{Q}^1 Z^1 \mathfrak{R}^2 \mathfrak{t}^1 p^1 \rangle \tilde{q}^2 R^1 r^2 X^3 q_1^2 P^2 S(q^2) \\
&\quad \times y^2 \tilde{Q}_1^2 Z^2 \mathfrak{R}^1 \mathfrak{t}^2 p^2 S(y^3 \tilde{Q}_2^2 Z^3), \\
(1.21, 1.3, 4.13, 1.27, 4.14) &= \langle \chi, S(y^1 X^1) \tilde{q}^1 R^2 r^1 y_1^2 x^1 X^2 \rangle \langle \psi, S(Y^1 p_1^1) \tilde{Q}^1 \mathfrak{R}^2 \mathfrak{t}^1 Y^2 p_2^1 \rangle \\
&\quad \times \tilde{q}^2 R^1 r^2 y_2^2 x^2 X_1^3 \tilde{Q}^2 \mathfrak{R}^1 \mathfrak{t}^2 Y^3 p^2 S(y^3 x^3 X_2^3), \\
(1.20, 1.27, 1.1, 1.19) &= \langle \chi, S(y^1 X^1) \tilde{q}^1 R^2 r^1 y_1^2 x^1 X^2 \rangle \\
&\quad \times \langle \psi, S(Y^1 p_1^1 X_1^3) \tilde{Q}^1 \mathfrak{R}^2 \mathfrak{t}^1 Y^2 p_2^1 X_2^3 \rangle \\
&\quad \times \tilde{q}^2 R^1 r^2 y_2^2 x^2 \tilde{Q}^2 \mathfrak{R}^1 \mathfrak{t}^2 Y^3 p^2 S(y^3 x^3).
\end{aligned}$$

By the above it follows that \mathcal{Q} is multiplicative. Since $\mathcal{Q}(1_{H^*}) = \mathcal{Q}(\varepsilon) = \beta = 1_{\underline{H}}$, we conclude that \mathcal{Q} is an algebra map. Thus, one has only to show that \mathcal{Q} is a coalgebra map. To this end, observe first that (1.3), (5.5) imply

$$X_1^1 p^1 \otimes X_2^1 p^2 S(X^2) \otimes X^3 = x^1 \otimes x^2 S(x_1^3 \tilde{p}^1) \otimes x_2^3 \tilde{p}^2. \quad (4.15)$$

Also, it is not hard to see that (4.4, 4.3, 1.25, 1.27) and (1.32) imply

$$\underline{\Delta}_H(h) = x^1 X^1 h_1 r^2 g^2 S(x^2 Y^1 R^2 y^2 X_1^3) \otimes x_1^3 Y^2 R^1 y^1 X^2 h_2 r^1 g^1 S(x_2^3 Y^3 y^3 X_2^3). \quad (4.16)$$

Therefore, by (4.16) and (2.3), for any $\chi \in H^*$ we have

$$\begin{aligned}
\underline{\Delta}_H(\mathcal{Q}(\chi)) &= \langle \chi, \tilde{q}^1 Z^1 \mathfrak{R}^2 \mathfrak{t}^1 p^1 \rangle x^1 X^1 \tilde{q}_{(1,1)}^2 Z_1^2 \mathfrak{R}_1^1 \mathfrak{t}_1^2 p_1^2 S(\tilde{q}_2^2 Z^3) r^2 g^2 S(x^2 Y^1 R^2 y^2 X_1^3) \\
&\quad \otimes x_1^3 Y^2 R^1 y^1 X^2 \tilde{q}_{(1,2)}^2 Z_2^2 \mathfrak{R}_2^1 \mathfrak{t}_2^2 p_2^2 S(\tilde{q}_2^2 Z^3) r^1 g^1 S(x_2^3 Y^3 y^3 X_2^3),
\end{aligned}$$

$$\begin{aligned}
(1.27, 1.11, 1.23, 1.1) &= \langle \chi, \tilde{q}^1 Z^1 \mathfrak{R}^2 \mathfrak{t}^1 V^1(T_1^1 p^1)_1 P^1 \rangle \\
&\quad \times x^1 X^1 (\tilde{q}_1^2 Z^2)_1 \mathfrak{R}_1^1 r^2 \mathfrak{t}_2^2 V^3 T_2^1 p^2 S(x^2 Y^1 R^2 y^2 (X^3 \tilde{q}_2^2)_1 Z_1^3 T^2) \\
&\quad \otimes x_1^3 Y^2 R^1 y^1 X^2 (\tilde{q}_1^2 Z^2)_2 \mathfrak{R}_2^1 r^1 \mathfrak{t}_1^2 V^2(T_1^1 p^1)_2 \\
&\quad \times P^2 S(x_2^3 Y^3 y^3 (X^3 \tilde{q}_2^2)_2 Z_2^3 T^3),
\end{aligned}$$

$$\begin{aligned}
 (4.15, 1.26, 1.25, 4.13) &= \langle \chi, S(v^1) \tilde{q}^1 v_1^2 \mathfrak{A}^2 t^3 \tau_2^1 \mathbf{R}^1 z_1^1 P^1 \rangle \\
 &\quad \times x^1 X^1 \tilde{q}_1^2 v_{(2,1)}^2 \mathfrak{A}_1^1 t^1 \tau^2 z^2 S(x^2 Y^1 R^2 y^2 X_1^3 v_1^3 z_1^3 \tilde{p}^1) \\
 &\quad \otimes x_1^3 Y^2 R^1 y^1 X^2 \tilde{q}_2^2 v_{(2,2)}^2 \mathfrak{A}_2^1 t^2 \tau_1^1 \mathbf{R}^2 z_2^1 P^2 S(x_2^3 Y^3 y^3 X_2^3 v_2^3 z_2^3 \tilde{p}^2), \\
 (1.25, 1.27, 1.25) &= \langle \chi, S(v^1) \tilde{q}^1 v_1^2 T^1 \mathfrak{A}^2 \tau^1 t^1 V^1 \mathfrak{R}^2 \mathbf{R}^1 z_1^1 P^1 \rangle \\
 &\quad \times x^1 X^1 \tilde{q}_1^2 v_{(2,1)}^2 T^2 \mathfrak{A}^1 \tau^2 t^2 r^2 V^3 z^2 S(x^2 Y^1 R^2 y^2 X_1^3 v_1^3 z_1^3 \tilde{p}^1) \\
 &\quad \otimes x_1^3 Y^2 R^1 y^1 X^2 \tilde{q}_2^2 v_{(2,2)}^2 T^3 t^3 r^1 V^2 \mathfrak{R}^1 \mathbf{R}^2 z_2^1 P^2 S(x_2^3 Y^3 y^3 X_2^3 v_2^3 z_2^3 \tilde{p}^2), \\
 (1.1, 4.13, 1.20, 1.3) &= \langle \chi, S(v^1) \tilde{q}^1 v_1^2 X_1^1 \mathfrak{A}^2 \tau^1 t^1 V^1 \mathfrak{R}^2 \mathbf{R}^1 z_1^1 P^1 \rangle \\
 &\quad \times x^1 \tilde{q}^2 v_2^2 X_2^1 \mathfrak{A}^1 \tau^2 t^2 r^2 V^3 z^2 S(x^2 Y^1 R^2 y^2 v_{(2,1)}^3 X_1^3 z_1^3 \tilde{p}^1) \\
 &\quad \otimes x_1^3 Y^2 R^1 y^1 v_1^3 X^2 t^3 r^1 V^2 \mathfrak{R}^1 \mathbf{R}^2 z_2^1 P^2 S(x_2^3 Y^3 y^3 v_{(2,2)}^3 X_2^3 z_2^3 \tilde{p}^2), \\
 (1.27, 1.3, 1.20) &= \langle \chi, S(v^1) \tilde{q}^1 v_1^2 \mathfrak{A}^2 \tau^1 t^1 Z^1 \mathfrak{R}^2 \mathbf{R}^1 P^1 \rangle \\
 &\quad \times x^1 \tilde{q}^2 v_2^2 \mathfrak{A}^1 \tau^2 t^2 X^1 r^2 z^2 S(x^2 Y^1 R^2 y^2 (v^3 t^3)_{(2,1)} X_1^3 z_1^3 \tilde{p}^1) \\
 &\quad \otimes x_1^3 Y^2 R^1 y^1 (v^3 t^3)_1 X^2 r^1 z^1 Z^2 \mathfrak{R}^1 \mathbf{R}^2 P^2 S(x_2^3 Y^3 y^3 (v^3 t^3)_{(2,2)} X_2^3 z_2^3 \tilde{p}^2 Z^3), \\
 (1.1, 1.27, 1.3, 1.26) &= \langle \chi, S(v^1) \tilde{q}^1 v_1^2 \mathfrak{A}^2 \tau^1 t^1 Z^1 \mathfrak{R}^2 \mathbf{R}^1 P^1 \rangle \\
 &\quad \times x^1 \tilde{q}^2 v_2^2 \mathfrak{A}^1 \tau^2 t^2 T^1 X^1 R_1^2 V^2 y_2^1 z^2 S(x^2 (v^3 t^3)_1 Y^1 T_1^2 X^2 R_2^2 V^3 y^2 z_1^3 \tilde{p}^1) \\
 &\quad \otimes x_1^3 (v^3 t^3)_{(2,1)} Y^2 T_2^2 X^3 R^1 V^1 y_1^1 z^1 Z^2 \mathfrak{R}^1 \mathbf{R}^2 P^2 \\
 &\quad \times S(x_2^3 (v^3 t^3)_{(2,2)} Y^3 T^3 y^3 z_2^3 \tilde{p}^2 Z^3), \\
 (1.3, 1.18, 1.5) &= \langle \chi, S(v^1) \tilde{q}^1 v_1^2 \mathfrak{A}^2 \tau^1 t^1 Z^1 \mathfrak{R}^2 \mathbf{R}^1 P^1 \rangle x^1 \tilde{q}^2 v_2^2 \mathfrak{A}^1 \tau^2 t^2 X^1 \beta S(x^2 (v^3 t^3)_1 X^2) \\
 &\quad \otimes x_1^3 (v^3 t^3)_{(2,1)} X_1^3 Z^2 \mathfrak{R}^1 \mathbf{R}^2 P^2 S(x_2^3 (v^3 t^3)_{(2,2)} X_2^3 Z^3), \\
 (1.3, 1.5, 1.20, 1.27) &= \langle \chi, S(t^1 x^1) \tilde{q}^1 \mathfrak{A}^2 \tau^1 t_1^2 x_{(1,1)}^2 z^1 Z^1 \mathfrak{R}^2 \mathbf{R}^1 P^1 \rangle \tilde{q}^2 \mathfrak{A}^1 \tau^2 t_2^2 x_{(1,2)}^2 z^2 \beta S(t^3 x_2^2 z^3) \\
 &\quad \otimes x_1^3 Z^2 \mathfrak{R}^1 \mathbf{R}^2 P^2 S(x_2^3 Z^3), \\
 (1.1, 1.5, 1.27, 2.3) &= \mathcal{Q}(\chi_1 \leftarrow S(x^1)) \otimes \chi_2 (x^2 Z^1 \mathfrak{R}^2 \mathbf{R}^1 P^1) x_1^3 Z^2 \mathfrak{R}^1 \mathbf{R}^2 P^2 S(x_2^3 Z^3).
 \end{aligned}$$

On the other hand, by (4.10) we have

$$\begin{aligned}
 (\mathcal{Q} \otimes \mathcal{Q})(\underline{\Delta}_{H^*}(\chi)) &= \mathcal{Q}(\chi_1 \leftarrow S(x^1)) \otimes \mathcal{Q}(x_2^3 X^3 \leftarrow \chi_2 \leftarrow x^2 X^1 \beta S(x_1^3 X^2)), \\
 (2.3, 1.19, 1.27) &= \mathcal{Q}(\chi_1 \leftarrow S(x^1)) \otimes \langle \chi_2, x^2 X^1 \beta S(x_1^3 X^2) \tilde{Q}^1 Z^1 (x_2^3 X^3)_{(1,1)} \mathfrak{R}^2 \mathbf{R}^1 P^1 \rangle \\
 &\quad \times \tilde{Q}_1^2 Z^2 (x_2^3 X^3)_{(1,2)} \mathfrak{R}^1 \mathbf{R}^2 P^2 S(\tilde{Q}_2^2 Z^3 (x_2^3 X^3)_2),
 \end{aligned}$$

$$(1.1, 1.20, 1.18) = \mathcal{Q}(\chi_1 \leftarrow S(x^1)) \otimes \langle \chi_2, x^2 S(\tilde{p}^1) \tilde{Q}^1 \tilde{p}_1^2 Z^1 \mathcal{R}^2 \mathbf{R}^1 P^1 \rangle \\ \times x_1^3 (\tilde{Q}^2 \tilde{p}_2^2)_1 Z^2 \mathcal{R}^1 \mathbf{R}^2 P^2 S(x_2^3 (\tilde{Q}^2 \tilde{p}_2^2)_2 Z^3),$$

$$(1.22) = \mathcal{Q}(\chi_1 \leftarrow S(x^1)) \otimes \langle \chi_2, x^2 Z^1 \mathcal{R}^2 \mathbf{R}^1 P^1 \rangle x_1^3 Z^2 \mathcal{R}^1 \mathbf{R}^2 P^2 S(x_2^3 Z^3).$$

So \mathcal{Q} is a coalgebra map since $(\underline{\varepsilon}_H \circ \mathcal{Q})(\chi) = \chi(\alpha) = \underline{\varepsilon}_{H^*}$ and this finishes our proof. \square

Let \mathcal{C} be a braided category with left duality. For any two objects $M, N \in \mathcal{C}$ there exists a canonical isomorphism in $\mathcal{C}, M^* \otimes N^* \xrightarrow{\sigma_{M,N}^*} (M \otimes N)^*$. In fact, $\sigma_{M,N}^* = \phi_{N,M}^* \circ c_{N^*, M^*}^{-1}$, where $\phi_{N,M}^* : N^* \otimes M^* \rightarrow (M \otimes N)^*$ is given by the following compositions:

$$\begin{aligned} N^* \otimes M^* &\xrightarrow{id_{N^* \otimes M^*} \otimes l_{N^* \otimes M^*}} (N^* \otimes M^*) \otimes \underline{1} \\ &\xrightarrow{id_{N^* \otimes M^*} \otimes coev_{M \otimes N}} (N^* \otimes M^*) \otimes ((M \otimes N) \otimes (M \otimes N)^*) \\ &\xrightarrow{a_{N^* \otimes M^*, M \otimes N, (M \otimes N)}^{-1}} ((N^* \otimes M^*) \otimes (M \otimes N)) \otimes (M \otimes N)^* \\ &\xrightarrow{a_{N^*, M^*, M \otimes N} \otimes id_{(M \otimes N)^*}} (N^* \otimes (M^* \otimes (M \otimes N))) \otimes (M \otimes N)^* \\ &\xrightarrow{(id_{N^*} \otimes a_{M^*, M, N}^{-1}) \otimes id_{(M \otimes N)^*}} (N^* \otimes ((M^* \otimes M) \otimes N)) \otimes (M \otimes N)^* \\ &\xrightarrow{(id_{N^*} \otimes (ev_M \otimes id_N)) \otimes id_{(M \otimes N)^*}} (N^* \otimes (\underline{1} \otimes N)) \otimes (M \otimes N)^* \\ &\xrightarrow{(id_{N^*} \otimes r_N^{-1}) \otimes id_{(M \otimes N)^*}} (N^* \otimes N) \otimes (M \otimes N)^* \\ &\xrightarrow{ev_N \otimes id_{(M \otimes N)^*}} \underline{1} \otimes (M \otimes N)^* \xrightarrow{r_{(M \otimes N)^*}^{-1}} (M \otimes N)^*. \end{aligned} \quad (4.17)$$

The morphism $\phi_{N,M}^*$ is an isomorphism. One can compute its inverse in the same manner as above, see [2, 29, 5]. Hence, $\sigma_{M,N}^*$ is an isomorphism and $\sigma_{M,N}^{*-1} = c_{N^*, M^*} \circ \phi_{N,M}^{*-1}$.

Also, following [18] for any morphism $v : M \rightarrow N$ in \mathcal{C} , we can define the transpose of v as being

$$\begin{aligned} v^* : N^* &\xrightarrow{l_{N^*}} N^* \otimes \underline{1} \xrightarrow{id_{N^*} \otimes coev_M} N^* \otimes (M \otimes M^*) \xrightarrow{id_{N^*} \otimes (v \otimes id_{M^*})} N^* \otimes (N \otimes M^*) \\ &\xrightarrow{a_{N^*, N, M^*}^{-1}} (N^* \otimes N) \otimes M^* \xrightarrow{ev_N \otimes id_{M^*}} \underline{1} \otimes M^* \xrightarrow{r_{M^*}^{-1}} M^*. \end{aligned} \quad (4.18)$$

Let now $(B, m_B, \underline{\Delta}_B, \underline{S}_B)$ be a braided Hopf algebra in \mathcal{C} and B^* the categorical left dual of B in \mathcal{C} . Then B^* is also a braided Hopf algebra in \mathcal{C} with multiplication m_{B^*} , comultiplication $\underline{\Delta}_{B^*}$, antipode \underline{S}_{B^*} , unit \underline{u}_{B^*} and counit $\underline{\varepsilon}_{B^*}$ defined by (see [22], p. 489)

$$m_{B^*} : B^* \otimes B^* \xrightarrow{\phi_{B,B}^*} (B \otimes B)^* \xrightarrow{\underline{\Delta}_B^*} B^*, \quad (4.19)$$

$$\underline{\Delta}_{B^*} : B^* \xrightarrow{m_B^*} (B \otimes B)^* \xrightarrow{\phi_{B,B}^{*-1}} B^* \otimes B^*, \quad (4.20)$$

$$\underline{S}_{B^*} = \underline{S}_B^*, \underline{u}_{B^*} = \varepsilon_B^*, \varepsilon_{B^*} = \underline{u}_B^* \tag{4.21}$$

Suppose now that (H, R) is a QT quasi-Hopf algebra, and that M, N are two finite dimensional left H -modules. Denote by $\{i m\}_{i=1, \overline{s}}$ and $\{i m\}_{i=1, \overline{s}}$ dual bases in M and M^* , and by $\{j n\}_{j=1, \overline{t}}$ and $\{j n\}_{j=1, \overline{t}}$ dual bases in N and N^* , respectively. By [5], in this particular case we have that morphism (4.17) is given by

$$\phi_{N, M}^*(n^* \otimes m^*)(m \otimes n) = \langle m^*, f^1 \cdot m \rangle \langle n^*, f^2 \cdot n \rangle \tag{4.22}$$

for all $m^* \in M^*, n^* \in N^*, m \in M$ and $n \in N$. Its inverse is defined by

$$\phi_{N, M}^{*-1}(\mu) = \langle \mu, g^1 \cdot i m \otimes g^2 \cdot j n \rangle^j n \otimes i m \tag{4.23}$$

for any $\mu \in (M \otimes N)^*$. Also, the morphism v^* defined by (4.18) coincide with the usual transpose map of v , i.e. $v^*(n^*) = n^* \circ v$.

Therefore, if (H, R) is finite dimensional then the categorical left dual of \underline{H} has a braided Hopf algebra structure in ${}_H\mathcal{M}$. We denote \underline{H}^* with this dual Hopf algebra structure by $(\underline{H})^*$. By the above, $(\underline{H})^*$ is a left H -module via

$$(h \succ \chi)(h') = \chi(S(h) \triangleright h'), \quad \forall h, h' \in H, \quad \chi \in H^* \tag{4.24}$$

By (4.19)–(4.21) the structure of $(\underline{H})^*$ as a Hopf algebra in ${}_H\mathcal{M}$ is given by the formulas

$$(\chi * \psi)(h) = \langle \chi, f^2 \triangleright h_2 \rangle \langle \psi, f^1 \triangleright h_1 \rangle, \tag{4.25}$$

$$1_{(\underline{H})^*} = \varepsilon, \tag{4.26}$$

$$\underline{\Delta}_{(\underline{H})^*}(\chi) = \langle \chi, (g^1 \triangleright i e) \bullet (g^2 \triangleright j e) \rangle^j e \otimes i e, \tag{4.27}$$

$$\underline{\varepsilon}_{(\underline{H})^*}(\chi) = \chi(\beta), \tag{4.28}$$

$$\underline{S}_{(\underline{H})^*}(\chi) = \chi \circ \underline{S}, \tag{4.29}$$

where $\{i e\}_{i=1, \overline{n}}$ and $\{i e\}_{i=1, \overline{n}}$ are dual bases in H and H^* .

Following [5], if (H, R) is a finite dimensional QT quasi-Hopf algebra then the usual linear dual of H has in ${}_H\mathcal{M}$ three braided Hopf algebra structures. Two of them are the left and the right categorical dual of \underline{H} in the sense of Takeuchi, [29] (in the Hopf case, the same point of view was used in [2]), and the third one is obtained in [4] by using the structure of a quasi-Hopf algebra with a projection given in [6]. Now, by the above, H^* has a fourth braided Hopf algebra structure in ${}_H\mathcal{M}$.

Proposition 4.2. *Let (H, R) be a finite dimensional QT quasi-Hopf algebra, \underline{H} the associated enveloping algebra braided group of H , $(\underline{H})^*$ the dual Hopf algebra structure of \underline{H} in ${}_H\mathcal{M}$, and \underline{H}^* the function algebra braided group associated to H^* . Then the map $\lambda: (\underline{H})^* \rightarrow \underline{H}^*$ given for all $\chi \in H^*$ by*

$$\lambda(\chi) = S^{-1}(g^1) \leftarrow \chi \circ S \leftarrow g^2 \tag{4.30}$$

is a braided Hopf algebra isomorphism. Here $g^1 \otimes g^2$ is the inverse of the Drinfeld twist f , see (1.14).

Proof. We can easily check that λ is H -linear. Next, we show that λ is an algebra and coalgebra morphism, and that it is bijective. Firstly, for all $\chi, \psi \in H^*$ and $h \in H$ we compute

$$\begin{aligned} \lambda(\chi * \psi)(h) &= \langle \chi, f^2 \triangleright (g^1 S(g^2 h))_2 \rangle \langle \psi, f^1 g^1 \triangleright (S(g^2 h))_1 \rangle, \\ (4.4, 4.3, 1.11) &= \langle \chi, f^2 x^3 R^1 \triangleright y^1 X^2 g_2^1 G^2 S(y^2 X_1^3 g_1^1 h_1) \rangle \\ &\quad \times \langle \psi, f_1^1 x^1 X^1 g_1^1 G^1 S(f_2^1 x^2 R^2 y^3 X_2^3 g_1^1 h_2) \rangle, \\ (1.9, 1.16, 1.1, 1.11) &= \langle \chi, f^2 x^3 R^1 \triangleright G_{(1,1)}^2 y^1 g^1 S(G_{(1,2)}^2 y^2 g_1^2 \mathfrak{S}^1 S(X_2^1) F^1 h_1 X^2) \rangle \\ &\quad \times \langle \psi, f_1^1 x^1 G^1 S(f_2^1 x^2 R^2 G_2^2 y^3 g_2^2 \mathfrak{S}^2 S(X_1^1) F^2 h_2 X^3) \rangle, \\ (1.9, 1.16, 4.3, 1.27) &= \langle \chi, f^2 x^3 G_2^2 R^1 \mathfrak{S}^1 \triangleright g^1 S(g^2 S(X_2^1 y^2) F^1 h_1 X^2 y^3) \rangle \\ &\quad \times \langle \psi, f_1^1 x^1 G^1 S(f_2^1 x^2 G_1^2 R^2 \mathfrak{S}^2 S(X_1^1 y^1) F^2 h_2 X^3) \rangle, \\ (1.32, 1.9, 1.16, 4.3, 1.11) &= \langle \chi, g^1 S(g^2 S(X_2^1 y^2 R_1^1 x_1^1) F^1 h_1 X^2 y^3 R_2^1 x_2^1) \rangle \\ &\quad \times \langle \psi, G^1 S(G^2 S(X_1^1 y^1 R^2 x^2) F^2 h_2 X^3 x^3) \rangle, \end{aligned}$$

and, on the other hand, by (4.8) we have

$$\begin{aligned} (\lambda(\chi) \cdot \lambda(\psi))(h) &= \langle \chi, g^1 S(g^2 S(x^1 X^1) f^2 R^1 h_1 x_1^3 Y^2 r^1 y^1 X^2) \rangle \\ &\quad \times \langle \psi, G^1 S(G^2 S(x^2 Y^1 r^2 y^2 X_1^3) f^1 R^2 h_2 x_2^3 Y^3 y^3 X_2^3) \rangle, \\ (1.32, 1.3, 1.27) &= \langle \chi, g^1 S(g^2 S(Y_2^1 R^1 z^1 x^1 X^1) f^1 h_1 Y^2 z^3 r^1 x_1^2 y^1 X^2) \rangle \\ &\quad \times \langle \psi, G^1 S(G^2 S(Y_1^1 R^2 z^2 r^2 x_2^2 y^2 X_1^3) f^2 h_2 Y^3 x^3 y^3 X_2^3) \rangle, \\ (1.3, 1.25) &= \langle \chi, g^1 S(g^2 S(Y_2^1 z^2 R_1^1 x_1^1) f^1 h_1 Y^2 z^3 R_2^1 x_2^1) \rangle \\ &\quad \times \langle \psi, G^1 S(G^2 S(Y_1^1 z^1 R^2 y^2) f^2 h_2 Y^3 y^3) \rangle, \end{aligned}$$

as needed. It is not hard to see that $\lambda(1_{(H)^*}) = 1_{H^*}$, so λ is an algebra morphism. In order to prove that λ is a coalgebra map we need the following formula

$$S(g^1) \alpha g^2 = S(\beta) \tag{4.31}$$

which can be found in [6]. Now, λ is a coalgebra morphism since

$$\begin{aligned} (\lambda \otimes \lambda)(\underline{\Delta}_{(H)^*}(\chi)) &= \langle \chi, (g^1 \triangleright_i e) \bullet (g^2 \triangleright_j e) \rangle \lambda(i e) \otimes \lambda(j e), \\ (4.30, 4.1, 4.3) &= \langle \chi, X^1 g_1^1 \mathfrak{S}^1 S(x^1 X^2 g_2^1 \mathfrak{S}^2_i e) \alpha x^2 X_1^3 g_1^2 G^1 S(x^3 X_2^3 g_2^2 G^2_j e) \rangle^j e \otimes i e, \\ (1.9, 1.16, 1.1, 1.5) &= \langle \chi, \mathfrak{S}^1 S(g^1 S(X^2 x^3)_i e X^3) \alpha g^2 S(\mathfrak{S}^2 S(X_1^1 x^1)_j e X_2^1 x^2) \rangle^j e \otimes i e, \\ (4.31, 4.30) &= \langle \lambda(\chi), S(X_1^1 x^1)_j e X_2^1 x^2 \beta S(X^2 x^3)_i e X^3 \rangle^j e \otimes i e, \\ (1.3, 1.5, 4.10) &= \lambda(\chi)_1 \leftarrow S(X_1^1 x^1) \otimes X^3 \rightarrow \lambda(\chi)_2 \leftarrow X_2^1 x^2 \beta S(X^2 x^3) = \underline{\Delta}_{(H)^*}(\lambda(\chi)) \end{aligned}$$

for all $\chi \in H^*$ and since the definitions of counits imply $\varepsilon_{H^*} \circ \lambda = \varepsilon_{(H)^*}$. It is easy to see that λ is bijective with inverse $\lambda^{-1}(\chi) = S(f^2) \rightharpoonup \chi \circ S^{-1} \leftarrow f^1$, for all $\chi \in H^*$. Thus, the proof is complete. \square

Summarizing the results of this Section we can give the true meaning of the map $\mathcal{Q}: H^* \rightarrow H$ defined in (2.1). It is a morphism of braided groups from $\underline{H^*}$, the function algebra braided group associated to H^* , to \underline{H} , the associated enveloping algebra braided group of H . When H is factorizable in the sense that the map \mathcal{Q} is bijective then $\mathcal{Q}: \underline{H^*} \cong \underline{H}$ as braided Hopf algebras. In other words, the function algebra braided group associated to H^* and the associated enveloping algebra braided group of H are categorical self dual, cf. Proposition 4.2.

5. $D(H)$ when H is factorizable

Schneider’s Theorem [27] asserts that the quantum double of a finite dimensional factorizable Hopf algebra H is a 2-cocycle twist of the usual (componentwise) tensor product Hopf algebra $H \otimes H$. We note that this result also appears in [26] without an explicit proof. The aim of this section is to give a proof of a similar result in the quasi-Hopf case. Our approach is based on the methods developed in [27,3].

Throughout, (H, R) will be a finite dimensional QT quasi-Hopf algebra and $D(H)$ its quantum double. When there is no danger of confusion the elements $\chi \bowtie h$ of $D(H)$ will be simply denoted by χh . Since H^* can be viewed only as a k -linear subspace of $D(H)$ we will denote by

$$\begin{aligned} \chi_{(1)} \otimes \chi_{(2)} &:= \Delta_D(\chi \bowtie 1_H) \\ (2.12) &= (\varepsilon \bowtie X^1 Y^1)(p_1^1 x^1 \rightharpoonup \chi_2 \leftarrow Y^2 S^{-1}(p^2) \bowtie p_2^1 x^2) \otimes X_1^2 \rightharpoonup \chi_1 \\ &\leftarrow S^{-1}(X^3) \bowtie X_2^2 Y^3 x^3. \end{aligned}$$

In our notation, for all $\chi \in H^*$ and $h \in H$, the comultiplication Δ_D of $D(H)$ comes out as

$$\Delta_D(\chi h) = \chi_{(1)} h_1 \otimes \chi_{(2)} h_2.$$

By [4, Lemma 3.1], there exists a quasi-Hopf algebra projection $\pi : D(H) \rightarrow H$ covering the canonical inclusion $i_D : H \rightarrow D(H)$. More precisely, if $R = R^1 \otimes R^2$ is the R -matrix of H then π is defined by

$$\pi(\chi \bowtie h) = \chi(q^2 R^1) q^1 R^2 h, \tag{5.1}$$

where $q_R = q^1 \otimes q^2$ is the element defined by (1.17). We have that π is a quasi-Hopf algebra morphism and $\pi \circ i_D = id_H$.

It is not hard to see that $\tilde{R} := R_{21}^{-1} = \overline{R^2} \otimes \overline{R^1}$ is another R -matrix for H . So, as in the Hopf case, there is always a second projection $\tilde{\pi} : D(H) \rightarrow H$ covering the canonical inclusion i_D . Explicitly, the morphism $\tilde{\pi}$ is given by

$$\tilde{\pi}(\chi \bowtie h) = \chi(q^2 \overline{R^2}) q^1 \overline{R^1} h \tag{5.2}$$

for all $\chi \in H^*$ and $h \in H$.

Let H and A be two quasi-bialgebras. Recall that a quasi-bialgebra map between H and A is an algebra map $v: H \rightarrow A$ which intertwines the quasi-coalgebra structures, respects the counits, and satisfies $(v \otimes v \otimes v)(\Phi) = \Phi_A$. Following [6], when H is a quasi-Hopf algebra we define

$$H^{co(v)} = \{h \in H \mid h_1 \otimes v(h_2) = x^1 h S(x_2^3 X^3) f^1 \otimes v(x^2 X^1 \beta S(x_1^3 X^2) f^2)\} \quad (5.3)$$

as being the set of coinvariants of H relative to v . Finally, if A is also a quasi-Hopf algebra then v is a quasi-Hopf algebra morphism if, in addition, $v(\alpha) = \alpha_A$, $v(\beta) = \beta_A$ and $S_A \circ v = v \circ S_A$.

Lemma 5.1. *Let (H, R) be a finite dimensional QT quasi-Hopf algebra, and π and $\tilde{\pi}$ the quasi-Hopf algebra morphisms defined by (5.1) and (5.2), respectively. Let $j: D(H)^{co(\pi)} \rightarrow D(H)$ be the inclusion map and $\Psi: H^* \rightarrow D(H)^{co(\pi)}$ defined by*

$$\Psi(\chi) = \chi_{(1)} \beta S(\pi(\chi_{(2)})) \quad (5.4)$$

for all $\chi \in H^*$. Then the following assertions hold:

- (1) Ψ is well defined and bijective.
- (2) If $\bar{\mathcal{Q}}$ is the map defined by (2.4) then $S \circ \bar{\mathcal{Q}} = \tilde{\pi} \circ j \circ \Psi$. In particular, $\bar{\mathcal{Q}}$ is bijective if and only if $\tilde{\pi}|_{D(H)^{co(\pi)}}$ is bijective.

Proof. (1) For all $\chi \in H^*$ we have

$$(id \otimes \pi) \Delta_D(\Psi(\chi)) = \chi_{((1),(1))} \beta_1 S(\pi(\chi_{(2)}))_1 \otimes \chi_{((1),(2))} \beta_2 S(\pi(\chi_{(2)}))_2,$$

where we use the Sweedler type notation

$$(\Delta_D \otimes id)(\Delta_D(\chi)) = \chi_{((1),(1))} \otimes \chi_{((1),(2))} \otimes \chi_{(2)},$$

$$(id \otimes \Delta_D)(\Delta_D(\chi)) = \chi_{(1)} \otimes \chi_{((2),(1))} \otimes \chi_{((2),(2))}.$$

Now, since H is a quasi-Hopf subalgebra of $D(H)$ and π is a quasi-Hopf algebra morphism such that $\pi(h) = h$ for any $h \in H$, by similar computations as in [6, Lemma 4.2] or [3, Lemma 3.6], one can prove that $\Psi(\chi) \in D(H)^{co(\pi)}$, so Ψ is well defined. We claim that the inverse of Ψ , $\Psi^{-1}: D(H)^{co(\pi)} \rightarrow H^*$, is given for all $\mathbf{D} \in D(H)^{co(\pi)}$ by the formula

$$\Psi^{-1}(\mathbf{D}) = (id \otimes \varepsilon)(\mathbf{D}).$$

Indeed, Ψ^{-1} is a left inverse since

$$\begin{aligned} (\Psi^{-1} \circ \Psi)(\chi) &= \langle id \otimes \varepsilon, \chi_{(1)} \beta S(\pi(\chi_{(2)})) \rangle = (id \otimes \varepsilon)(\chi_{(1)}) \varepsilon_D(\chi_{(2)}) \\ &= (id \otimes \varepsilon)(\chi \otimes 1) = \chi, \end{aligned}$$

for all $\chi \in H^*$. It is also a right inverse. If $\mathbf{D} = {}_i \chi_i h \in D(H)^{co(\pi)}$ then

$${}_i \chi_{(1)} h_1 \otimes \pi({}_i \chi_{(2)}) h_2 = x^1 [{}_i \chi_i h] S(x_2^3 X^3) f^1 \otimes x^2 X^1 \beta S(x_1^3 X^2) f^2$$

in $D(H) \otimes H$. Therefore,

$$\begin{aligned} (\Psi \circ \Psi^{-1})(\mathbf{D}) &= \varepsilon(ih)\Psi(i\chi) = \varepsilon(ih)_{i\chi(1)}\beta S(\pi(i\chi(2))) \\ &= {}_{i\chi(1)}i h_1 \beta S(\pi(i\chi(2))_i h_2) \\ &= x^1 [{}_{i\chi}i h] S(x_2^3 X^3) f^1 \beta S(x^2 X^1 \beta S(x_1^3 X^2) f^2) \\ &= {}_{i\chi}i h = \mathbf{D}, \end{aligned}$$

because of $f^1 \beta S(f^2) = S(\alpha)$, and (1.5), (1.6).

(2) By (5.4, 2.12), (2.12) and (5.1), for any $\chi \in H^*$ we find that

$$\begin{aligned} \Psi(\chi) &= (X^1 Y^1)_{(1,1)} p_1^1 x^1 \leftarrow \chi \leftarrow S^{-1}(X^3) q^2 R^1 X_1^2 Y^2 S^{-1}((X^1 Y^1)_2 p^2) \\ &\bowtie (X^1 Y^1)_{(1,2)} p_2^1 x^2 \beta S(q^1 R^2 X_2^2 Y^3 x^3) \end{aligned}$$

and, if we denote by $Q^1 \otimes Q^2$ another copy of q_R , and by $r^1 \otimes r^2$ another copy of R , then by (5.2) we compute that

$$\begin{aligned} (\tilde{\pi} \circ j \circ \Psi)(\chi) &= \langle \chi, S^{-1}(X^3) q^2 R^1 X_1^2 Y^2 S^{-1}((X^1 Y^1)_2 p^2) Q^2 \bar{R}^2 (X^1 Y^1)_{(1,1)} p_1^1 x^1 \rangle \\ &\quad Q^1 \bar{R}^1 (X^1 Y^1)_{(1,2)} p_2^1 x^2 \beta S(q^1 R^2 X_2^2 Y^3 x^3), \\ (1.27, 1.19) &= \langle \chi, S^{-1}(X^3) q^2 R^1 X_1^2 Y^2 S^{-1}(p^2) Q^2 \bar{R}^2 p_1^1 x^1 \rangle \\ &\quad \times X^1 Y^1 Q^1 \bar{R}^1 p_2^1 x^2 \beta S(q^1 R^2 X_2^2 Y^3 x^3), \\ (1.27, 1.21) &= \langle \chi, S^{-1}(X^3) q^2 R^1 X_1^2 Y^2 \bar{R}^2 x^1 \rangle X^1 Y^1 \bar{R}^1 x^2 \beta S(q^1 R^2 X_2^2 Y^3 x^3), \\ (1.27, 1.25) &= \langle \chi, S^{-1}(X^3) q^2 X_2^2 R^1 r^2 Y^3 \bar{R}^2 \rangle X^1 Y^1 \bar{R}^1 \beta S(q^1 X_1^2 R^2 r^1 Y^2 \bar{R}_2^1), \\ (1.5, 1.28, 1.27) &= \langle \chi, S^{-1}(X^3) q^2 R^1 r^2 X_2^2 Y^3 \rangle S(q^1 R^2 r^1 X_1^2 Y^2 S^{-1}(X^1 Y^1 \beta)), \\ (1.18, 2.4) &= (S \circ \bar{\mathcal{Q}})(\chi), \end{aligned}$$

as needed. Since H is finite dimensional the antipode S is bijective. So $\bar{\mathcal{Q}}$ is bijective if and only if $\tilde{\pi} \circ j$ is bijective. Thus, the proof is complete. \square

For the next result we need the concept of right quasi-Hopf bimodule introduced in [14], and the second Structure Theorem for right quasi-Hopf bimodules proved in [3].

Let H be a quasi-bialgebra, M and H -bimodule and $\rho : M \rightarrow M \otimes H$ an H -bimodule map. Then (M, ρ) is called a right quasi-Hopf H -bimodule if the following relations hold:

$$(id \otimes \varepsilon) \circ \rho = id, \tag{5.5}$$

$$\Phi(\rho \otimes id)(\rho(m)) = (id \otimes \Delta)(\rho(m))\Phi, \quad \forall m \in M. \tag{5.6}$$

A morphism between two right quasi-Hopf H -bimodules is an H -bimodule map which is also right H -colinear (just like in the Hopf case). ${}_H\mathcal{M}_H^H$ is the category of right quasi-Hopf H -bimodules and morphisms of right quasi-Hopf H -bimodules.

Let H be a quasi-Hopf algebra and $M \in {}_H\mathcal{M}_H^H$. Following [3], we define

$$M^{\overline{\text{co}(H)}} := \{n \in M \mid \rho(n) = x^1 \cdot n \cdot S(x_2^3 X^3) f^1 \otimes x^2 X^1 \beta S(x_1^3 X^2) f^2\} \quad (5.7)$$

and $\bar{E} : M \rightarrow M$, by

$$\bar{E}(m) = m_{(0)} \cdot \beta S(m_{(1)}), \quad (5.8)$$

for all $m \in M$, where $\rho(m) := m_{(0)} \otimes m_{(1)}$. From [3, Lemma 3.6] we have that $\text{Im}(\bar{E}) = M^{\overline{\text{co}(H)}}$, and that $M^{\overline{\text{co}(H)}}$ is a left H -submodule of M , where M is considered a left H -module via the left adjoint action, that is $h \triangleright m = h_1 \cdot m \cdot S(h_2)$, for all $h \in H$ and $m \in M$. Moreover, if $M^{\overline{\text{co}(H)}} \otimes H$ is viewed as a right quasi-Hopf H -bimodule via the structure

$$h \cdot (n \otimes h') \cdot h'' = h_1 \triangleright n \otimes h_2 h' h'', \quad \rho'(n \otimes h) = x^1 \triangleright n \otimes x^2 h_1 \otimes x^3 h_2,$$

then the map

$$\bar{v}_M : M^{\overline{\text{co}(H)}} \otimes H \rightarrow M, \quad \bar{v}_M(n \otimes h) = X^1 \cdot n \cdot S(X^2) \alpha X^3 h$$

is an isomorphism of quasi-Hopf H -bimodules, cf. [3, Theorem 3.7]. The inverse of \bar{v}_M is given by the formula

$$\bar{v}_M^{-1}(m) = \bar{E}(m_{(0)}) \otimes m_{(1)}.$$

Suppose now that H is a quasi-Hopf algebra and $\mu : M \rightarrow N$ is a morphism between two right quasi-Hopf H -bimodules. It is not hard to see that the restriction of μ defines a left H -linear map between $M^{\overline{\text{co}(H)}}$ and $N^{\overline{\text{co}(H)}}$. Moreover, if we denote this (co)restriction by μ_0 then the following diagram is commutative.

$$\begin{array}{ccc} M^{\overline{\text{co}(H)}} \otimes H & \xrightarrow{\bar{v}_M} & M \\ \mu_0 \otimes \text{id} \downarrow & & \downarrow \mu \\ N^{\overline{\text{co}(H)}} \otimes H & \xrightarrow{\bar{v}_N} & N \end{array}$$

Consequently, the map μ is bijective if and only if the map $\mu_0 : M^{\overline{\text{co}(H)}} \rightarrow N^{\overline{\text{co}(H)}}$ is bijective.

Lemma 5.2. *Let D, A and B three quasi-bialgebras and ϑ, v, κ three quasi-bialgebra morphisms as in the diagram below*

$$\begin{array}{ccc} D & \xrightarrow{\vartheta} & A \\ \downarrow v & \uparrow \kappa & \searrow \zeta := (v \otimes v) \circ \Delta_D \\ B & & A \otimes B. \end{array}$$

Suppose that $v \circ \kappa = id_B$ and define ζ as above. Then the following assertion hold:

(1) D and $A \otimes B$ are right quasi-Hopf B -bimodules via the following structures

$$D \in {}_B\mathcal{M}_B^B : \begin{cases} b \cdot d \cdot b' = \kappa(b)d\kappa(b') \\ \rho_D(d) = d_1 \otimes v(d_2), \end{cases}$$

$$A \otimes B \in {}_B\mathcal{M}_B^B : \begin{cases} b' \cdot (a \otimes b) \cdot b'' = \vartheta(\kappa(b'_1))a\vartheta(\kappa(b''_1)) \otimes b'_2bb''_2 \\ \rho_{A \otimes B}(a \otimes b) = \vartheta(\kappa(x^1))a\vartheta(\kappa(X^1)) \otimes x^2b_1X^2 \otimes x^3b_2X^3, \end{cases}$$

$a \in A, b, b', b'' \in B, d \in D$, and ζ becomes a quasi-Hopf B -bimodule morphism.

(2) If D, A and B are quasi-Hopf algebras and ϑ, v and κ are quasi-Hopf algebra maps then $D^{\overline{co(B)}} = D^{co(v)}$ and

$$(A \otimes B)^{\overline{co(B)}} = \{ \vartheta(\kappa(x^1))a\vartheta(\kappa(S(x_2^3X^3)f^1)) \otimes x^2X^1\beta S(x_1^3X^2)f^2 \mid a \in A \}.$$

Proof. Since no confusion is possible we will write without subscripts D, A or B in the tensor components of the reassociators of D, A or B , respectively. The same thing we will do when we write their inverses.

(1) It is straightforward to show that with the above structures D is an object of ${}_B\mathcal{M}_B^B$, and that $A \otimes B$ is a B -bimodule. The map $\rho_{A \otimes B}$ is a B -bimodule map since

$$\begin{aligned} \rho_{A \otimes B}(b' \cdot (a \otimes b) \cdot b'') &= \rho_{A \otimes B}(\vartheta(\kappa(b'_1))a\vartheta(\kappa(b''_1)) \otimes b'_2bb''_2) \\ &= \vartheta(\kappa(x^1b'_1))a\vartheta(\kappa(b''_1X^1)) \otimes x^2b'_{(2,1)}b_1b''_{(2,1)}X^2 \\ &\quad \otimes x^3b'_{(2,2)}b_2b''_{(2,2)}X^3, \\ (1.1) &= b'_1 \cdot (\vartheta(\kappa(x^1))a\vartheta(\kappa(X^1)) \otimes x^2b_1X^2) \cdot b''_1 \otimes b'_2x^3b_2X^3b''_2 \\ &= \Delta(b')\rho_{A \otimes B}(a \otimes b)\Delta(b''), \end{aligned}$$

for all $a \in A$ and $b, b', b'' \in B$. Similar computations show that

$$\Phi^{-1}(id \otimes \Delta)(\rho_{A \otimes B}(a \otimes b))\Phi = (\rho_{A \otimes B} \otimes id)(\rho_{A \otimes B}(a \otimes b)),$$

for all $a \in A$ and $b \in B$, so $A \otimes B \in {}_B\mathcal{M}_B^B$. Also, we can check directly that ζ becomes a morphism in ${}_B\mathcal{M}_B^B$, the details are left to the reader.

(2) By definitions we have

$$\begin{aligned} D^{\overline{co(B)}} &= \{ d \in D \mid \rho_D(d) = x^1 \cdot d \cdot S(x_2^3X^3)f^1 \otimes x^2X^1\beta S(x_1^3X^2)f^2 \} \\ &= \{ d \in D \mid d_1 \otimes v(d_2) = \kappa(x^1)d\kappa(S(x_2^3X^3)f^1) \otimes v(\kappa(x^2X^1\beta S(x_1^3X^2)f^2)) \} \\ &= D^{co(v)}. \end{aligned}$$

Observe now that $a \otimes b \in (A \otimes B)^{\overline{co(B)}}$ if and only if

$$\begin{aligned} &\vartheta(\kappa(x^1))a\vartheta(\kappa(X^1)) \otimes x^2b_1X^2 \otimes x^3b_2X^3 \\ &= \vartheta(\kappa(x^1_1))a\vartheta(\kappa((S(x_2^3X^3)f^1)_1)) \otimes x^2_1b(S(x_2^3X^3)f^1)_2 \\ &\quad \otimes x^2X^1\beta S(x_1^3X^2)f^2. \end{aligned} \tag{5.9}$$

If $a \otimes b \in (A \otimes B)^{\overline{\text{co}(B)}}$ applying $id \otimes \varepsilon \otimes id$ to equality (5.9) we obtain

$$a \otimes b = \varepsilon(b)\vartheta(\kappa(x^1))a\vartheta(\kappa(S(x_2^3 X^3) f^1)) \otimes x^2 X^1 \beta S(x_1^3 X^2) f^2,$$

so $(A \otimes B)^{\overline{\text{co}(B)}} \subseteq \{\vartheta(\kappa(x^1))a\vartheta(\kappa(S(x_2^3 X^3) f^1)) \otimes x^2 X^1 \beta S(x_1^3 X^2) f^2 \mid a \in A\}$. Conversely, if $\delta = \delta^1 \otimes \delta^2$ is the element defined by (1.12) and $F^1 \otimes F^2 = \mathbf{F}^1 \otimes \mathbf{F}^2$ are other copies of f then for all $a \in A$ we compute

$$\vartheta(\kappa(y^1))[\vartheta(\kappa(x^1))a\vartheta(\kappa(S(x_2^3 X^3) f^1))]\vartheta(\kappa(Y^1)) \otimes y^2[x^2 X^1 \beta S(x_1^3 X^2) f^2]_1 Y^2$$

$$\otimes y^3[x^2 X^1 \beta S(x_1^3 X^2) f^2]_2 Y^3$$

$$(1.15, 1.11) = \vartheta(\kappa(y^1 x^1))a\vartheta(\kappa(S(x_2^3 X^3) f^1 Y^1))$$

$$\otimes y^2 x_1^2 X_1^1 z^1 \beta S(x_{(1,2)}^3 X_2^2 z_2^3 Z^3) F^1 f_1^2 Y^2$$

$$\otimes y^3 x_2^3 X_2^1 z^2 Z^1 \beta S(x_{(1,1)}^3 X_1^2 z_1^3 Z^2) F^2 f_2^2 Y^3,$$

$$(\text{twice } 1.3) = \vartheta(\kappa(x_1^1 y^1))a\vartheta(\kappa(S((x^3 y_2^3)_2 X^3 T^3) f^1 Y^1))$$

$$\otimes x_2^1 y^2 T^1 \beta S((x^3 y_2^3)_{(1,2)} X_2^2 Z^3 T^2) F^1 f_1^2 Y^2$$

$$\otimes x^2 y_1^3 X^1 Z^1 \beta S((x^3 y_2^3)_{(1,1)} X_1^2 Z^2) F^2 f_2^2 Y^3,$$

$$(1.3, 1.5, 1.1) = \vartheta(\kappa(x_1^1 y^1))a\vartheta(\kappa(S(x_2^3 z^3 (y_{(2,2)}^3 X^3)_2 T^3 f^1 Y^1))$$

$$\otimes x_2^1 y^2 T^1 \beta S(x_{(1,2)}^3 z^2 (y_{(2,2)}^3 X^3)_1 T^2) F^1 f_1^2 Y^2$$

$$\otimes x^2 y_1^3 X^1 \beta S(x_{(1,1)}^3 z^1 y_{(2,1)}^3 X^2) F^2 f_2^2 Y^3,$$

$$(1.1, 1.5) = \vartheta(\kappa(x_1^1 y^1))a\vartheta(\kappa(S(x_2^3 z^3 X_2^3 y_2^3 T^3) f^1 Y^1))$$

$$\otimes x_2^1 y^2 T^1 \beta S(x_{(1,2)}^3 z^2 X_1^3 z^2 X_1^3 y_1^3 T^2) F^1 f_1^2 Y^2$$

$$\otimes x^2 X^1 \beta S(x_{(1,1)}^3 z^1 X^2) F^2 f_2^2 Y^3,$$

$$(1.1, 1.9, 1.16) = \vartheta(\kappa(x_1^1))[\vartheta(\kappa(y^1))a\vartheta(\kappa(S(y_2^3 T^3) \mathbf{F}^1))]\vartheta(\kappa(g^1 S(x_{(2,2)}^3 X_2^3) F^1 f_1^1))$$

$$\otimes x_2^1 [y^2 T^1 \beta S(y_1^3 T^2) \mathbf{F}^2] g^2 S(x_{(2,1)}^3 X_1^3) F^2 f_2^1 \otimes x^2 X^1 \beta S(x_1^3 X^2) f^2,$$

$$(1.11) = \vartheta(\kappa(x_1^1))[\vartheta(\kappa(y^1))a\vartheta(\kappa(S(y_2^3 T^3) \mathbf{F}^1))]\vartheta(\kappa((S(x_2^3 X^3) f^1)_1))$$

$$\otimes x_2^1 [y^2 T^1 \beta S(y_1^3 T^2) \mathbf{F}^2] (S(x_2^3 X_1^3) f^1)_2 \otimes x^2 X^1 \beta S(x_1^3 X^2) f^2,$$

as needed. Therefore, $\{\vartheta(\kappa(x^1))a\vartheta(\kappa(S(x_2^3 X^3) f^1)) \otimes x^2 X^1 \beta S(x_1^3 X^2) f^2 \mid a \in A\} \subseteq (A \otimes B)^{\overline{\text{co}(B)}}$, and this finishes our proof. \square

Proposition 5.3. *Let D be a quasi-Hopf algebra, A and B two quasi-bialgebras and $\vartheta : D \rightarrow A$, $v : D \rightarrow B$ two quasi-bialgebra maps. Consider $\zeta : D \rightarrow A \otimes B$ given by $\zeta(d) = \vartheta(d_1) \otimes v(d_2)$, for all $d \in D$.*

- (1) Suppose that (D, R) is quasitriangular and define $\mathfrak{F} = \mathfrak{F}^1 \otimes \mathfrak{F}^2 \in (A \otimes B)^{\otimes 2}$, by
- $$\mathfrak{F} = \vartheta(Y_1^1 x^1 X^1 y_1^1) \otimes v(Y_2^1 x^2 \bar{R}^1 X^3 y^2) \otimes \vartheta(Y^2 x^3 \bar{R}^2 X^2 y_2^1) \otimes v(Y^3 y^3), \quad (5.10)$$
- where, as usual, $\bar{R}^1 \otimes \bar{R}^2$ is the inverse of the R -matrix R of D . Then \mathfrak{F} is a twist on $A \otimes B$ (here $A \otimes B$ has the componentwise quasi-bialgebra structure) and $\zeta : D \rightarrow (A \otimes B)_{\mathfrak{F}}$ is a quasi-bialgebra morphism. Moreover, if A and B are quasi-Hopf algebras and ϑ and v are quasi-Hopf algebra morphisms, then $\zeta : D \rightarrow (A \otimes B)_{\mathfrak{F}}^{\mathfrak{U}}$ is a quasi-Hopf algebra morphism, where $\mathfrak{U} = \vartheta(R^2 g^2) \otimes v(R^1 g^1)$.
- (2) Suppose that A and B are quasi-Hopf algebras, ϑ and v are quasi-Hopf algebra morphisms, and that there exists a quasi-Hopf algebra map $\kappa : B \rightarrow D$ such that $v \circ \kappa = id_B$. Then ζ is a bijective map if and only if the restriction of ϑ provides a bijection from $D^{co(v)}$ to A .

Proof. (1) We have that $\zeta = (\vartheta \otimes v) \circ \Delta_D$, so clearly ζ is an algebra map. It also respects the comultiplications. Indeed, applying (1.8), twice (1.1), (1.27), and then again (1.1) two times, it is not hard to see that

$$(\Delta_{(A \otimes B)_{\mathfrak{F}}} \circ \zeta)(d) = ((\zeta \otimes \zeta) \circ \Delta_D)(d)$$

for all $d \in D$. Obviously, $\varepsilon_{A \otimes B} \circ \zeta = \varepsilon_D$, so ζ respects the counits. It remains to show that

$$(\zeta \otimes \zeta \otimes \zeta)(\Phi_D) = \Phi_{(A \otimes B)_{\mathfrak{F}}}.$$

This follows from a long, technical but straightforward computation, we leave the details to the reader. Suppose now that A and B are quasi-Hopf algebras and that ϑ and v are quasi-Hopf algebra morphisms. In this case, $\zeta : D \rightarrow (A \otimes B)_{\mathfrak{F}}^{\mathfrak{U}}$ is also a quasi-bialgebra morphism since $(A \otimes B)_{\mathfrak{F}}^{\mathfrak{U}} = (A \otimes B)_{\mathfrak{F}}$ as quasi-bialgebras. Thus, we are left to show that

$$\zeta(\alpha) = \mathfrak{U} \alpha_{(A \otimes B)_{\mathfrak{F}}}, \quad \zeta(\beta) = \beta_{(A \otimes B)_{\mathfrak{F}}} \mathfrak{U}^{-1}, \quad (\zeta \circ S_D)(d) = \mathfrak{U} S_{A \otimes B}(\zeta(d)) \mathfrak{U}^{-1}$$

for all $d \in D$. Take $\mathfrak{F}^{-1} = \mathfrak{G}^1 \otimes \mathfrak{G}^2$ as being the inverse of the twist \mathfrak{F} . By (1.10) and (5.10) we compute:

$$\begin{aligned} \alpha_{(A \otimes B)_{\mathfrak{F}}} &= S_{A \otimes B}(\mathfrak{G}^1) \alpha_{A \otimes B} \mathfrak{G}^2 \\ &= \vartheta(S(Y_1^1 x^1 X^1 y_1^1) \alpha Y_2^1 x^2 R^2 X^3 y^2) \otimes v(S(Y^2 x^3 R^1 X^2 y_2^1) \alpha Y^3 y^3), \\ (1.5, 1.26) &= \vartheta(S(R_1^2 X^2 \bar{R}^2 y_1^1) \alpha R_2^2 X^3 y^2) \otimes v(S(R^1 X^1 \bar{R}^1 y_2^1) \alpha y^3), \\ (1.5, 1.28, 1.27) &= \vartheta(S(X^2 y_2^1 \bar{R}^2) \alpha X^3 y^2) \otimes v(S(X^1 y_1^1 \bar{R}^1) \alpha y^3), \\ (1.12, 1.15) &= \vartheta(S(\bar{R}^2) \gamma^1) \otimes v(S(\bar{R}^1) \gamma^2) = \vartheta(S(\bar{R}^2) f^1 \alpha_1) \otimes v(S(\bar{R}^1) f^2 \alpha_2), \\ (1.32) &= \vartheta(f^2 \bar{R}^2 \alpha_1) \otimes v(f^1 \bar{R}^1 \alpha_2) = \mathfrak{U}^{-1} \zeta(\alpha), \end{aligned}$$

as needed. In a similar manner one can prove that $\beta_{(A \otimes B)_{\mathfrak{F}}} = \zeta(\beta) \mathfrak{U}$, the details are left to the reader. Finally, for all $d \in D$ we have

$$\begin{aligned} \mathfrak{U} S_{A \otimes B}(\zeta(d)) \mathfrak{U}^{-1} &= \vartheta(R^2 g^2 S(d_1) f^2 \bar{R}^2) \otimes v(R^1 g^1 S(d_2) f^1 \bar{R}^1), \\ (1.11, 1.27) &= \vartheta(S(d)_1) \otimes v(S(d)_2) = \zeta(S(d)). \end{aligned}$$

(2) We are in the same hypothesis as in the Lemma 5.2, so $\zeta : D \rightarrow A \otimes B$ is a right quasi-Hopf B -bimodule morphism. As we have already explained before Lemma 5.2, the morphism ζ is bijective if and only if ζ_0 , the restriction of ζ , defines an isomorphism between $D^{\overline{co(B)}}$ and $(A \otimes B)^{\overline{co(B)}}$. But $D^{\overline{co(B)}} = D^{co(v)}$, so if $d \in D^{\overline{co(B)}}$ then

$$\begin{aligned} \zeta(d) &= \vartheta(d_1) \otimes v(d_2) \\ &= \vartheta(\kappa(x^1))\vartheta(d)\vartheta(\kappa(S(x_2^3 X^3) f^1)) \otimes x^2 X^1 \beta S(x_1^3 X^2) f^2, \end{aligned}$$

because of $v \circ \kappa = id_B$. Hence, by Lemma 5.2, ζ is bijective if and only if the map

$$\zeta_0 : D^{co(v)} \rightarrow \{\vartheta(\kappa(x^1))a\vartheta(\kappa(S(x_2^3 X^3) f^1)) \otimes x^2 X^1 \beta S(x_1^3 X^2) f^2 \mid a \in A\},$$

$$\zeta_0(d) = \vartheta(\kappa(x^1))\vartheta(d)\vartheta(\kappa(S(x_2^3 X^3) f^1)) \otimes x^2 X^1 \beta S(x_1^3 X^2) f^2$$

is bijective. Now, it follows that ζ is bijective if and only if the restriction of ϑ defines a bijection between $D^{co(v)}$ and A . \square

We can now state the structure theorem of $D(H)$ when H is factorizable. The next result generalizes [27, Theorem 4.3].

Theorem 5.4. *Let (H, R) be a finite dimensional QT quasi-Hopf algebra, $D(H) \xrightarrow{\pi} H$ the quasi-Hopf algebra morphisms defined by (5.1) and (5.2), respectively, and define $\zeta : D(H) \rightarrow H \otimes H$, given by $\zeta(\mathbf{D}) = \tilde{\pi}(\mathbf{D}_1) \otimes \pi(\mathbf{D}_2)$, for all $\mathbf{D} \in D(H)$, and*

$$\mathbf{F} = Y_1^1 x^1 X^1 y_1^1 \otimes Y_2^1 x^2 R^2 X^3 y^2 \otimes Y^2 x^3 R^1 X^2 y_2^1 \otimes Y^3 y^3, \quad (5.11)$$

where $R^1 \otimes R^2$ is the R -matrix R of H . Then the following assertions hold:

- (1) $\zeta : D(H) \rightarrow (H \otimes H)_{\mathbf{F}}^{\mathbf{U}}$ is a quasi-Hopf algebra morphism, where $\mathbf{U} := \overline{R}^1 g^2 \otimes \overline{R}^2 g^1$.
- (2) ζ is bijective if and only if (H, R) is factorizable.

Proof. We consider in Proposition 5.3 $D = D(H)$, $A = B = H$, $\vartheta = \tilde{\pi}$, $v = \pi$ and $\kappa = i_D$. So the map ζ in the statement is the map ζ in Proposition 5.3 specialized for our case. Moreover, from definition (2.17) of the R -matrix \mathcal{R} of $D(H)$ we have

$$\begin{aligned} \pi(\mathcal{R}^1) \otimes \tilde{\pi}(\mathcal{R}^2) &= S^{-1}(p^2)_i e p_1^1 \otimes \langle {}^i e, q^2 \overline{R}^2 \rangle q^1 \overline{R}^1 p_2^1 \\ &= S^{-1}(p^2) q^2 \overline{R}^2 p_1^1 \otimes q^1 \overline{R}^1 p_2^1, \end{aligned}$$

$$(1.27, 1.21) = S^{-1}(p^2) q^2 p_2^1 \overline{R}^2 \otimes q^1 p_1^1 \overline{R}^1 = \overline{R}^2 \otimes \overline{R}^1.$$

Since π and $\tilde{\pi}$ are algebra maps we obtain that $\pi(\overline{\mathcal{R}}^1) \otimes \tilde{\pi}(\overline{\mathcal{R}}^2) = R^2 \otimes R^1$, so the twist (5.11) is the twist \mathfrak{F} defined in (5.10) specialized for our situation. Also, the element \mathbf{U} is the element \mathfrak{U} defined in Proposition 5.3 specialized for our context and this prove the first assertion.

Applying again Proposition 5.3 we have that ζ is bijective if and only if the restriction of $\tilde{\pi}$ provides a bijection from $D(H)^{co(\pi)}$ to H . By Lemma 5.1 this is equivalent to $\overline{\mathfrak{F}}$ bijective. Finally, by Proposition 2.2 we obtain that ζ is bijective if and only if (H, R) is factorizable, and this finishes our proof. \square

6. Factorizable implies unimodular

In [25] it is proved that a finite dimensional factorizable Hopf algebra is unimodular. In this section we will show that this also holds for a finite dimensional factorizable quasi-Hopf algebra. In particular, we obtain that for any finite dimensional quasi-Hopf algebra H its Drinfeld double $D(H)$ is always a unimodular quasi-Hopf algebra.

Throughout, H will be a finite dimensional quasi-Hopf algebra. Recall that $t \in H$ is called a left (respectively right) integral in H if $ht = \varepsilon(h)t$ (respectively $th = \varepsilon(h)t$) for all $h \in H$. We denote by \int_l^H (\int_r^H) the space of left (right) integrals in H . It follows from the bijectivity of the antipode with $S(\int_l^H) = \int_r^H$ and $S(\int_r^H) = \int_l^H$. If there is a non-zero left integral in H which is at the same time a right integral, then H is called unimodular. Hausser and Nill [14] proved that for a finite dimensional quasi-Hopf algebra the space of left or right integrals has dimension 1.

Let t be a non-zero integral in H . Since the space of left integrals is a two-sided ideal it follows from the uniqueness of integrals in H that there exists $\mu \in H^*$ such that

$$th = \mu(h)t, \quad \forall t \in \int_l^H \quad \text{and} \quad h \in H. \tag{6.1}$$

It was noted in [14] that μ is an element of $Alg(H, k)$, i.e. μ is an algebra morphism from H to k . Moreover, $Alg(H, k)$ is a group with multiplication given by $\varrho \circ \varsigma = (\varrho \otimes \varsigma) \circ \Delta$, unit ε , and inverse $\varrho^{-1} = \varrho \circ S = \varrho \circ S^{-1}$. Observe that $\mu = \varepsilon$ if and only if H is unimodular. As in the case of a Hopf algebra we will call μ the distinguished group-like element of H^* .

Hasser and Nill [14] also introduced left cointegrals on a finite dimensional quasi-Hopf algebra. These cointegrals are the elements λ of the dual space H^* which satisfy for all $h \in H$,

$$\lambda(V^2 h_2 U^2) V^1 h_1 U^1 = \mu(x^1) \lambda(hS(x^2)) x^3. \tag{6.2}$$

Here $U = U^1 \otimes U^2$ is the element defined by (2.2), μ is the distinguished group-like element of H^* , and if $p_R = p^1 \otimes p^2$ and $f = f^1 \otimes f^2$ are the elements defined by (1.17) and (1.13), respectively, then $V = V^1 \otimes V^2$ is given by

$$V = S^{-1}(f^2 p^2) \otimes S^{-1}(f^1 p^1). \tag{6.3}$$

Using another structure theorem for right quasi-Hopf H -bimodules, Hausser and Nill prove that the space of left cointegrals \mathcal{L} is one dimensional, and that the dual pairing $\mathcal{L} \otimes \int_r^H \ni \lambda \otimes r \mapsto \langle \lambda, r \rangle \in k$ is non-degenerated. Let λ be a non-zero left cointegral and r a non-zero right integral in H such that $\lambda(r) = 1$. Following [14], we call

$$\underline{g} := \lambda(V^1 r_1 U^1) V^2 r_2 U^2 \tag{6.4}$$

the comodulus of H . It was proved in [14] that \underline{g} is invertible, and that its inverse is given by

$$\underline{g}^{-1} = \lambda(S(V^2 r_2 U^2)) S^2(V^1 r_1 U^1). \tag{6.5}$$

The results in the next two Lemmas also appear in a recent preprint of Kadison [17]. We prefer here to give direct proofs because they provide new formulas, which are of independent interest.

The following result expresses \underline{g} and \underline{g}^{-1} in terms of left integrals.

Lemma 6.1. *Let H be a finite dimensional quasi-Hopf algebra, λ a left cointegral on H and $0 \neq r \in \int_r^H$ such that $\lambda(r) = 1$. If we set $r = S^{-1}(t)$ for a certain left integral t in H , then*

$$\underline{g} = \lambda(S^{-1}(q^2 t_2 p^2)) S^{-1}(q^1 t_1 p^1), \quad (6.6)$$

$$\underline{g}^{-1} = \lambda(q^1 t_1 p^1) S(q^2 t_2 p^2), \quad (6.7)$$

where $p_R = p^1 \otimes p^2$ and $q_R = q^1 \otimes q^2$ are the elements defined by (1.17).

Proof. Let $q_L = \tilde{q}^1 \otimes \tilde{q}^2$ be the element defined by (1.18). We prove first that

$$q^1 t_1 \otimes q^2 t_2 = \tilde{q}^1 t_1 \otimes \tilde{q}^2 t_2, \quad (6.8)$$

for all $t \in \int_l^H$. To this end, we need the following relations

$$q_R = (\tilde{q}^2 \otimes 1) V \Delta(S^{-1}(\tilde{q}^1)), \quad (6.9)$$

$$p_R = \Delta(S(\tilde{p}^1)) U(\tilde{p}^2 \otimes 1), \quad (6.10)$$

$$U^1 \otimes U^2 S(h) = \Delta(S(h_1)) U(h_2 \otimes 1), \quad \forall h \in H, \quad (6.11)$$

which can be found in [14] (here $\tilde{p}^1 \otimes \tilde{p}^2$ is the element p_L defined by (1.18)). Now, $t \in \int_l^H$ and (6.9) imply that

$$q^1 t_1 \otimes q^2 t_2 = V^1 t_1 \otimes V^2 t_2. \quad (6.12)$$

Together with a quasi-Hopf algebra $H = (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ we also have H^{cop} as quasi-Hopf algebra, where cop means opposite comultiplication. The quasi-Hopf algebra structure is obtained by putting $\Phi_{\text{cop}} = (\Phi^{-1})^{321} = x^3 \otimes x^2 \otimes x^1$, $S_{\text{cop}} = S^{-1}$, $\alpha_{\text{cop}} = S^{-1}(\alpha)$ and $\beta_{\text{cop}} = S^{-1}(\beta)$. It is not hard to see that in H^{cop} we have $(q_R)_{\text{cop}} = \tilde{q}^2 \otimes \tilde{q}^1$, $(p_R)_{\text{cop}} = \tilde{p}^2 \otimes \tilde{p}^1$ and $f_{\text{cop}} = (S^{-1} \otimes S^{-1})(f)$, and therefore $V_{\text{cop}} = S(\tilde{p}^1) f^2 \otimes S(\tilde{p}^2) f^1$. Specializing (6.12) for H^{cop} , we obtain

$$\tilde{q}^1 t_1 \otimes \tilde{q}^2 t_2 = S(\tilde{p}^2) f^1 t_1 \otimes S(\tilde{p}^1) f^2 t_2.$$

On the other hand, one can easily check that (1.18, 1.9, 1.16) and $S^{-1}(f^2) \beta f^1 = S^{-1}(\alpha)$ imply

$$S(\tilde{p}^2) f^1 \otimes S(\tilde{p}^1) f^2 = q^1 g_1^1 \otimes S^{-1}(g^2) q^2 q_2^1,$$

where, as usual, we denote $f^{-1} = g^1 \otimes g^2$. From the above, we conclude that

$$\tilde{q}^1 t_1 \otimes \tilde{q}^2 t_2 = q^1 g_1^1 t_1 \otimes S^{-1}(g^2) q^2 g_2^1 t_2,$$

$$\left(t \in \int_l^H \right) = q^1 t_1 \otimes q^2 t_2,$$

as needed. We claim now that

$$U^1 \otimes U^2 = \tilde{q}_1^1 p^1 \otimes \tilde{q}_2^1 p^2 S(\tilde{q}^2). \tag{6.13}$$

Indeed, by (6.10) we have

$$\begin{aligned} \tilde{q}_1^1 p^1 \otimes \tilde{q}_2^1 p^2 S(\tilde{q}^2) &= [\tilde{q}^1 S(\tilde{p}^1)]_1 U^1 \tilde{p}^2 \otimes [\tilde{q}^1 S(\tilde{p}^1)]_2 U^2 S(\tilde{q}^2), \\ (6.11) &= [\tilde{q}^1 S(\tilde{q}_1^2 \tilde{p}^1)]_1 U^1 \tilde{q}_2^2 \tilde{p}^2 \otimes [\tilde{q}^1 S(\tilde{q}_1^2 \tilde{p}^1)]_2 U^2, \\ (1.22) &= U^1 \otimes U^2. \end{aligned}$$

We write $p_R = p^1 \otimes p^2 = P^1 \otimes P^2$, $f = f^1 \otimes f^2 = F^1 \otimes F^2$ and $f^{-1} = g^1 \otimes g^2$. Then the above relations allow us to compute

$$\begin{aligned} V^1 r_1 U^1 \otimes V^2 r_2 U^2 &= V^1 r_1 \tilde{q}_1^1 p^1 \otimes V^2 r_2 \tilde{q}_2^1 p^2 S(\tilde{q}^2), \\ \left(r \in \int_r^H, 6.3 \right) &= S^{-1}(f^2 P^2) S^{-1}(t)_1 p^1 \otimes S^{-1}(f^1 P^1) S^{-1}(t)_2 p^2, \\ (1.11, 1.17) &= S^{-1}(S(x^1) f^2 t_2 P^2) \otimes S^{-1}(S(x^2) f^1 t_1 P^1) \beta S(x^3), \\ (1.9, 1.16) &= S^{-1}(F^2 x^3 g_2^2 t_2 P^2) \otimes S^{-1}(f^2 F_2^1 x^2 g_1^2 t_1 P^1) \beta f^1 F_1^1 x^1, \\ \left(S^{-1}(f^2) \beta f^1 = S^{-1}(\alpha), 1.5, t \in \int_1^H \right) &= S^{-1}(x^3 t_2 P^2) \otimes S^{-1}(S(x^1) \alpha x^2 t_1 P^1), \\ (1.18, 6.8) &= S^{-1}(\tilde{q}^2 t_2 P^2) \otimes S^{-1}(\tilde{q}^1 t_1 P^1) = S^{-1}(q^2 t_2 P^2) \otimes S^{-1}(q^1 t_1 P^1). \end{aligned}$$

Thus, we have proved that

$$V^1 r_1 U^1 \otimes V^2 r_2 U^2 = S^{-1}(q^2 t_2 P^2) \otimes S^{-1}(q^1 t_1 P^1).$$

It follows now that the above equality and (6.4), (6.5) imply (6.6) and (6.7), so our proof is complete. \square

Recall from [3, Remarks 2.6] that the map

$$\bar{\theta} : H^* \rightarrow H, \quad \bar{\theta}(\chi) = \chi(q^2 t_2 p^2) q^1 t_1 p^1 \quad \forall \chi \in H^*,$$

is bijective. Thus there is a unique $\lambda \in H^*$ such that

$$\lambda(q^2 t_2 p^2) q^1 t_1 p^1 = 1. \tag{6.14}$$

Lemma 6.2. *The linear map λ defined above is a non-zero left cointegral on H .*

Proof. The fact that λ is non-zero follows from $\bar{\theta}(\lambda) = 1$. Let λ_0 be a non-zero left cointegral on H . Then

$$\begin{aligned} \bar{\theta}(\lambda_0) &= \lambda_0(q^2 t_2 p^2) q^1 t_1 p^1, \\ (6.9, 6.10) &= \lambda_0(V^2 [S^{-1}(\tilde{q}^1) t S(\tilde{p}^1)]_2 U^2) \tilde{q}^2 V^1 [S^{-1}(\tilde{q}^1) t S(\tilde{p}^1)]_1 U^1 \tilde{p}^2, \end{aligned}$$

$$\left(t \in \int_l^H, 6.1 \right) = \mu^{-1}(\tilde{p}^1)\lambda_0(V^2t_2U^2)V^1t_1U^1\tilde{p}^2,$$

$$(6.2) = \mu(x^1)\mu^{-1}(\tilde{p}^1)\lambda_0(tS(x^2))x^3\tilde{p}^2,$$

$$(6.1, 1.18) = \mu(x^1)\mu(X^1\beta S(X^2))\mu(S(x^2))\lambda_0(t)x^3X^3 = \bar{\theta}(\mu(\beta)\lambda_0(t)\lambda).$$

Since $\bar{\theta}$ is bijective we deduce that $\lambda_0 = \mu(\beta)\lambda_0(t)\lambda$, and since $0 \neq \lambda_0 \in \mathcal{L}$, by the uniqueness of left cointegrals on H we conclude that λ is a non-zero left cointegral on H . \square

We finally need the following result.

Lemma 6.3. *Let H be a finite dimensional quasi-Hopf algebra, t a non-zero left integral in H and μ the distinguished group-like element of H^* . Then for any $h \in H$ the following relations hold:*

$$q^1t_1 \otimes S^{-1}(h)q^2t_2 = hq^1t_1 \otimes q^2t_2, \quad (6.15)$$

$$t_1 \otimes t_2 = \beta q^1t_1 \otimes q^2t_2 = q^1t_1 \otimes S^{-1}(\beta)q^2t_2, \quad (6.16)$$

$$t_1p^1 \otimes t_2p^2S(h \leftarrow \mu) = t_1p^1h \otimes t_2p^2, \quad (6.17)$$

where for all $h \in H$ and $\chi \in H^*$ we define $h \leftarrow \chi = \chi(h_1)h_2$.

Proof. The relations (6.15, 6.16) are proved in [3, Lemma 2.1]. The equality (6.17) follows from the following computation

$$t_1p^1 \otimes t_2p^2S(h \leftarrow \mu) = \mu(h_1)t_1p^1 \otimes t_2p^2S(h_2),$$

$$(6.1) = t_1h_{(1,1)}p^1 \otimes t_2h_{(1,2)}p^2S(h_2),$$

$$(1.19) = t_1p^1h \otimes t_2p^2,$$

for all $h \in H$, and this finishes the proof. \square

Theorem 6.4. *Let (H, R) be a finite dimensional QT quasi-Hopf algebra and μ the distinguished group-like element of H^* . Then the following assertion hold.*

- (1) *If $q_R = q^1 \otimes q^2 = Q^1 \otimes Q^2$ and $p_R = p^1 \otimes p^2 = P^1 \otimes P^2$ are the elements defined by (1.17) then*

$$\mu(Q^1)q^2t_2p^2S(Q^2(R^2P^2 \leftarrow \mu))R^1P^1 \otimes q^1t_1p^1 = S(u)q^1t_1p^1 \otimes q^2t_2p^2, \quad (6.18)$$

where $R = R^1 \otimes R^2$ is the R-matrix of H and u is the element defined in (1.29).

- (2) *If (H, R) is factorizable then H is unimodular.*

Proof. (1) Let us start by noting that $g^1S(g^2\alpha) = \beta$, (1.32, 1.33) and (1.31) imply

$$R^1\beta S(R^2) = S(\beta u). \quad (6.19)$$

Now, from (6.17) we have

$$\begin{aligned}
 & \mu(Q^1)q^2t_2p^2S(Q^2(R^2P^2 \leftarrow \mu))R^1P^1 \otimes q^1t_1p^1 \\
 & \quad = \mu(Q^1)q^2t_2p^2S(Q^2)R^1P^1 \otimes q^1t_1p^1R^2P^2, \\
 (6.1) & \quad = q^2t_2Q_2^1p^2S(Q^2)R^1P^1 \otimes q^1t_1Q_1^1p^1R^2P^2, \\
 (1.21) & \quad = q^2t_2R^1P^1 \otimes q^1t_1R^2P^2, \\
 (1.27) & \quad = q^2R^1t_1P^1 \otimes q^1R^2t_2P^2, \\
 (6.16) & \quad = q^2R^1\beta Q^1t_1P^1 \otimes q^1R^2Q^2t_2P^2, \\
 (6.15) & \quad = q^2R^1\beta S(q^1R^2)Q^1t_1P^1 \otimes Q^2t_2P^2, \\
 (6.19) & \quad = S(q^1\beta uS^{-1}(q^2))Q^1t_1P^1 \otimes Q^2t_2P^2, \\
 (1.31, 1.17, 1.16) & \quad = S(u)Q^1t_1P^1 \otimes Q^2t_2P^2,
 \end{aligned}$$

and this proves the first assertion.

(2) Let $\lambda \in H^*$ be the element defined by (6.14). By Lemma 6.2 we know that λ is a non-zero left cointegral on H . Consider now r a non-zero right integral in H such that $\lambda(r) = 1$, and take $r = S^{-1}(t)$ for some non-zero left integral t in H . Then, by Lemma 6.1 we have

$$S^{-1}(g^{-1}) = \lambda(q^1t_1p^1)q^2t_2p^2.$$

Applying $id \otimes \lambda$ to equality (6.18) we obtain

$$\mu(Q^1)S^{-1}(g^{-1})S(Q^2(R^2P^2 \leftarrow \mu))R^1P^1 = S(u),$$

and since $S^{-1}(g)S(u) = S(uS^{-2}(g)) = S(gu)$, it follows that the above relation is equivalent to

$$\mu(Q^1)S(Q^2(R^2P^2 \leftarrow \mu))R^1P^1 = S(u)S(g). \tag{6.20}$$

On the other hand, if we denote by $r^1 \otimes r^2$ another copy of R , we then have

$$\mu(Q^1)S(Q^2(R^2P^2 \leftarrow \mu))R^1P^1$$

$$(1.17) = \mu(X^1R_1^2P_1^2)S(X^2R_2^2P_2^2)\alpha X^3R^1P^1,$$

$$(1.26) = \mu(X^1R^2y^2P_1^2)S(r^2X^3y^3P_2^2)\alpha r^1X^2R^1y^1P^1,$$

$$(1.33, 1.31, 1.17) = \mu(q^1R^2y^2P_1^2)S(S(q^2)y^3P_2^2)uR^1y^1P^1,$$

$$(1.23, 1.27) = \mu(q^1X_{(1,1)}^1p_1^1R^2P^2S(X^3)f^1)S(S(q^2)X_2^1p^2S(X^2)f^2)uX_{(1,2)}^1p_2^1R^1P^1,$$

$$(1.31, 1.19) = \mu(X^1q^1p_1^1R^2P^2S(X^3)f^1)S(S(q^2p_2^1)p^2S(X^2)f^2)uR^1P^1,$$

$$(1.21, 1.31) = \mu(X^1R^2P^2S(X^3)f^1)uS^{-1}(S(X^2)f^2)R^1P^1.$$

From the above computation and (6.20) we obtain

$$\mu(X^1 R^2 P^2 S(X^3) f^1) S^{-1}(S(X^2) f^2) R^1 P^1 = u^{-1} S(u) S(\underline{g}). \quad (6.21)$$

But, as we have already seen, if (H, R) is QT then $\tilde{R} = R_{21}^{-1} = \bar{R}^2 \otimes \bar{R}^1$ is another R -matrix for H . Repeating the above computations for (H, \tilde{R}) instead of (H, R) , we find that

$$\mu(X^1 \tilde{r}^1 P^2 S(X^3) f^1) S^{-1}(S(X^2) f^2) \tilde{r}^2 P^1 = \tilde{u}^{-1} S(\tilde{u}) S(\underline{g}), \quad (6.22)$$

where we denote by \tilde{u} the element defined as in (1.29) for (H, \tilde{R}) instead of (H, R) , and where $\tilde{r}^1 \otimes \tilde{r}^2$ is another copy of R^{-1} . More precisely, we have that

$$\tilde{u} = S(u^{-1}). \quad (6.23)$$

Indeed, one can easily see that (6.19) and (1.31) imply

$$\tilde{r}^2 \beta S(\tilde{r}^1) = S^{-1}(\beta) u^{-1} = u^{-1} S(\beta). \quad (6.24)$$

Now, we compute

$$\begin{aligned} \tilde{u} &= S(\tilde{r}^1 x^2 \beta S(x^3)) \alpha \tilde{r}^2 x^1, \\ (S(\beta f^1) f^2 = \alpha) &= S(\beta f^1 \tilde{r}^1 x^2 \beta S(x^3)) f^2 \tilde{r}^2 x^1, \\ (1.32, 1.17) &= S(\tilde{r}^2 \beta S(\tilde{r}^1) f^2 p^2) f^1 p^1, \\ (6.24) &= S(S^{-1}(f^1 p^1) u^{-1} S(\beta) f^2 p^2), \\ (1.31, S(\beta f^1) f^2 = \alpha, 1.17, 1.6) &= S(u^{-1} S(p^1) \alpha p^2) = S(u^{-1}). \end{aligned}$$

Now, since $S^2(u) = u$ the relation (6.22) becomes

$$\mu(X^1 \tilde{r}^1 P^2 S(X^3) f^1) S^{-1}(S(X^2) f^2) \tilde{r}^2 P^1 = S(u) u^{-1} S(\underline{g}). \quad (6.25)$$

From (1.31) it follows that $u S^{-1}(u) = S(u) u$, and since $S^2(u) = u$ we conclude that $u S(u) = S(u) u$, so $u^{-1} S(u) = S(u) u^{-1}$. Hence, by (6.21) and (6.25) we obtain

$$\mu(X^1 R^2 P^2 S(X^3) f^1) S^{-1}(S(X^2) f^2) R^1 P^1 = \mu(X^1 \tilde{r}^1 P^2 S(X^3) f^1) S^{-1}(S(X^2) f^2) \tilde{r}^2 P^1.$$

This comes out explicitly as

$$\mu(R^2 P^2) R^1 P^1 = \mu(\tilde{r}^1 P^2) \tilde{r}^2 P^1$$

and implies

$$\mu(Q_1^1 R^2 P^2 S(Q^2)) Q_2^1 R^1 P^1 = \mu(Q_1^1 \tilde{r}^1 P^2 S(Q^2)) Q_2^1 \tilde{r}^2 P^1.$$

From (1.27) and (1.21) we deduce that

$$\mu(R^2) R^1 = \mu(\tilde{r}^1) \tilde{r}^2 \Leftrightarrow \mu(R^2 r^1) R^1 r^2 = 1. \quad (6.26)$$

Finally, the above relation allows us to compute

$$\begin{aligned} \mathcal{Q}(\mu) &= \mu(\tilde{q}^1 X^1 R^2 r^1 p^1) \tilde{q}_1^2 X^2 R^1 r^2 p^2 S(\tilde{q}_2^2 X^3), \\ (6.26, 1.17) &= \mu(\tilde{q}^1 X^1 x^1) \tilde{q}_1^2 X^2 x^2 \beta S(\tilde{q}_2^2 X^3 x^3), \\ (1.5, 1.18) &= \mu(\alpha) \beta = \mathcal{Q}(\mu(\alpha) \varepsilon). \end{aligned}$$

If (H, R) is factorizable then \mathcal{Q} is bijective, so $\mu = \mu(\alpha)\varepsilon$. In particular, $1 = \mu(1) = \mu(\alpha)\varepsilon(1) = \mu(\alpha)$. Hence $\mu = \varepsilon$, and this means that H is unimodular. \square

Theorem 6.5. *Let H be a finite dimensional quasi-Hopf algebra. Then the Drinfeld double $D(H)$ of H is a unimodular quasi-Hopf algebra.*

Proof. It is an immediate consequence of Proposition 2.3 and Theorem 6.4 \square

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