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# Collineations and dualities of partial geometries

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ABSTRACT

In this paper, we first prove some general results on the number of fixed points of collineations of finite partial geometries, and on the number of absolute points of dualities of partial geometries. In the second part of the paper, we establish the number of isomorphism classes of partial geometries arising from a Thas maximal arc constructed from a (finite) Suzuki–Tits ovoid in a classical projective plane. We also determine the full automorphism group of these structures, and show that every partial geometry arising from any Thas maximal arc is self-dual.



## 1. Introduction

Partial geometries were introduced by Bose [3] in 1963 as a geometric approach to many strongly regular graphs. Although a number of classes and sporadic examples of (finite) partial geometries are known, they do not seem to exist in great numbers. In order to understand the structure of partial geometries better, it seems reasonable to try to understand how collineations act on them. In particular, a general but pertinent question is: what can we say about the fixed points and fixed lines of an arbitrary collineation? And can one say something about the number of absolute points and lines of a duality of a partial geometry? This paper intends to answer these questions.

The formulae we find for self-dual partial geometries lead us to take a closer look at the examples of self-dual partial geometries. There are very few of these. The most prominent examples are the partial geometries arising from a Thas maximal arc of a Desarguesian projective plane constructed with a Suzuki–Tits ovoid. We show (1) that these examples are really self-dual; in fact we show that this holds when considering any ovoid of $\mathsf{PG}(3, q)$, with $q$ even. Our methods then allow us to (2) determine the full collineation groups of these geometries. As an application we show (3) that, for each Suzuki–Tits ovoid, there are exactly two isomorphism classes of Thas maximal arcs in the classical plane, and consequently also two isomorphism classes of corresponding partial geometries. Questions (2) and (3) were also answered by Hamilton and Penttila [7], tacitly assuming that, with the notation of Section 5.3, the vertex $x$ of the cone defining the maximal arcs in question is fixed under every collineation stabilizing the maximal arc. We include a complete proof. Question (1) was, as far as we know, never treated before and has been open since 1974, when Thas introduced these geometries.

Theorem 3 was also proved in [6], but we repeat the proof here, as it gives us the opportunity to introduce the technique we will use for the dualities.

## 2. Generalities

A *(finite) partial geometry* is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$, with an incidence relation satisfying the following axioms

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1. each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line;
2. each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point;
3. if $x$ is a point and $L$ is a line not incident with $x$, then there are exactly $\alpha(\alpha \geq 1)$ points $x_1, x_2, \ldots, x_\alpha$ and $\alpha$ lines $L_1, L_2, \ldots, L_\alpha$ such that $x \mathrm{I} L_i \mathrm{I} x_i \mathrm{I} L$, $i \in \{1, 2, \ldots, \alpha\}$.

We will say that such a partial geometry is of *order* $(s, t, \alpha)$. If $|\mathcal{P}| = v$ and $|\mathcal{L}| = b$, then $v = \frac{(s+1)(st+\alpha)}{\alpha}$ and $b = \frac{(t+1)(st+\alpha)}{\alpha}$.

In a finite projective plane of order $q$, any non-void set of $l$ points may be described as an $\{l; n\}$-arc, where $n \neq 0$ is the largest number of collinear points in the set. For given $q$ and $n$, $n \neq 0$, $l$ can never exceed $(n-1)(q+1)+1$, and an arc with that number of points will be called a *maximal arc* (cf. [2]). It is easily seen that a maximal arc meets every line in either 0 or $n$ points.

A *near polygon* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with the following property: if $x$ is a point and $L$ is a line not incident with $x$, then there exists a unique point $y$ incident with $L$ for which $\mathrm{dist}(x, y)$ is minimal. If the maximal distance between two elements is $n$, then the near polygon is also called a *near $n$-gon*.

We will say that a near polygon is of order $(s, t)$, if there are $s+1$ points on every line and $t+1$ lines through every point.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial geometry of order $(t, t, \alpha)$. Then we define the *double* of it as the following geometry: the point set is $\mathcal{P} \cup \mathcal{L}$, the line set is the set of *flags* of $\mathcal{S}$, where a flag is an incident point-line pair, and incidence is the natural one.

In this way each partial geometry of order $(t, t, \alpha)$ gives rise to a unique near octagon, that is, a near 8-gon, of order $(1, t)$, for which the following property holds: for every two points $x$ and $y$ which lie at distance 6 from each other, there exist precisely $\alpha$ paths of length 6 from $x$ to $y$, and for every two points $x'$ and $y'$ which lie at distance 4 from each other there exists precisely 1 shortest path from $x'$ to $y'$. We will say that such a near octagon is of *order* $(1, t; \alpha, 1)$. Conversely, each near octagon of order $(1, t; \alpha, 1)$ arises from a partial geometry of order $(t, t, \alpha)$.

We will need the following lemmas in Section 3. The proofs can be found in [11].

Let $A$ be an adjacency matrix of the point graph of a partial geometry $\mathcal{S}$ of order $(s, t, \alpha)$ with $v$ points, let $M = A+(t+1)I$, let $\theta$ be an automorphism of $\mathcal{S}$ of order $n$ and let $Q = (q_{ij})$ be the $v \times v$ matrix with $q_{ij} = 1$ if $x_i^\theta = x_j$ and $q_{ij} = 0$ otherwise.

**Lemma 1.** *Suppose that $\xi$ and $\xi'$ are both primitive dth roots of unity, with $d$ a divisor of $n$, and let $\lambda$ be an integer eigenvalue of $M$. If at least one of $\xi\lambda$ and $\xi'\lambda$ is an eigenvalue of $QM$, than they both are and they have the same multiplicity.*

**Lemma 2.** *Let $\xi$ be an nth root of unity and $\lambda$ an eigenvalue of $M$ such that $-\lambda$ is not an eigenvalue. Then the multiplicity of $\xi\lambda$ as an eigenvalue of $QM$ is equal to the multiplicity of $\xi\lambda^j$ as an eigenvalue of $QM^j$, with $j = 2, 3, \ldots$.*

## 3. A Benson-type theorem for partial geometries

We now introduce some further notation. Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a partial geometry of order $(s, t, \alpha)$. It is convenient to use the notion of collinearity only for distinct points. Let $D$ be an incidence matrix of $\mathcal{S}$. Then $M := DD^T = A+(t+1)I$, where $A$ is an adjacency matrix of the point graph of $\mathcal{S}$. Let $\theta$ be an automorphism of $\mathcal{S}$ of order $n$ and let $Q = (q_{ij})$ be the $v \times v$ matrix with $q_{ij} = 1$ if $x_i^\theta = x_j$ and $q_{ij} = 0$ otherwise; so $Q$ is a permutation matrix. Because $M = A + (t+1)I$, the eigenvalues of $M$ are as follows (cf. [3]):

| Eigenvalues of $M$ | Multiplicity |
|---|---|
| $0$ | $m_0 = \frac{s(s+1-\alpha)(st+\alpha)}{\alpha(s+t+1-\alpha)}$ |
| $(s+1)(t+1)$ | $m_1 = 1$ |
| $s+t+1-\alpha$ | $m_2 = \frac{(s+1)(t+1)st}{\alpha(s+t+1-\alpha)}$ |

**Theorem 3.** *Let $\mathcal{S}$ be a partial geometry of order $(s, t, \alpha)$ and let $\theta$ be an automorphism of $\mathcal{S}$. If $f_0$ is the number of points fixed by $\theta$ and if $f_1$ is the number of points $x$ for which $x^\theta \neq x \sim x^\theta$, then for some integer $k$*

$$\mathrm{tr}(QM) = k(s+t+1-\alpha) + (1+s)(1+t) = (t+1)f_0 + f_1.$$

**Proof.** Suppose that $\theta$ has order $n$, so that $(QM)^n = Q^n M^n = M^n$. It follows that the eigenvalues of $QM$ are the eigenvalues of $M$ multiplied by the appropriate roots of unity. Let $J$ be the $v \times v$ matrix with all entries equal to 1. Since $MJ = (1+s)(1+t)J$, we have $(QM)J = (1+s)(1+t)J$, so $(1+s)(1+t)$ is an eigenvalue of $QM$. Because $m_1 = 1$, it follows that this eigenvalue of $QM$ has multiplicity 1. Further it is clear that 0 is an eigenvalue of $QM$ with multiplicity $m_0$. For each divisor $d$ of $n$, let $\xi_d$ denote a primitive $d$th root of unity, and put $U_d = \sum \xi_d^i$, where the summation is over those integers $i \in \{1, 2, \ldots, d-1\}$ that are relatively prime to $d$. Then $U_d$ is an integer by [9]. For each divisor $d$ of $n$, the primitive $d$th roots of unity all contribute the same number of times to the eigenvalues $\varphi$ of $QM$ with $|\varphi| = s+t+1-\alpha$, because of Lemma 1. Let $a_d$ denote the multiplicity of $\xi_d(s+t+1-\alpha)$ as an eigenvalue of $QM$, with $d|n$, and $\xi_d$ a primitive $d$th root of unity. Then

$$\mathrm{tr}(QM) = \sum_{d|n} a_d(s+t+1-\alpha)U_d + (1+s)(1+t),$$

or

$$\mathrm{tr}(QM) = k(s + t + 1 - \alpha) + (1 + s)(1 + t),$$

with $k$ an integer.

Since the entry on the $i$th row and $i$th column of $QM$ is the number of lines incident with $x_i$ and $x_i^\theta$, we have $\mathrm{tr}(QM) = (1 + t)f_0 + f_1$. Hence

$$k(s + t + 1 - \alpha) + (1 + s)(1 + t) = (1 + t)f_0 + f_1,$$

with $k$ an integer. □

**Corollary 4.** *Let $\mathcal{S}$ be a partial geometry of order $(s, t, \alpha)$ and let $\theta$ be an automorphism of $\mathcal{S}$. If $s$, $t$ and $\alpha - 1$ have a common divisor distinct from 1, then there exists at least one fixed point or at least one point which is mapped to a point collinear to itself.*

**Proof.** Suppose that there are no fixed points and no points which are mapped to a collinear point, hence $f_0 = f_1 = 0$. Because of the previous theorem, $k(s+t+1-\alpha)+(1+s)(1+t)$ has to be equal to 0. Hence $k(s+t+1-\alpha)+s+t+st = -1$. But because $s$, $t$ and $\alpha - 1$ have a common divisor distinct from 1, there exists an integer $m$ which divides $s$, $t$ and $\alpha - 1$. Hence $m$ divides $k(s + t + 1 - \alpha) + s + t + st$, but $m$ does not divide $-1$ and we have a contradiction. □

**Corollary 5.** *Let $\mathcal{S}$ be a partial geometry of order $(s, t, \alpha)$ and let $\theta$ be an involution of $\mathcal{S}$. If $s$, $t$ and $\alpha - 1$ have a common divisor distinct from 1, then there exists at least one fixed point or at least one fixed line.*

**Proof.** This follows immediately from the previous corollary because if there is a point $x$ which is mapped to a point collinear to $x$ by the involution $\theta$, then the line $xx^\theta$ is a fixed line. □

We now have a look at the double of a partial geometry of order $(t, t, \alpha)$, which is a near octagon of order $(1, t; \alpha, 1)$.

If the matrix $M$ of this near octagon is defined as before, then it has the following eigenvalues (cf. [4]):

| Eigenvalues of $M$ | Multiplicity |
|---|---|
| $0$ | $m_0 = 1$ |
| $2t + 2$ | $m_1 = 1$ |
| $1 + t$ | $m_2 = \frac{2(2-\alpha)(t+\alpha)}{\alpha(t+2-\alpha)}$ |
| $t + 1 + \sqrt{2t + 1 - \alpha}$ | $m_3 = \frac{2(t+1)t}{\alpha(t+2-\alpha)}$ |
| $t + 1 - \sqrt{2t + 1 - \alpha}$ | $m_4 = \frac{2(t+1)t}{\alpha(t+2-\alpha)}$ |

Since these eigenvalues are not necessarily integers, we must first establish a lemma similar to Lemma 1 for eigenvalues of the form $a+b\sqrt{c}$, where $c$ is a square-free natural number, and $a$ and $b$ are integers. Using the action of the Galois group of the extension $[\mathbb{Q}(\sqrt{c}, \zeta_d) : \mathbb{Q}(\sqrt{c})]$, where $\zeta_d$ is a primitive $d$th root of unity, and $\sqrt{c}$ does not belong to the $d$th cyclotomic extension of $\mathbb{Q}$, the following is obvious.

**Lemma 6.** *Suppose that $\xi$ and $\xi'$ are both primitive dth roots of unity, with $d$ a divisor of $n$, and let $a + b\sqrt{c}$ be a non-integer eigenvalue of $M$, with $a, b, c$ as above. If at least one of $\xi\lambda$ and $\xi'\lambda$ is an eigenvalue of $QM$, and if $c$ is not a square in $\mathbb{Q}(\zeta_d)$, then they both are eigenvalues of $QM$ and they have the same multiplicity.*

It is an elementary exercise to calculate when precisely a square-free natural number $c$ is not a square in the $d$th cyclotomic extension of $\mathbb{Q}$. This has been done explicitly in [10]. This happens exactly when either (1) $c$ does not divide $d$, or (2) $c$ divides $d$, $c$ is even and $d$ is not a multiple of 8, or (3) $c$ divides $d$, $c \equiv 3 \bmod 4$, and $d$ is not a multiple of 4. If one of these conditions holds, then we say that $\lambda = a + b\sqrt{c}$ is cyclotomically independent of $d$. We extend this definition to integers by defining every integer to be cyclotomically independent of $d$. Note that, if $\lambda$ is cyclotomically independent of $d$, then it is also cyclotomically independent of any integer divisor of $d$.

**Theorem 7.** *Let $\mathcal{S}$ be a near octagon of order $(1, t; \alpha, 1)$ and let $\theta$ be an automorphism of $\mathcal{S}$ of order $n$. Suppose all eigenvalues of $M$ are cyclotomically independent of $n$. If $f_0$ is the number of points fixed by $\theta$ and $f_1$ is the number of points $x$ for which $x^\theta \neq x \sim x^\theta$, then for some integers $k_1$, $k_2$ and $k_3$*

$$k_1(1 + t) + k_2(1 + t + \sqrt{2t + 1 - \alpha}) + k_3(1 + t - \sqrt{2t + 1 - \alpha}) + 2(1 + t) = (1 + t)f_0 + f_1.$$

**Proof.** Clearly $(QM)^n = Q^nM^n = M^n$. It follows that the eigenvalues of $QM$ are the eigenvalues of $M$ multiplied by the appropriate roots of unity. Let $J$ be the $v\times v$ matrix with all entries equal to 1. Since $MJ = 2(1+t)J$, we have $(QM)J = 2(1+t)J$, so $2(1 + t)$ is an eigenvalue of $QM$. Because $m_1 = 1$, it follows that this eigenvalue of $QM$ has multiplicity 1. Further it is clear that 0 is an eigenvalue of $QM$ with multiplicity $m_0 = 1$. For each divisor $d$ of $n$, let $\xi_d$ denote a primitive $d$th root of unity, and put $U_d = \sum \xi_d^i$, where the summation is over those integers $i \in \{1, 2, \ldots, d - 1\}$ that are relatively prime to $d$.

Then $U_d$ is an integer by [9]. For each divisor $d$ of $n$, the primitive $d$th roots of unity all contribute the same number of times to the eigenvalues $\varphi$ of $QM$ with $|\varphi| = 1+t+\sqrt{2t+1-\alpha}$ and also the primitive $d$th roots of unity all contribute the same number of times to the eigenvalues $\varphi'$ of $QM$ with $|\varphi'| = 1+t-\sqrt{2t+1-\alpha}$, because of Lemma 6 and our assumptions. Let $a_d$ denote the multiplicity of $\xi_d(1+t+\sqrt{2t+1-\alpha})$ and let $b_d$ denote the multiplicity of $\xi_d(1+t-\sqrt{2t+1-\alpha})$ as eigenvalues of $QM$, with $d|n$ and $\xi_d$ a primitive $d$th root of unity. Then

$$\operatorname{tr}(QM) = \sum_{d|n} a_d(1+t+\sqrt{2t+1-\alpha})U_d + \sum_{d|n} b_d(1+t-\sqrt{2t+1-\alpha})U_d + 2(1+t),$$

or

$$\operatorname{tr}(QM) = k_1(1+t+\sqrt{2t+1-\alpha}) + k_2(1+t-\sqrt{2t+1-\alpha}) + 2(1+t),$$

with $k_1$ and $k_2$ integers. Since the entry on the $i$th row and $i$th column of $QM$ is the number of lines incident with $x_i$ and $x_i^\theta$, we have $\operatorname{tr}(QM) = (1+t)f_0 + f_1$. Hence

$$k_1(1+t+\sqrt{2t+1-\alpha}) + k_2(1+t-\sqrt{2t+1-\alpha}) + 2(1+t) = (1+t)f_0 + f_1,$$

with $k_1$ and $k_2$ integers. □

**Theorem 8.** *Let $\mathcal{S}$ be a near octagon of order $(1, t; \alpha, 1)$ and let $\theta$ be an automorphism of $\mathcal{S}$ of order $n$. Suppose all eigenvalues of $M$ are cyclotomically independent of $n$. If $f_0$ is the number of points fixed by $\theta$, $f_1$ is the number of points $x$ for which $x^\theta \neq x \sim x^\theta$ and $f_2$ is the number of points for which $\operatorname{dist}(x, x^\theta) = 4$, then for some integers $k_1$, $k_2$ and $k_3$*

$$\begin{aligned} &k_1(1+t)^2 + k_2(1+t+\sqrt{2t+1-\alpha})^2 + k_3(1+t-\sqrt{2t+1-\alpha})^2 + (2(1+t))^2 \\ &\quad = (2+t)(1+t)f_0 + (2+2t)f_1 + f_2. \end{aligned}$$

**Proof.** Suppose that $M$, $A$ and $Q$ are defined as before. Now we consider $M^2$ and we have $(QM^2)^n = Q^nM^{2n} = M^{2n}$. It follows that the eigenvalues of $QM^2$ are the eigenvalues of $M^2$ multiplied by the appropriate roots of unity. Since $M^2J = (2(1+t))^2J$, we have $(QM^2)J = (2(1+t))^2J$, so $(2(1+t))^2$ is an eigenvalue of $QM^2$. By Lemma 2 $m_1 = 1$ and it follows that this eigenvalue of $QM^2$ has multiplicity 1. Further it is clear that 0 is an eigenvalue of $QM^2$ with multiplicity $m_0$. For each divisor $d$ of $n$, let $\xi_d$ again denote a primitive $d$th root of unity, and put $U_d = \sum \xi_d^i$, where the summation is over those integers $i \in \{1, 2, \ldots, d-1\}$ that are relatively prime to $d$. Then $U_d$ is an integer [9]. For each divisor $d$ of $n$, the primitive $d$th roots of unity all contribute the same number of times to the eigenvalues $\varphi$, respectively $\varphi'$ and $\varphi''$, of $QM^2$ with $|\varphi| = (t+1+\sqrt{2t+1-\alpha})^2$, respectively $|\varphi'| = (t+1-\sqrt{2t+1-\alpha})^2$ and $|\varphi''| = (1+t)^2$, because of Lemma 6 and our assumptions. Let $a_d$ denote the multiplicity of $\xi_d(1+t+\sqrt{2t+1-\alpha})^2$, let $b_d$ denote the multiplicity of $\xi_d(1+t-\sqrt{2t+1-\alpha})^2$ and let $c_d$ denote the multiplicity of $\xi_d(1+t)^2$ as eigenvalues of $QM^2$, with $d|n$ and $\xi_d$ a primitive $d$th root of unity. Then

$$\operatorname{tr}(QM^2) = \sum_{d|n} a_d(1+t+\sqrt{2t+1-\alpha})^2U_d + \sum_{d|n} b_d(1+t-\sqrt{2t+1-\alpha})^2U_d + \sum_{d|n} c_d(1+t)^2U_d + (2(1+t))^2,$$

or

$$\operatorname{tr}(QM^2) = k_1(1+t)^2 + k_2(1+t+\sqrt{2t+1-\alpha})^2 + k_3(1+t-\sqrt{2t+1-\alpha})^2 + (2(1+t))^2,$$

with $k_1$, $k_2$ and $k_3$ integers. On the other hand, we have

$$\begin{aligned} M = A + (1+t)I &\Rightarrow QM = QA + (1+t)Q \\ &\Rightarrow \operatorname{tr}(QM) = \operatorname{tr}(QA) + (1+t)\operatorname{tr}(Q) \\ &\Rightarrow (1+t)f_0 + f_1 = \operatorname{tr}(QA) + (1+t)f_0 \\ &\Rightarrow \operatorname{tr}(QA) = f_1. \end{aligned}$$

The matrix $A^2 = (a_{ij})$ is the matrix with $(1+t)$ along the main diagonal and on the other entries we have $a_{ij} = 1$ if $\operatorname{dist}(x_i, x_j) = 4$ and $a_{ij} = 0$ otherwise. Hence $\operatorname{tr}(QA^2) = (1+t)f_0 + f_2$. It follows that

$$\begin{aligned} \operatorname{tr}(QM^2) &= \operatorname{tr}(Q(A+(1+t)I)^2) \\ &= \operatorname{tr}(QA^2) + 2(1+t)\operatorname{tr}(QA) + (1+t)^2\operatorname{tr}(Q) \\ &= (1+t)f_0 + f_2 + 2(1+t)f_1 + (1+t)^2f_0 \\ &= (2+t)(1+t)f_0 + 2(1+t)f_1 + f_2. \quad \square \end{aligned}$$

**Theorem 9.** *Let $\mathcal{S}$ be a near octagon of order $(1, t; \alpha, 1)$ and let $\theta$ be a nontrivial automorphism of $\mathcal{S}$ of order $n$. Suppose that all eigenvalues of $M$ are cyclotomically independent of $n$. If $f_0$ is the number of points fixed by $\theta$, $f_1$ is the number of points $x$ for* *which* $x^\theta \neq x \sim x^\theta$, $f_2$ *is the number of points for which* $\mathrm{dist}(x, x^\theta) = 4$ *and* $f_3$ *is the number of points for which* $\mathrm{dist}(x, x^\theta) = 6$, *then for some integers* $k_1$, $k_2$ *and* $k_3$ *holds*

$$\begin{aligned}&k_1(1+t)^3 + k_2(1+t+\sqrt{2t+1-\alpha})^3 + k_3(1+t-\sqrt{2t+1-\alpha})^3 + ((1+s)(1+t))^3\\ &\quad = (3(1+t)^2 + (1+t)^3)f_0 + (1+2t+3(1+t)^2)f_1 + 3(1+t)f_2 + \alpha f_3.\end{aligned}$$

**Proof.** Suppose that $M$, $A$ and $Q$ are defined as before. In the same way as in the proof of Theorems 7 and 8 we can prove that $\mathrm{tr}(QM^3) = k_1(1+t)^3 + k_2(1+t+\sqrt{2t+1-\alpha})^3 + k_3(1+t-\sqrt{2t+1-\alpha})^3 + (2(1+t))^3$, with $k_1$, $k_2$ and $k_3$ integers. On the other hand, we can calculate that $A^3 = (a_{ij})$ is the matrix with 0 along the main diagonal while on the other entries we have $a_{ij} = 1+2t$ if $x_i \sim x_j$, $a_{ij} = \alpha$ if $\mathrm{dist}(x_i, x_j) = 6$ and $a_{ij} = 0$ otherwise. Hence $\mathrm{tr}(QA^3) = (1+2t)f_1 + \alpha f_3$. Because of the proof of Theorem 8 we know that $\mathrm{tr}(QA^2) = (1+t)f_0 + f_2$, $\mathrm{tr}(QA) = f_1$ and $\mathrm{tr}(Q) = f_0$. Hence

$$\begin{aligned}\mathrm{tr}(QM^3) &= \mathrm{tr}(Q(A+(1+t)I)^3)\\ &= \mathrm{tr}(QA^3) + 3(1+t)\mathrm{tr}(QA^2) + 3(1+t)^2\mathrm{tr}(QA) + (1+t)^3\mathrm{tr}(Q)\\ &= (1+2t)f_1 + \alpha f_3 + 3(1+t)((1+t)f_0 + f_2) + 3(1+t)^2 f_1 + (1+t)^3 f_0\\ &= (3(1+t)^2 + (1+t)^3)f_0 + ((1+2t) + 3(1+t)^2)f_1 + 3(1+t)f_2 + \alpha f_3. \quad \square\end{aligned}$$

Note that the integers $k_1$, $k_2$ and $k_3$ in Theorems 7–9 are the same by Lemma 2.

Suppose that we have a duality in the underlying partial geometry, then we know that $f_0 = 0$ and $f_2 = 0$. Because of Theorems 7–9, we have the following equations, under the assumption that all eigenvalues of $M$ are cyclotomically independent of the order of the duality.

$$\begin{aligned}&k_1(1+t) + k_2(1+t+\sqrt{2t+1-\alpha}) + k_3(1+t-\sqrt{2t+1-\alpha}) + 2(1+t) = f_1,\\ &k_1(1+t)^2 + k_2(1+t+\sqrt{2t+1-\alpha})^2 + k_3(1+t-\sqrt{2t+1-\alpha})^2 + (2(1+t))^2 = (2+2t)f_1,\\ &k_1(1+t)^3 + k_2(1+t+\sqrt{2t+1-\alpha})^3 + k_3(1+t-\sqrt{2t+1-\alpha})^3 + (2(1+t))^3\\ &\quad = (1+2t+3(1+t)^2)f_1 + \alpha f_3.\end{aligned}$$

Because $f_0$ and $f_2$ are 0, we know that $f_1 + f_3 = \frac{2(t+1)(\alpha+t^2)}{\alpha}$. Hence

$$\begin{aligned}k_1 &= 0,\\ k_2 &= \frac{-2(t+1)+f_1}{2\sqrt{2t+1-\alpha}},\\ k_3 &= -\frac{-2(t+1)+f_1}{2\sqrt{2t+1-\alpha}},\\ f_3 &= \frac{2(t+1)(\alpha+t^2)}{\alpha} - f_1.\end{aligned}$$

So $\frac{-2(t+1)+f_1}{2\sqrt{2t+1-\alpha}}$ has to be an integer. In the case that $2t+1-\alpha$ is not a square, this only holds if $f_1 = 2(t+1)$. Suppose that $2t+1-\alpha$ is a square, then $f_1 - 2(t+1)$ should be a multiple of $2\sqrt{2t+1-\alpha}$. If $\alpha$ is odd, then $f_1 \equiv 1+\alpha \mod 2\sqrt{2t+1-\alpha}$. If $\alpha$ is even, then $f_1 \equiv 1+\alpha+\sqrt{2t+1-\alpha} \mod 2\sqrt{2t+1-\alpha}$.

**Corollary 10.** *If* $\theta$ *is a duality of a partial geometry of order* $(t, t, \alpha)$, *with* $2t+1-\alpha$ *not a square, but such that it is cyclotomically independent of the order of* $\theta$, *then it has* $1+t$ *absolute points and* $1+t$ *absolute lines, and there are* $(1+t)t^2/\alpha$ *points which are mapped to a line at distance* 3 *and* $(1+t)t^2/\alpha$ *lines which are mapped to a point at distance* 3.

**Corollary 11.** *Suppose that* $\theta$ *is a duality of a partial geometry of order* $(t, t, \alpha)$, *with* $2t+1-\alpha$ *a square. If* $\alpha$ *is odd, then it has* $(1+\alpha)/2 \mod \sqrt{2t+1-\alpha}$ *absolute points and equally many absolute lines. If* $\alpha$ *is even, then it has* $(1+\alpha+\sqrt{2t+1-\alpha})/2 \mod \sqrt{2t+1-\alpha}$ *absolute points and equally many absolute lines.*

## 4. Partial geometries which arise from maximal arcs

We are able to construct a partial geometry from a maximal arc (cf. [13]). Suppose that we have a maximal $\{qn-q+n, n\}$-arc $K$, $1 < n < q$, of a projective plane $\pi$ of order $q$. Define the points of the partial geometry $\mathcal{S}$ as the points of $\pi$ which are not contained in $K$. The lines of $\mathcal{S}$ are the lines of $\pi$ which are incident with $n$ points of $K$ and the incidence is the incidence of $\pi$. This gives us a partial geometry of order $(q-n, q-q/n, q-q/n-n+1)$.

Consider an ovoid $\mathcal{O}$ and a 1-spread $\mathcal{R}$ of $\mathrm{PG}(3, 2^m)$, $m > 0$, such that each line of $\mathcal{R}$ has one and only one point in common with $\mathcal{O}$. Let $\mathrm{PG}(3, 2^m)$ be embedded as a hyperplane $H$ in $\mathrm{PG}(4, 2^m) = P$, and let $x$ be a point of $P \setminus H$. Call $C$ the set of the points of $P \setminus H$ which are on a line $xy$, with $y \in \mathcal{O}$. Then the point set $C$ is a maximal $\{2^{3m} - 2^{2m} + 2^m, 2^m\}$-arc of the projective plane $\pi$ defined by the 1-spread $\mathcal{R}$ (cf. [13]). We will call such maximal arcs *Thas maximal arcs*. As described above, we can construct a partial geometry $\mathsf{pg}(C)$ from this arc $C$ having order $(2^{2m} - 2^m, 2^{2m} - 2^m, 2^{2m} - 2^{m+1} + 1)$.

An interesting example of this situation occurs when the spread is a regular spread (so there arises a Desarguesian projective plane of order $2^{2m}$) and the ovoid is a Suzuki–Tits ovoid (hence the maximal arc is not a Denniston maximal arc; see [13]). In the following we determine the isomorphism classes of such maximal arcs and of the corresponding partial geometries. We also determine the full automorphism groups and correlation groups of these structures.

In order to do so, and in particular in order to prove that the partial geometries are self-dual, we first give an alternative description of the maximal arcs in a more homogeneous setting.

Consider the projective space $\mathsf{PG}(5, q)$ and suppose that we have a regular spread $\mathcal{S}$ of lines in this space. The lines of this spread can be considered as the points of a projective plane $\mathsf{PG}(2, q^2)$ while the 3-spaces of $\mathsf{PG}(5, q)$ containing $q^2 + 1$ spread lines are the lines of this projective plane. Fix such a 3-space $\mathsf{PG}(3, q)$ and denote by $L_\infty$ the corresponding line of $\mathsf{PG}(2, q^2)$. Let $\mathcal{O}$ be an ovoid in $\mathsf{PG}(3, q)$ such that every point of $\mathcal{O}$ is incident with a unique line of $\mathcal{S}$. Take a line $L$ of $\mathcal{S}$ outside $\mathsf{PG}(3, q)$ and a point $x$ incident with $L$. Let $\mathsf{PG}(4, q)$ be the hyperplane generated by $\mathsf{PG}(3, q)$ and $x$. Then there is a bijective correspondence $\beta$ between the points of $\mathsf{PG}(4, q) \setminus \mathsf{PG}(3, q)$ and the lines of $\mathcal{S}$ not in $\mathsf{PG}(3, q)$ given by containment. It is also obvious that a 3-space distinct from $\mathsf{PG}(3, q)$ containing $q^2 + 1$ spread lines intersects $\mathsf{PG}(4, q)$ in a plane $\pi$ which on its turn intersects $\mathsf{PG}(3, q)$ in a member of $\mathcal{S}$. Hence the bijection $\beta$ described above defines an isomorphism between the two models of $\mathsf{PG}(2, q^2)$.

Using $\beta$, we now see that in $\mathsf{PG}(5, q)$, the spread lines corresponding to points of the Thas maximal arc $C$ defined by $\mathcal{O}$ and $x$ are the elements of $\mathcal{S}$ not in $\mathsf{PG}(3, q)$ that meet a line $xp$ in a point, where $p \in \mathcal{O}$.

## 5. Collineations and dualities of the partial geometry $\mathsf{pg}(C)$

### 5.1. Duality problem

In this section we show that the partial geometry $\mathsf{pg}(C)$, with $C$ a Thas maximal arc in the Desarguesian projective plane $\mathsf{PG}(2, q)$, is self-dual.

Note that, for a given maximal arc $C$ in any projective plane, the set of external lines of $C$ is a maximal arc $C^*$ in the dual projective plane, and it has the complementary parameters, i.e., if $C$ is a maximal $\{qn - q + n, n\}$-arc, then $C^*$ is a (dual) $\{qh - q + h, h\}$-arc, with $nh = q$. In the case of a Thas maximal arc considered above, we see that $n = h = 2^m$.

So, in order to prove that the partial geometry related to a Thas maximal arc is self-dual, it suffices to show that the corresponding Thas maximal arc is "self-dual", i.e., a Thas maximal arc $C$ is projectively equivalent with the set $C^*$ of external lines in the dual projective plane.

So let $C$ be a Thas maximal arc in $\mathsf{PG}(2, q^2)$, constructed as above using the ovoid $\mathcal{O}$. First of all, we remark that the set of tangent planes of $\mathcal{O}$ is an ovoid $\mathcal{O}^*$ in the dual of $\mathsf{PG}(3, q)$. Indeed, the set of tangent lines of $\mathcal{O}$ is the line set of a symplectic generalized quadrangle $\mathsf{W}(q)$, which arises from a (symplectic) polarity $\rho$ of $\mathsf{PG}(3, q)$. This symplectic polarity maps each point of $\mathsf{PG}(3, q)$ onto the plane spanned by the lines of $\mathsf{W}(q)$ through $x$. Hence it maps each point of $\mathcal{O}$ onto its tangent plane. Now it is also clear that $\mathcal{O}$ and $\mathcal{O}^*$ are isomorphic.

Next we consider the following construction of $C$. We dualize in $\mathsf{PG}(5, q)$ the construction of $\mathsf{PG}(2, q^2)$ outlined above. The line $L$ not in $\mathsf{PG}(3, q)$ of the spread plays the role of the space $\mathsf{PG}(3, q)$; the ovoid $\mathcal{O}$, as a set of points in $\mathsf{PG}(3, q)$ is replaced by the set of hyperplanes (which we will call the *dual ovoid* in the sequel) spanned by $L$ and the tangent planes to $\mathcal{O}$ in $\mathsf{PG}(3, q)$. The space $\mathsf{PG}(3, q)$ plays the role of $L$. The point $x$ plays the role of the hyperplane $X$ generated by $x$ and $\mathsf{PG}(3, q)$. The spread lines in $\mathsf{PG}(3, q)$ and the 3-spaces containing $L$ and $q^2 + 1$ spread lines are also interchanged. Let $H$ be an element of the dual ovoid. We claim that $H$ contains a unique 3-space $K$ containing $L$ and $q^2 + 1$ spread lines. Indeed, $K$ is the 3-space generated by $L$ and the spread line incident with the point of $\mathcal{O}$ obtained by intersecting the tangent plane of $\mathcal{O}$ corresponding to $H$ with $\mathcal{O}$. Now, interpreting the Thas maximal arc in this dual setting in the $\mathsf{PG}(5, q)$-model of $\mathsf{PG}(2, q^2)$, this maximal arc consists of those 3-spaces $S$ containing $q^2 + 1$ spread lines and contained in a hyperplane which contains $\langle x, \pi \rangle$ but not $L$, where $\pi$ is a tangent plane of $\mathcal{O}$. Then $S$ contains the spread line $T$ in $\pi$. It is clear that $S$ has no point in common with the cone $x\mathcal{O}$, and hence defines a line of $C^*$.

Hence we have shown the following result.

**Theorem 12.** *Let $C$ be a Thas maximal arc in $\mathsf{PG}(2, q^2)$, arising from an ovoid $\mathcal{O}$ in $\mathsf{PG}(3, q)$ by considering the points of the cone $x\mathcal{O}$ not in $\mathsf{PG}(3, q)$. Then $C$ is isomorphic to its dual $C^*$, and there is a duality of $\mathsf{PG}(2, q^2)$ that interchanges the point $x$ with the line $L_\infty = \mathsf{PG}(3, q)$. In particular, the partial geometry which arises from this maximal arc is self-dual.*

We will now apply the Benson-type formulae to this example. We have a partial geometry of order $(s, t, \alpha)$, with (cf. [13])

$$s = t = 2^{2m} - 2^m \quad \text{and} \quad \alpha = 2^{2m} - 2^{m+1} + 1.$$

So the maximal arc is a $\{2^{3m} - 2^{2m} + 2^m, 2^m\}$-arc. Hence $2t + 1 - \alpha = 2^{2m}$, which is a square. In this case $\alpha$ is odd; hence

$$\begin{aligned} f_1 &\equiv 1 + \alpha \bmod 2^{m+1} \\ &\equiv 1 + 2^{2m} - 2^{m+1} + 1 \bmod 2^{m+1} \\ &\equiv 2 \bmod 2^{m+1}. \end{aligned}$$

We can conclude that if we have a duality in this partial geometry, then there will be at least one absolute point and one absolute line.

### 5.2. Automorphism problem

Consider the construction which we described in Section 4. So we have a projective plane $\mathrm{PG}(2, q^2)$ and a maximal arc $C$. Consider the partial geometry which arises from this arc and a collineation of this partial geometry. Now we will have a look at this collineation in the projective plane. The points outside the maximal arc are permuted and also the lines which intersect the maximal arc are permuted. Consider a line outside the maximal arc. This is a set of $q^2 + 1$ points, with the condition that any two of them are non-collinear in the partial geometry. Hence this line is mapped to a set $B$ of $q^2 + 1$ mutually non-collinear points. Consider a point $z$ of $B$. From the foregoing, we deduce that every line containing $z$ which intersects $C$ non-trivially is a tangent line to $B$. Hence every point of the maximal arc is a nucleus of $B$ and because of Theorem 13.43 in [8] and the fact that $|C| > q - 1$, $B$ should be a line of the projective plane. Hence also the lines outside $C$ and the points inside $C$ are permuted and incidence is preserved, because we look at external lines as sets of points and at maximal arc points as sets of lines. We conclude that a collineation of the partial geometry, which arises from $C$, induces a collineation of the projective plane $\mathrm{PG}(2, q^2)$.

So we have the following result.

**Theorem 13.** *The collineation group of* $\mathrm{pg}(C)$ *is induced by the collineation group of* $\mathrm{PG}(2, q^2)$.

**Remark 14.** The previous theorem holds for all maximal arcs and corresponding partial geometries.

We will apply this theorem in the next section to give a description of the complete correlation groups of the partial geometries arising from the maximal arcs in $\mathrm{PG}(2, q^2)$ related to the Suzuki–Tits ovoids. But first we determine the isomorphism classes of such partial geometries.

### 5.3. Isomorphism problem

The arguments below will require that $m > 1$ (equivalently, $q > 2$). Henceforth, we assume $m > 1$. At the end we make a remark about the case $m = 1$.

Consider again the projective space $\mathrm{PG}(5, q)$ and a regular spread of lines in this space. Take a 3-space $\mathrm{PG}(3, q)$ containing $q^2 + 1$ spread lines in this 5-space and take a Suzuki–Tits ovoid $\mathcal{O}$ in this 3-space with the property that each point of $\mathcal{O}$ is on a unique spread line. The tangent lines to $\mathcal{O}$ form the lines of a symplectic quadrangle $W(q)$ (cf.[8]). The lines of the spread which lie in this $\mathrm{PG}(3, q)$ are lines of $W(q)$. Hence these lines form a spread $S$ of $W(q)$.

The Suzuki–Tits ovoid determines a unique polarity $\rho$ of $W(q)$, see [15]; here we require $q > 2$. Hence we obtain a set of absolute lines which corresponds with $\rho$. This set of lines forms a Lüneburg–Suzuki–Tits spread $T$.

By [1], see also [5], there are two possibilities for the size of $S \cap T$, namely $q + \sqrt{2q} + 1$ and $q - \sqrt{2q} + 1$.

It will turn out that the maximal arcs, which we obtain by taking a Suzuki–Tits ovoid, and for which we obtain $q+\sqrt{2q}+1$ as intersection number are not isomorphic to those for which we obtain $q-\sqrt{2q}+1$ as intersection number. To prove this, we determine the collineation groups of each maximal arc. Now, by [1], the subgroup of $\mathrm{PGL}_4(q)$ stabilizing $S$ and $T$ is dihedral of order $4|S \cap T|$. Taking into account all generalized homologies with center $x$ and axis $\mathrm{PG}(3, q)$ in $\mathrm{PG}(4, q)$, one easily sees that the stabilizer of $x$ and $L_\infty$ inside the stabilizer of the maximal arc $C$ in the group $\mathrm{PGL}_2(q^2) : 2$ (the extension of order 2 is due to the unique involution of $\mathrm{GF}(q^2)$, which is linear over $\mathrm{GF}(q)$ in $\mathrm{PG}(4, q)$), acting on $\mathrm{PG}(2, q^2)$) is a group of order $4|S \cap T|(q - 1)$ isomorphic to the direct product of the dihedral group of order $4|S \cap T|$ and a cyclic group of order $q - 1$. We now claim that every collineation stabilizing $C$ must fix $x$.

We will first prove the following lemma.

**Lemma 15.** *Let $\mathcal{O}$ be a Suzuki–Tits ovoid in $\mathrm{PG}(3, q)$, $q > 2$, and let $\pi$ be a plane that intersects $\mathcal{O}$ in an oval $O$. Let $T$ be the corresponding Lüneburg–Suzuki–Tits spread. If $q > 8$ and $p \in O$, then $O \setminus \{p\}$ is no non degenerate conic minus a point. If $q = 8$ and $p \in O \setminus \{p'\}$, with $p'$ the point of $O$ incident with the line of $T$ in $\pi$, then $O \setminus \{p\}$ is no non degenerate conic minus a point.*

**Proof.** By [14] 7.6.13 we can choose the coordinates such that $\mathcal{O} = \{(1, 0, 0, 0)\} \cup \{(a^{\theta+2} + aa' + a'^{\theta}, 1, a', a) : a, a' \in \mathbb{K}\}$, with $\theta$ a Tits automorphism, i.e. $(x^\theta)^\theta = x^2$, $\forall x \in \mathrm{GF}(q)$. Since all plane intersections play the same role, we can choose the plane $X_3 = 0$. The oval $O$ is the point set of the algebraic curve $\mathcal{C}' : X_0 X_1^{\theta-1} + X_2^\theta = X_3 = 0$. Let $p \in O$, $q > 8$ and assume, by way of contradiction that $O \setminus \{p\}$ is a non degenerate conic $\mathcal{C}$ minus a point. Then $\mathcal{C}$ and $\mathcal{C}'$ have at least $q$ common points. As $q > 2\theta$, by the Theorem of Bézout, $\mathcal{C}$ is a component of $\mathcal{C}'$. Hence $O$ is a conic, contradiction. Next, let $q = 8$, $p \in O$, $p \neq p'$, and assume, by way of contradiction, that $O \setminus \{p\}$ is a non degenerate conic $\mathcal{C}$ minus a point. Here $p' = (1, 0, 0, 0)$ and $O \setminus \{p'\}$ is a conic $\mathcal{C}''$ minus a point. The conics $\mathcal{C}$ and $\mathcal{C}''$ have at least 7 points in common, so coincide. Hence $O$ is a conic, a contradiction. □

Note that the previous lemma is also true for the infinite case.

Now, all lines of $\mathrm{PG}(2, q^2)$ through $x$ meet $C$ in an affine Baer subline minus one point. Consider a point $z \in C$, $z \neq x$, and let $\pi$ be a plane through $z$ and through a line of $S \setminus T$. Put $C' = \pi \cap C$. Then the projection from $x$ of $C'$ onto $\mathrm{PG}(3, q)$

is a plane intersection of $\mathcal{O}$ minus a point of the Suzuki–Tits ovoid $\mathcal{O}$ satisfying the assumption of Lemma 15. Hence $C'$ is not a Baer subline minus a point in $\mathsf{PG}(2, q^2)$. So, there are lines through each other point of $C$ meeting $C$ in a set different from a Baer subline minus one point. The claim that every collineation of $\mathsf{PG}(2, q^2)$ stabilizing $C$ must fix $x$ is proved. By Theorem 12 also $L_\infty$ must be fixed by such a collineation.

At this point we could refer to [7] for the remainder of the proof. But since we have come that far, it only takes a few paragraphs to finish.

From the foregoing it follows that the full stabilizer of $C$ is a group with a normal subgroup as described above, and the corresponding factor group a group of order $m$, corresponding to the field automorphisms of $\mathsf{GF}(q)$.

This not only shows that the order of the full collineation group of $C$, and hence also of $\mathsf{pg}(C)$, is equal to $4m|S \cap T|(q-1)$, $q = 2^m$, but it also shows that the two partial geometries related to the two different intersections are not isomorphic.

At last we show that two partial geometries $\mathsf{pg}(C)$ and $\mathsf{pg}(C')$ related to two maximal arcs $C$ and $C'$ corresponding to respective Suzuki–Tits ovoids $\mathcal{O}$ and $\mathcal{O}'$, for which the corresponding respective spreads $T$ and $T'$ satisfy $|S \cap T| = |S \cap T'|$, are isomorphic.

First we claim that for a given intersection $S \cap T$, with $T$ a Lüneburg–Suzuki–Tits spread, $T$ is the only Lüneburg–Suzuki–Tits spread intersecting $S$ in $S \cap T$. Indeed, we count the number of all possible intersections of $S$ with some Lüneburg–Suzuki–Tits spread that occur. As above, it follows from [1] (see also [5]) that, for $\epsilon \in \{+1, -1\}$, the intersection of size $q + \epsilon\sqrt{2q} + 1$ occurs at least

$$\frac{|\mathsf{PGL}_2(q^2)|}{2(q + \epsilon\sqrt{2q} + 1)} = \frac{(q^2+1)q^2(q^2-1)}{2(q + \epsilon\sqrt{2q} + 1)} = \frac{1}{2}(q - \epsilon\sqrt{2q} + 1)q^2(q^2-1)$$

times. Hence, in total, we have at least $(q+1)q^2(q^2-1)$ possible intersections that occur. But this is equal to the index of the Suzuki group in the symplectic group, namely

$$\frac{q^4(q^4-1)(q^2-1)}{(q^2+1)q^2(q-1)},$$

which is precisely the number of Lüneburg–Suzuki–Tits spreads. Our claim follows.

Now since every two intersections of the same size can be mapped onto each other, while preserving $S$, and there are unique Suzuki–Tits ovoids corresponding with them, we conclude that the corresponding maximal arcs are isomorphic.

Hence the following result has been shown.

**Theorem 16.** *There are exactly two isomorphism classes of partial geometries $\mathsf{pg}(C)$ in $\mathsf{PG}(2, 2^m)$, where $C$ is a Thas maximal arc in $\mathsf{PG}(2, q^2)$ corresponding to a Suzuki–Tits ovoid, with $m > 1$ odd. Each such partial geometry is self-dual and each collineation and duality of $\mathsf{pg}(C)$ is induced by a collineation or duality of the projective plane $\mathsf{PG}(2, q^2)$. The size of the full correlation group is $8m(2^m + \epsilon 2^{\frac{m+1}{2}} + 1)(2^m - 1)$, with $\epsilon \in \{+1, -1\}$.*

**Remark 17.** If $q = 2$, then any maximal arc in $\mathsf{PG}(2, 4)$ is a hyperoval obtained by adding the nucleus to a conic. The corresponding partial geometry is the unique generalized quadrangle of order $(2, 2)$, which is isomorphic to $\mathsf{W}(2)$. Also in this case, the full collineation group and correlation group are induced by the collineation and correlation groups of $\mathsf{PG}(2, 4)$, see for instance [12].

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