Maximum Principles for Fourth Order Ordinary Differential Inequalities

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1. INTRODUCTION

It is well known that if \( u \) is a real-valued function of class \( C^2 \) on a closed interval \( I \) and if \( u \) satisfies the inequality \( u'' \geq 0 \) in the interior of \( I \) then \( u \) satisfies the maximum principle. More precisely, if \( u \) attains its maximum at an interior point of \( I \), then \( u \) is identically constant on \( I \). This result, however, is not true for functions satisfying higher order inequalities. For example, choosing \( I = [-1, 1] \) and \( u = -x^2 \), it follows that \( u \) satisfies the inequality \( u^{(4)} > 0 \) and yet \( u \) assumes its maximum at \( x = 0 \). Similarly, \( u = x^3 \) satisfies the same inequality, and yet assumes its minimum at \( x = 0 \).

On the other hand, the following result was proved in [1]: Let \( u \) be a real-valued function of class \( C^4 \) on the interval \( [a, b] \). Suppose \( u \) satisfies the inequalities

\[
 u^{(4)} \geq 0, \quad x \in (a, b),
\]

\[
 u'(a) \geq 0, \quad u'(b) \leq 0,
\]

and, moreover, attains its minimum at a point \( x_0 \in (a, b) \). Then \( u \) is identically constant on \( [a, b] \).

In this paper we obtain an extension of this result which is also applicable to more general fourth order differential inequalities. Moreover, we shall obtain an extremum principle near the end points which gives information on the sign of \( u^{(4)} \) at the end points where an extremum is attained.

In the following section we formulate our assumptions and state the main results. Moreover, we give counterexamples to show that the results are best possible in some sense. Section 3 contains the statements of some applications of the main theorems to questions of uniqueness and continuous dependence of solutions. Section 4 contains the proofs of the stated results in Sections 2 and 3.
2. Statement of Main Results

Our main results are given by the following two theorems.

**Theorem 1.** Let \( u \in C^4(a, b) \cap C^2[a, b] \) satisfy the differential inequalities

\[
\begin{align*}
    u^{(4)} + g(x) u'' + h(x) u'' & \geq 0, \quad x \in (a, b), \\
    u'(a) & \geq 0, \quad u'(b) \leq 0,
\end{align*}
\]

(2.1)

where the given functions \( g(x) \) and \( h(x) \) are bounded on every closed subinterval of \((a, b)\). If there exists a function \( w \in C^2[a, b] \) such that

\[
\begin{align*}
    w(x) & > 0, \quad x \in [a, b], \\
    w'' + g(x) w' + h(x) w & \leq 0, \quad x \in (a, b),
\end{align*}
\]

then \( u \) cannot assume a minimum value at an interior point of \((a, b)\) unless \( u \) is identically constant.

**Theorem 2.** Let \( u \in C^4(a, b) \cap C^2[a, b] \) be a nonconstant function which has one-sided third derivatives at \( a \) and \( b \) and which satisfies the system

\[
\begin{align*}
    u^{(4)} + g(x) u''' + h(x) u'' & \geq 0, \quad x \in (a, b), \\
    u'(a) & = 0, \quad u'(b) = 0,
\end{align*}
\]

where the given functions \( g(x) \) and \( h(x) \) are bounded on every closed subinterval of \((a, b)\). Suppose there exists a function \( w \in C^2[a, b] \) satisfying

\[
\begin{align*}
    w(x) & > 0, \quad x \in [a, b], \\
    w'' + g(x) w' + h(x) w & \leq 0, \quad x \in (a, b), \\
    w'(a) & = 0, \quad w'(b) = 0.
\end{align*}
\]

If \( u \) assumes its minimum value at \( x = a \), then \( u''''(a) < 0 \), whereas if \( u \) assumes its minimum value at \( x = b \), then \( u''''(b) > 0 \).

Remark 1. The preceding results continue to hold if all the inequalities involving \( u \) are reversed, provided the word minimum is replaced by the word maximum.

Remark 2. If \( h(x) \leq 0 \) on \((a, b)\), then the function \( w(x) \equiv 1 \) satisfies the required inequalities as stated in Theorems 1 and 2.
Remark 3. The following examples show that Theorem 1 is false if lower order terms are allowed in the first inequality in (2.1):

Indeed, the function \( u = - \sin x \) attains its minimum value at \( x = \pi/2 \), and yet satisfies

\[
\begin{align*}
    u^{(4)} - u &= 0, \quad x \in (-\pi, 2\pi), \\
    u'(-\pi) &= 1, \quad u'(2\pi) = -1.
\end{align*}
\]

Similarly, for suitably chosen \( a \) and \( b \), the function \( u = -e^x \sin x \) satisfies the system

\[
\begin{align*}
    u^{(4)} + 4u &= 0, \quad x \in (a, b), \\
    u'(a) &\geq 0, \quad u'(b) \leq 0,
\end{align*}
\]

and yet attains its minimum value (in fact, a negative minimum) at some point \( x_0 \in (a, b) \).

Finally, the systems

\[
\begin{align*}
    u^{(4)} + u' &= 0, \quad x \in (a, b), \\
    u'(a) &\geq 0, \quad u'(b) \leq 0,
\end{align*}
\]

have as solutions \( u = -e^{x^2} \sin \sqrt{3} x/2 \) which attain their minimum values at an interior point of \( (a, b) \) for suitably chosen \( a \) and \( b \).

Remark 4. Theorem 2 is false, in general, if the boundary conditions are replaced by either \( u'(a) > 0, u'(b) \leq 0 \) or \( u'(a) \geq 0, u'(b) < 0 \). For example, the function \( u = 1 - (x - 1)^2 \), which satisfies

\[
\begin{align*}
    u^{(4)} &= 0, \quad x \in (0, 1), \\
    u'(0) &= 2, \quad u'(1) = 0,
\end{align*}
\]

attains its minimum value at \( x = 0 \), yet \( u'''(0) = 0 \).

Similarly, the function \( u = x^3 - 4 \), which satisfies

\[
\begin{align*}
    u^{(4)} &= 0, \quad x \in (-1, 0), \\
    u'(-1) &= 3, \quad u'(0) = 0,
\end{align*}
\]

attains its minimum value at \( x = -1 \), yet \( u''''(-1) = 6 \).

3. Applications

As a consequence of Theorems 1 and 2 we obtain the following uniqueness result.
Corollary 1. Suppose \( u_1 \) and \( u_2 \) satisfy

\[
u_1^{(4)} + g(x) u_1^{(3)} + h(x) u_1'' = f(x), \quad x \in (a, b).
\]

\[
\alpha_1 u(a) + \alpha_2 u''(a) = \gamma_1, \quad u'(a) = \gamma_2,
\]

\[
\beta_1 u(b) - \beta_2 u''(b) = \gamma_3, \quad u'(b) = \gamma_4,
\]

where \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are nonnegative constants such that \( \alpha_1^2 + \alpha_2^2 > 0 \), \( \beta_1^2 + \beta_2^2 > 0 \). If there exists a function \( w \in C^2[a, b] \) such that

\[
w'' + g(x) w' + h(x) w < 0, \quad x \in (a, b),
\]

\[
w'(a) = w'(b) = 0,
\]

then \( u_1 \equiv u_2 \) unless \( \alpha_1 = \beta_1 = 0 \) in which case \( u_1 - u_2 = \text{constant} \).

Remark. If \( \alpha_2 = \beta_2 = 0 \) then the boundary conditions \( w'(a) = 0, \ w'(b) = 0 \) can be omitted.

Our next application of Theorem 1 yields an a priori estimate from which readily follows the continuous dependence of solutions.

Corollary 2. Suppose the boundary value problem

\[
u^{(4)} + g(x) u^{(3)} + h(x) u'' = f(x), \quad x \in (a, b)
\]

\[
u(a) = \gamma_1, \quad u'(a) = \gamma_2, \quad u(b) = \gamma_3, \quad u'(b) = \gamma_4.
\]

(3.1)

can be solved for arbitrary continuous \( f \) and arbitrary constants \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \).

If there exists a function \( w \in C^2[a, b] \) such that

\[
w(x) > 0, \quad x \in [a, b],
\]

\[
w'' + g(x) w' + h(x) w \leq 0, \quad x \in (a, b),
\]

and if \( u \) is a solution of (3.1), then for all \( x \in [a, b] \),

\[
|u(x)| \leq c \max \{ \max |f|, \max |\gamma_1|, |\gamma_2|, |\gamma_3|, |\gamma_4| \}
\]

(3.2)

where the positive constant \( c \) depends only on the coefficients \( g \) and \( h \).

Remark. In particular, if the coefficients \( g, h \) are continuous and \( h \leq 0 \) on \([a, b]\), then inequality (3.2) holds. In fact, in this case, we can choose \( w \equiv 1 \), and by the general theory, the system (3.1) is solvable for arbitrary continuous \( f \) and arbitrary \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \) if uniqueness holds. The latter condition is guaranteed by Corollary 1.
4. Proofs

The proofs of Theorems 1 and 2 rely on the following result [3]:

**Theorem A.** Let \( v \in C^2(a, b) \cap C[a, b] \) satisfy the differential inequality

\[
v'' + g(x) v' + h(x) v \geq 0, \quad x \in (a, b),
\]

where the given functions \( g(x) \) and \( h(x) \) are bounded on every closed subinterval of \((a, b)\). Suppose there exists a function \( w \in C^2[a, b] \) such that

\[
w(x) > 0, \quad x \in [a, b],
\]

\[
w'' + g(x) w' + h(x) w \leq 0, \quad x \in (a, b).
\]

Then, (a) \( v/w \) cannot attain a nonnegative maximum in \((a, b)\) unless it is a constant; (b) if \( v/w \) attains its nonnegative maximum at \( x = a \) \((x = b)\) and if \( v/w \) is nonconstant, the inequality \( (v/w)'(a) < 0, (v/w)'(b) > 0 \) holds.

**Proof of Theorem 1.** Suppose \( u \) assumes a minimum value \( m \) at a point of \((a, b)\). Consider the set \( m \) of all minimum points \( x \in (a, b) \):

\[
m = \{ x \in (a, b) : u(x) = m \}.
\]

By hypothesis, \( m \) is nonempty and by continuity of \( u \), \( m \) is closed relative to \((a, b)\). Since \((a, b)\) is connected it suffices to show that \( m \) is open relative to \((a, b)\), for then, \( m = (a, b) \) and by continuity \( u \equiv m \) in \([a, b]\). Thus, suppose \( x_0 \in m \). In view of the boundary condition at \( x = a \), there exists a point \( x_0 \in [a, x_0) \) such that \( u'(x_0) = 0 \) and hence, there exists a point \( x_0 \in [x_0, x_0) \) such that \( u''(x_0) = 0 \). Similarly there exist points \( \eta \in (x_0, b] \), \( \eta^* \in (x_0, \eta] \) such that \( u'(\eta) = u''(\eta^*) = 0 \).

Restricting ourselves to the interval \((\xi^*, \eta^*)\), it readily follows that the function \( v(x) = u''(x) \) satisfies the system

\[
v'' + g(x) v' + h(x) v \geq 0, \quad x \in (\xi^*, \eta^*),
\]

\[
v(\xi^*) = v(\eta^*) = 0.
\]

In view of the hypotheses concerning \( w(x) \), it follows from Theorem A \( a) \) that either \( v \equiv 0 \) or \( v < 0 \) in \((\xi^*, \eta^*)\), that is, either \( u'' \equiv 0 \) or \( u'' < 0 \) in \((\xi^*, \eta^*)\). Since \( u \) has an interior minimum at \( x_0 \in (\xi^*, \eta^*) \), we cannot have \( u'' < 0 \) in \((\xi^*, \eta^*)\). Thus \( u'' = 0 \) in \((\xi^*, \eta^*)\) and this together with \( u(x_0) = m \), \( u'(x_0) = 0 \) yields \( u = m \) in \((\xi^*, \eta^*)\). We conclude that \( m \) is open relative to \((a, b)\) and the proof is complete.
Proof of Theorem 2. If $u$ assumes its minimum value at $x = a$, then clearly $u''(a) \geq 0$. Moreover, in view of the boundary conditions, there exists a point $\xi \in (a, b)$ such that $u''(\xi) = 0$. Now the function $v(x) = u''(x)$ satisfies

$$
v'' + g(x)v' + h(x)v \geq 0, \quad x \in (a, \xi),
$$

$$
v(a) \geq 0, \quad v(\xi) = 0.
$$

Furthermore, it is easily seen that $v$ is nonconstant. In fact, the only constant solution of (4.1) is $v \equiv 0$ which together with the boundary condition $u'(a) = 0$ and Theorem 1 yield the contradiction that $u$ is constant on $[a, b]$.

Now from Theorem A (a) it follows that $v/w$ assumes a nonnegative maximum value at $x = a$, and thus it follows from Theorem A (b) that either $(v/w)'(a) = v'(a)/w(a) < 0$ (recall that $w'(a) = 0$) or $v/w$ is a constant. However, if the latter condition holds then it is easily seen that $v$ itself must be identically zero and this contradicts the remark above. Thus $v'(a) = u'''(a) < 0$. Similarly it can be shown that $u'''(b) > 0$ if $u$ attains its minimum value at $x = b$.

This completes the proof.

Proof of Corollary 1. Define $u(x) = u_1(x) - u_2(x)$. Then $u$ satisfies

$$
u^{(4)} + g(x) u''' + h(x) u'' = 0, \quad x \in (a, b),
$$

$$
\alpha_1 u(a) + \alpha_2 u'''(a) = 0, \quad u'(a) = 0,
$$

$$
\beta_1 u(b) - \beta_2 u'''(b) = 0, \quad u'(b) = 0.
$$

If $u$ were ever positive, it would have a positive maximum. By Theorem 1 this maximum must occur at either $x = a$ or $x = b$. If $u(a) > 0$ and $u$ is nonconstant, then Theorem 2 implies $u''(a) > 0$ which contradicts the boundary condition at $x = a$. Similarly, if $u(b) > 0$ then $u''(b) < 0$ which contradicts the boundary condition at $x = b$. Thus either $u$ is constant or $u \leq 0$ in $[a, b]$. Applying the same argument to $-u$ we see that $u$ must be constant. Finally, no constant other than 0 satisfies (4.2) unless $\alpha_1 = \beta_1 = 0$, in which case any constant satisfies the system.

Remark. If $\alpha_2 = \beta_2 = 0$, then Corollary 1 follows immediately from Theorem 1 and therefore the boundary conditions $w'(a) = w'(b) = 0$ need not be imposed.

Proof of Corollary 2. Since we are assuming the solvability of (3.1) for arbitrary continuous $f$ and arbitrary $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, let $v$ be a solution of

$$
v^{(4)} + g(x) v''' + h(x) v'' = 1, \quad x \in (a, b),
$$

$$
v(a) = 1, \quad v'(a) = 1, \quad v(b) = 1, \quad v'(b) = -1.
$$

Note that $v > 0$ on $[a, b]$ in view of Theorem 1.
Let $u$ be a solution of (3.1) and set

$$M = \max \{ \max \mid f \mid, \max (\mid \gamma_1 \mid, \mid \gamma_2 \mid, \mid \gamma_3 \mid, \mid \gamma_4 \mid) \}.$$  

Then it follows that the function $z(x) = u(x) - Mu(x)$ satisfies

$$z^{(4)} + g(x)z''' + h(x)z'' \leq 0, \quad x \in (a, b),$$

$$z(a) \leq 0, \quad z(b) \leq 0, \quad z'(a) \leq 0, \quad z'(b) \geq 0,$$

and Theorem 1 implies $z \leq 0$, that is, $u(x) \leq Mu(x)$. Similarly it follows that $u(x) + Mu(x) \geq 0$, and thus $|u(x)| \leq Mu(x)$. Setting $c = \max v(x), x \in [a, b]$, the estimate (3.2) follows.

**Final remark.** The maximum principle not only has applications to the questions of uniqueness and continuous dependence for linear problems, but also to the question of existence for nonlinear problems using monotone methods. In this direction, one can consult [2].

**References**

