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Hexagon quadrangle systems

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Abstract

A hexagon quadrangle system of order n and index ρ [$\mathrm{HQS}_\rho(n)$] is a pair (X, H), where X is a finite set of n vertices and H is a collection of edge disjoint hexagon quadrangles (called *blocks*) which partitions the edge set of ρK_n, with vertex set X. *A hexagon quadrangle system* is said to be a 4-*nesting* [$N(4) - \mathrm{HQS}$] if the collection of all the 4-cycles contained in the hexagon quadrangles is a $\rho/2$-fold 4-cycle system. It is said to be a 6-*nesting* [$N(6) - \mathrm{HQS}$] if the collection of 6-cycles contained in the hexagon quadrangles is a $(\frac{3\varrho}{4})$-fold 6-cycle system. It is said to be a (4, 6)-*nesting*, briefly a $N(4, 6) - \mathrm{HQS}$, if it is both a 4-*nesting* and a 6-*nesting*.

In this paper we determine completely the spectrum of $N(4, 6) - \mathrm{HQS}$ for $\lambda = 6h$, $\mu = 4h$ and $\rho = 8h$, h positive integer.

Keywords: Designs; Graphs; G-decompositions; Nesting

1. Introduction

A λ-fold *m-cycle system* of order n is a pair (X, C), where X is a finite set of n elements, called *vertices*, and C is a collection of edge disjoint m-cycles which partitions the edge set of λK_n, complete graph with vertex set X and where every pair of vertices is joined by λ edges. In this case, $|C| = \lambda n(n-1)/2m$. When $\lambda = 1$, we will simply say *m-cycle system*. A 3-cycle is also be called a *triple* and so a λ-fold 3-*cycle system* will also be called a λ-fold 3-*triple system*. When $\lambda = 1$, we have the well known definition of *Steiner triple system* (or, simply, *triple system*).

Fairly recently the spectrum (i.e., the set of all n such that a *m-cycle system* of order n exists) has been determined to be [1,3]:

(1) $n \geqslant m$ if $n > 1$;
(2) n is odd and
(3) $\frac{n(n-1)}{2m}$ is an integer.

The spectrum for λ-fold *m-cycle system* for $\lambda \geqslant 2$ is still an open problem.

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In what follows, a *hexagon quadrangle* will be a graph obtained from a 6-cycle $(x_1, x_2, x_3, x_4, x_5, x_6)$ with the two chords of vertices x_1, x_3 and x_4, x_6.

A hexagon quadrangle system of order n and index ρ [$\mathrm{HQS}_\rho(n)$, or simply $\mathrm{HQS}(n)$ when $\rho = 1$] is a pair (X, H), where X is a finite set of n vertices and H is a collection of edge disjoint hexagon quadrangles (called *blocks*) which partitions the edge set of ρK_n, with vertex set X.

A hexagon quadrangle system (X, H) of *order* n and *index* ρ is said to be a 4-*nesting* [$N(4) - \mathrm{HQS}$] if the collection of all the 4-cycles contained in the hexagon quadrangles is a $\rho/2$-fold 4-cycle system. We will say that this ($\mu = \frac{\rho}{2}$)-fold 4-cycle system is *nested* in the HQS (X, H).

A hexagon quadrangle system (X, H) of *order* n and *index* ρ is said to be a 6-*nesting* [$N(6) - \mathrm{HQS}$] if the collection of 6-cycles contained in the hexagon quadrangles is a ($\lambda = \frac{3\varrho}{4}$)-fold 6-cycle system. This 6-cycle system is said to be *nested* in (X, H).

A hexagon quadrangle system of *order* n and *index* ϱ is said to be a (4, 6)-*nesting*, briefly a $N(4, 6) - \mathrm{HQS}$, if it is both a 4-*nesting* and a 6-*nesting*. In these cases, we say that the hexagon quadrangle system has indices (λ, ρ, μ), which we will indicate by $N(4, 6) - \mathrm{HQS}_{(\lambda,\rho,\mu)(n)}$.

Remark 1. Observe that, in a $N(4, 6) - \mathrm{HQS}_{(\lambda,\rho,\mu)}(n)$ of blocks $[(x_1, x_2, x_3), (x_4, x_5, x_6)]$, the triples (x_1, x_2, x_3), (x_4, x_5, x_6) give necessarily a Steiner triple system of index $\sigma = \frac{3}{4}\rho$. So, the *hexagon quadrangle system* is 3-nesting and it could also be indicated by $N(3, 4, 6) - \mathrm{HQS}_{\lambda,\rho,\mu,\sigma}(n)$.

In the following examples the vertex set is Z_7.

Example 1. The following HQS(7) is neither a 4-nesting or a 6-nesting.

Base blocks: [(0, 2, 1), (3, 4, 5)], [(0, 3, 4), (1, 2, 5)], [(0, 5, 3), (6, 2, 1)].

Example 2. The following HQS(7) is a 4-nesting but not a 6-nesting.

Base blocks: [(0, 6, 1), (3, 4, 2)], [(0, 3, 6), (2, 1, 5)], [(0, 6, 2), (5, 3, 1)].

Example 3. The following HQS(7) is a 6-nesting but not a 4-nesting.

Base blocks: [(0, 1, 3), (6, 4, 5)], [(0, 1, 3), (6, 5, 4)], [(1, 5, 2), (0, 4, 3)].

Example 4. The following HQS(7) is both a 4 and 6 nesting.

Base blocks: [(0, 4, 1), (2, 6, 3)], [(0, 1, 2), (4, 5, 6)], [(0, 5, 3), (6, 4, 2)].

In this paper we determine completely the spectrum of $N(4, 6) - \mathrm{HQS}(n)$ for $\lambda = 6h$, $\mu = 4h$ and $\rho = 8h$; h positive integer.

2. Necessary conditions for $N(4, 6) - \mathrm{HQS}_{(\lambda,\rho,\mu)}(n)$

In this section we prove some necessary conditions for the existence of $\mathrm{HQS}_{(\lambda,\rho,\mu)}(n)s$.

Theorem 2. *Let (X, H) be a $N(4, 6) - \mathrm{HQS}_{(\lambda,\rho,\mu)}(n)$. Then:*

(1) $3\rho = 4\lambda$, $2\lambda = 3\mu$, $\rho = 2\mu$;
(2) $\rho \equiv 0 \bmod 4$, $\mu \equiv 0 \bmod 2$, $\lambda \equiv 0 \bmod 3$; *and*
(3) $\rho = 4h$, $\lambda = 3h$, $\mu = 2h$, *h is a positive integer.*

Proof. Let (X, H) be a $N(4, 6) - \mathrm{HQS}_{(\lambda,\rho,\mu)}(n)$ of index ρ and let (X, C') be the 6-cycle system of index λ and (X, C'') the 4-cycle system of index μ, nested in it.

(1) It is immediate that:

$$|H| = |C'| = |C''|,$$

$$|H| = \frac{\binom{n}{2}}{8}\rho, \quad |C'| = \frac{\binom{n}{2}}{6}\lambda, \quad |C''| = \frac{\binom{n}{2}}{4}\mu.$$

It follows that $3\rho = 4\lambda$, $\rho = 2\mu$ and $2\lambda = 3\mu$

(2) From (1), the index $\mu = \frac{2\lambda}{3}$ must be an even number. So, since $\rho = 2\mu$ and $\lambda = \frac{3\mu}{2}$, it follows that $\rho \equiv 0 \pmod 4$ and $\lambda \equiv 3 \pmod 0$.

(3) From (2), directly. □

Theorem 3. *Let* (X, H) *be a* $N(4,6)-\mathrm{HQS}_{(\lambda,\rho,\mu)}(n)$. *If* $(\rho, \lambda, \mu)=(4, 3, 2)$, *then* $n \equiv 0, 1 \pmod 4$ *if* $(\rho, \lambda, \mu)=(8, 6, 4)$ *then* $n \equiv 0, 1 \pmod 2$.

Proof. This follows from Theorem 2. □

3. $N(4,6) - \mathrm{HQS}_{(8,6,4)}(p)$ and $N(4,6) - \mathrm{HQS}_{(8,6,4)}(p+1)$ for p a prime number. $N(4,6) - \mathrm{HQS}_{(8,6,4)}(d)$ and $N(4,6) - \mathrm{HQS}_{(8,6,4)}(d+1)$ for d odd and not divisible by 3 or 5

In this section we will examine the existence of $N(4,6) - \mathrm{HQS}_{(\rho,\lambda,\mu)}(n)$ for $(\rho, \lambda, \mu) = (8, 6, 4)$ and n a prime number or an odd number not divisible by 3 or 5.

Theorem 4. *For every prime number* p, $p \geqslant 7$, *there exist a* $N(4,6) - \mathrm{HQS}(p)$ *with indices* $(\rho, \lambda, \mu) = (8, 6, 4)$.

Proof. Let $X = \{0, 1, 2, \ldots, p-1\} = Z_p$. Observe that if $x, y \in H$, $x < y$, then: $y - x = \Delta = \{1, 2, \ldots, \frac{p-1}{2}\}$.
Consider the following families of hexagon quadrangles, 6-cycles and 4-cycles, respectively:

$$H = \left\{ b_{j,i} = \left[\left(j, \frac{i}{2} + j, i + j \right), \left(2i + j, \frac{5i}{2} + j, 3i + j \right) \right] : i \in \Delta,\ i \text{ even},\ j \in Z_p \right\}$$

$$\cup \left\{ b_{j,i} = \left[\left(j, \frac{p+i}{2} + j, i + j \right), \left(2i + j, \frac{p+5i}{2} + j, 3i + j \right) \right] : i \in \Delta,\ i \text{ odd},\ j \in Z_p \right\};$$

$$C' = \left\{ c_{j,i} = \left(j, \frac{i}{2} + j, i + j, 2i + j, \frac{5i}{2} + j, 3i + j \right) : i \in \Delta,\ i \text{ even},\ j \in Z_p \right\}$$

$$\cup \left\{ c_{j,i} = \left(j, \frac{p+i}{2} + j, i + j, 2i + j, \frac{p+5i}{2} + j, 3i + j \right) : i \in \Delta,\ i \text{ odd},\ j \in Z_p \right\};$$

$$C'' = \{ q_{j,i} = (j, i + j, 2i + j, 3i + j) : i \in \Delta,\ j \in Z_p \}.$$

We prove that (Z_p, C'') is a 4-cycle system of index $\mu = 4$. In fact, for every pair $x, y \in Z_p$, $x < y$, if $y - x = u$, then $u \in \Delta$ and the following blocks of C'' contain the edge $\{x, y\}$:

$$q_{x,u} = (x, y = x + u, x + 2u, x + 3u),$$

$$q_{x-u,x} = (x - u, x, y = x + u, x + 2u),$$

$$q_{x-2u,x-u} = (x - 2u, x - u, x, y = x + u).$$

Further, since p is a prime number

$$\{3i : i \in \Delta\} = \left\{ 3, 6, \ldots, \frac{3(p-1)}{3} \right\} = \Delta.$$

This implies that there exists an $u' \in \Delta$ such that $3u' = u$ and

$$q_{x,u/3} = (x, x + u', x + 2u', x + 3u' = y) \in C''.$$

Since:

$$|C''| = \frac{p(p-1)}{2} = \frac{\binom{p}{2}}{4}4,$$

the pair (Z_p, C'') is a 4-cycle system of index $\mu = 4$.

We prove that (Z_p, C') is a 6-cycle system of index $\lambda = 6$. In fact, for every pair $x, y \in Z_p$, $x < y$, if $y - x = u$, then $u \in \Delta$. Further, there are six blocks of C' containing the edge $\{x, y\}$.

For $u \leqslant \lfloor \frac{p-1}{4} \rfloor$ there are the cycles $c_{x,2u}, c_{x-u,2u}, c_{x-4u,2u}, c_{x-8u,2u}$ among the blocks:

$$\left\{ c_{j,i} = \left(j, \frac{i}{2} + j, i + j, 2i + j, \frac{5i}{2} + j, 3i + j \right) : i \in \Delta,\ i \textit{ even},\ j \in Z_p \right\}.$$

For $u > \lceil \frac{p-1}{4} \rceil$ there are the cycles $c_{x,p-u}, c_{x-u,p-u}, c_{x-4u,p-u}, c_{x-5u,p-u}$ among the blocks:

$\{c_{j,i} = (j, \frac{p+i}{2} + j, i + j, 2i + j, \frac{p+5i}{2} + j, 3i + j) : i \in \Delta,\ i \textit{ odd},\ j \in Z_p\}$. Further, since p is a prime number, there are two cycles $(x_1, x_2, x_3, x_4, x_5, x_6)$ such that $\{x_1, x_6\} = \{x, y\}$ and $\{x_3, x_4\} = \{x, y\}$, respectively. Since: $|C'| = \frac{p(p-1)}{2} = \left(\frac{\binom{p}{2}}{6}\right)6$, the pair (Z_p, C') is a 6-cycle system of index $\lambda = 6$. It follows that (Z_p, H) is a $N(4, 6) - \mathrm{HQS}_{(8,6,4)}(p)$. If we delete, in every $b_{j,i} \in H$, the edges $\{j, i + j\}$ and $\{2(i + j), 3(i + j)\}$, we obtain the 6-cycle system (Z_p, C') of index $\lambda = 6$. If we delete, in every $b_{j,i} \in H$, the edges $\{j, \frac{i}{2} + j\}$, $\{\frac{i}{2} + j, i + j\}$ and $\{2i + j, \frac{5i}{2} + j\}$, $\{\frac{5i}{2} + j, 3i + j\}$, in the case i even, and the edges $\{j, \frac{p+i}{2} + j\}$, $\{\frac{p+i}{2} + j, i + j\}$ and $\{2i + j, \frac{p+5i}{2} + j\}$, $\{\frac{p+5i}{2} + j, 3i + j\}$, in the case i odd, we obtain the 4-cycle system (Z_p, C'') of index $\mu = 4$. This completes the proof. □

Theorem 5. *For every prime number p, $p \geqslant 7$, there exist $N(4, 6) - \mathrm{HQS}(p + 1)$ having order $p + 1$ and indices $(\rho, \lambda, \mu) = (8, 6, 4)$.*

Proof. Let $X = \{0, 1, 2 \ldots, p - 1\} = Z_p$, $X^* = X \cup \{\infty\}$, $\Delta = \{1, 2 \ldots, \frac{p-1}{2}\}$, (X, H), where

$$H = \left\{ b_{j,i} = \left[\left(j, \frac{i}{2} + j, i + j \right), \left(2i + j, \frac{5i}{2} + j, 3i + j \right) \right] : i \in \Delta,\ i \textit{ even},\ j \in Z_p \right\}$$

$$\cup \left\{ b_{j,i} = \left[\left(j, \frac{p+i}{2} + j, i + j \right), \left(2i + j, \frac{p+5i}{2} + j, 3i + j \right) \right] : i \in \Delta,\ i \textit{ odd},\ j \in Z_p \right\}.$$

From Theorem 4, (X, H) is a $N(4, 6) - \mathrm{HQS}_{(8,6,4)}(p)$ of order p, p a prime, which defines a 6-cycle system (X, C') and a 4-cycle system (X, C''), where:

$$C' = \left\{ c_{j,i} = \left(j, \frac{i}{2} + j, i + j, 2i + j, \frac{5i}{2} + j, 3i + j \right) : i \in \Delta,\ i \textit{ even},\ j \in Z_p \right\}$$

$$\cup \left\{ c_{j,i} = \left(j, \frac{p+i}{2} + j, i + j, 2i + j, \frac{p+5i}{2} + j, 3i + j \right) : i \in \Delta,\ i \textit{ odd},\ j \in Z_p \right\};$$

$$C'' = \{q_{j,i} = (j, i + j, 2i + j, 3i + j) : i \in \Delta,\ j \in Z_p\}.$$

Consider $b_{j,1}, b_{j,(p-1)/2} \in H$, for $j \in Z_p$ and define the following blocks:

$$b_{j,\infty,1} = \left[(j, \infty, j + 1), \left(2 + j, \frac{p+5}{2} + j, 3 + j \right) \right],\ j \in Z_p;$$

$$b_{j,\infty,2} = \left[\left(j, j + \frac{p-1}{2}, \infty \right), \left(j + p - 1, j + \alpha, j + 3\frac{p-1}{2} \right) \right] : j \in Z_p;$$

$$b_{j,\infty,3} = \left[\left(j, \beta + j, j + \frac{p-1}{2} \right), \left(p - 1 + j, j + 3\frac{p-1}{2}, \infty \right) \right] : j \in Z_p;$$

where

$$\alpha = \begin{cases} \dfrac{5(p-1)}{4} & \text{if } \dfrac{p-1}{2} \text{ is even,} \\ \dfrac{7p-5}{4} & \text{if } \dfrac{p-1}{2} \text{ is odd,} \end{cases}$$

$$\beta = \begin{cases} \dfrac{p-1}{4} & \text{if } \dfrac{p-1}{2} \text{ is even,} \\ \dfrac{3p-1}{4} & \text{if } \dfrac{p-1}{2} \text{ is odd.} \end{cases}$$

Observe that, if we indicate by $b = [(x_1, x_2, x_3), (x_4, x_5, x_6)]$ the blocks of H, then the blocks $b_{j,\infty,1}, b_{j,\infty,2}, b_{j,\infty,3}$ are constructed starting from the blocks $b_{j,1}, b_{j,(p-1)/2}$ of H, by the same edges, with the same multiplicity and such that the edges $\{\infty, j\}$, for $j \in Z_p$, are repeated 6 times in the cycles $(x_1, x_2, x_3, x_4, x_5, x_6)$ of $b_{j,\infty,1}, b_{j,\infty,2}, b_{j,\infty,3}$, 4 times in the cycles (x_1, x_3, x_4, x_6) and 8 times in $b_{j,\infty,1}, b_{j,\infty,2}, b_{j,\infty,3}$. So, if $H^* = H \backslash \{b_{j,1}, b_{j,(p-1)/2}\} \cup \{b_{j,\infty,1}, b_{j,\infty,2}, b_{j,\infty,3}\}$, it is possible to verify that (X^*, H^*) is a $N(4,6) - \mathrm{HQS}_{(\rho,\lambda,\mu)}(p+1)$, completing the proof. □

The results of Theorems 4 and 5 can be extended to $N(4,6) - \mathrm{HQS}(n)$ of indices $(8h, 6h, 4h)$, by repetition of blocks.

Theorem 6. *For every odd number d, not divisible by 3 or 5, there exist $N(4,6) - \mathrm{HQS}(\rho, \lambda, \mu)$ having order d and indices $(\rho, \lambda, \mu) = (8, 6, 4)$.*

Proof. Consider the same families of hexagon quadrangles defined in Theorem 4, where $\Delta = \{1, 2, \ldots, \frac{d-1}{2}\}$:

$$H = \left\{ b_{j,i} = \left[\left(j, \frac{i}{2} + j, i + j \right), \left(2i + j, \frac{5i}{2} + j, 3i + j \right) \right] : i \in \Delta,\ i \ \mathit{even},\ j \in Z_p \right\}$$

$$\cup\, b_{j,i} = \left[\left(j, \frac{p+i}{2} + j, i + j \right), \left(2i + j, \frac{p+5i}{2} + j, 3i + j \right) \right] : i \in \Delta,\ i \ \mathit{odd},\ j \in Z_p \Bigg\};$$

$$C' = \left\{ c_{j,i} = \left(j, \frac{i}{2} + j, i + j, 2i + j, \frac{5i}{2} + j, 3i + j \right) : i \in \Delta,\ i \ \mathit{even},\ j \in Z_p \right\}$$

$$\cup \left\{ c_{j,i} = \left(j, \frac{p+i}{2} + j, i + j, 2i + j, \frac{p+5i}{2} + j, 3i + j \right) : i \in \Delta,\ i \ \mathit{odd},\ j \in Z_p \right\};$$

$$C'' = \{ q_{j,i} = (j, i + j, 2i + j, 3i + j) : i \in \Delta,\ j \in Z_p \}.$$

These families define a $N(4,6) - \mathrm{HQS}_{(8,6,4)}(n)$, (Z_d, C), nesting both the 6-cycle system (Z_d, C') and the 4-cycle system (Z_d, C''). Observe that all the edges of the hexagon quadrangles are obtained by difference methods, starting from the following base blocks:

$$b_{0,1}, b_{0,2}, \ldots, b_{0,d-1/2},$$

$$c_{0,1}, c_{0,2}, \ldots, c_{0,d-1/2},$$

$$q_{0,1}, q_{0,2}, \ldots, q_{0,d-1/2}.$$

It is necessary to observe that, since d is not divisible by 3 or 5, there is not any repetition of vertices in the blocks:

$$\left[\left(j, \frac{i}{2} + j, i + j \right), \left(2i + j, \frac{5i}{2} + j, 3i + j \right) \right] \quad \text{for } i \ \mathit{even},\ i \in \Delta \text{ and } j \in Z_d,$$

$$\left[\left(j, \frac{d+i}{2} + j, i + j \right), \left(2i + j, \frac{d+5i}{2} + j, 3i + j \right) \right] \quad \text{for } i \ \mathit{odd},\ i \in \Delta \text{ and } j \in Z_d.$$

Therefore, the conclusion follows as in Theorem 4. □

Theorem 7. *For every odd number d, not divisible by 3 or 5, there exist* $N(4,6)-\mathrm{HQS}_{(8,6,4)}$ *of order* $d+1$.

Proof. The statement follows from Theorems 4 and 5, directly. □

4. Construction $v \to 3v$ and construction $v \to 3v-2$

In this section we give two constructions for $N(4,6)-\mathrm{HQS}_{(8,6,4)}(n)$. In this case these constructions can be extended to $N(4,6)-\mathrm{HQS}(n)$ of indices $(8h, 6h, 4h)$.

Theorem 8. $N(4,6)-\mathrm{HQS}_{(8,6,4)}(3n)$*s can be constructed from* $N(4,6)-\mathrm{HQS}_{(8,6,4)}(n)$.

Proof. Let (Z_n, H) be a $N(4,6)-\mathrm{HQS}_{(8,6,4)}(n)$ of order n, $n \geqslant 7$. Let $X = Z_n \times \{1,2,3\}$ and let $(Z_{n,i}, H_i)$ be the $\mathrm{HQS}_{(8,6,4)}(n)$, for $i=1,2,3$, such that $Z_{n,i} = Z_n \times \{i\}$, and $[((a,i),(b,i),(c,i)),((\alpha,i),(\beta,i),(\gamma,i))] \in H_i$ if and only if $[(a,b,c),(\alpha,\beta,\gamma)] \in H$. Let H^* be the collection of hexagon quadrangles defined on X by

$$H_1 \subseteq H^*, \quad H_2 \subseteq H^*, \quad H_3 \subseteq H^*.$$

Further, if

$$\Phi_{123} = \{[((i,1),(j,2),(u,3)),((i+1,1),(j+1,2),(u+1,3))] : i,j,\ u \in Z_n\},$$

$$\Phi_{231} = \{[((i,2),(j,3),(u,1)),((i+1,2),(j+1,3),(u+1,1))] : i,j,u\ \in Z_n\},$$

$$\Phi_{312} = \{[((i,3),(j,1),(u,2)),((i+1,3),(j+1,1),(u+1,2))] : i,j,u \in Z_n\},$$

then

$$\Phi_{123} \subseteq H^*, \quad \Phi_{231} \subseteq H^*, \quad \Phi_{312} \subseteq H^*.$$

To begin with (X, H^*) is a $N(4,6)-\mathrm{HQS}_{(8,6,4)}(3n)$. It is easy to see that all the edges of type $\{(x,i),(y,i)\}$ are contained in H_i with the correct repetition. In fact, $(Z_{n,i}, H_i)$ is a $N(4,6)-\mathrm{HQS}_{(8,6,4)}(n)$ and no edge $\{(x,i),(y,i)\}$ is contained in any of the blocks of $\Phi_{123} \cup \Phi_{213} \cup \Phi_{312}$, which contains blocks with edges of type $\{(x,i),(y,i)\}$, for $i \neq j$.

If we consider an edge of type $\{(x,i),(y,i)\}$ with $i \neq j$, then:
if the edge is of type $\{(x,1),(y,2)\}$:

- it is contained 2-times in the blocks of Φ_{123} and in both cases it is an edge of a 6-cycle of type: $((i,1),(j,2),(u,3),(i+1,1),(j+1,2),(u+1,3))$;
- it is contained 2-times in the blocks of Φ_{312} and in both cases it is an edge of a 6-cycle of type: $((i,3),(j,1),(u,2),(i+1,3),(j+1,1),(u+1,2))$;
- it is contained 4-times in the blocks of Φ_{213} and in all of these cases it is an edge of a 4-cycle of type $((i,2),(u,1),(i+1,2),(u+1,2))$.

If the edge is of type $\{(x,2),(y,3)\}$ or $\{(x,3),(y,1)\}$ an analogous argument holds.

We observe that the number of blocks of H^* is

$$|H^*| = |H_1| + |H_2| + |H_3| + |\Phi_{123}| + |\Phi_{231}| + |\Phi_{312}|$$
$$= \frac{3\binom{n}{2}}{8}8 + 3n^2 = \frac{3n(n-1)}{2} + 3n^2$$
$$= \frac{6n^2 + 3n^2 - 3n}{2} = \frac{9n^2 - 3n}{2}$$

which is exactly the number of blocks of a $N(4,6)-\mathrm{HQS}_{(8,6,4)}(3n)$:

$$\frac{\binom{3n}{2}}{8}8 = \frac{3n(3n-1)}{2} = \frac{9n^2-3n}{2}.$$

So, the proof is completed. □

Theorem 9. *$N(4,6)-\mathrm{HQS}_{(8,6,4)}(3n-2)$s can be constructed from $N(4,6)-\mathrm{HQS}_{(8,6,4)}(n)$.*

Proof. Let (Z_n, H) be a $N(4,6)-\mathrm{HQS}_{(8,6,4)}(n)$, of order n, and let $x = n-1 \in Z_n$. If $Z_{n-1,i} = Z_{n-1} \times \{i\}$ and $X = (Z_{n-1} \times \{1,2,3\}) \cup \{x\}$, then $|X| = 3n-2$. Further, let $(x,1) = (x,2) = (x,3) = x$ and let $(Z_{n-1,i} \cup \{x\}, H_i)$ be the $\mathrm{HQS}_{(8,6,4)}(n)$ for $i = 1,2,3$, such that $[((a,i),(b,i),(c,i)),((\alpha,i),(\beta,i),(\gamma,i))] \in H_i$ if and only if $[(a,b,c),(\alpha,\beta,\gamma)] \in H$.

We define a collection H^* of hexagon qradrangles on X, as follows:

$$H_1 \subseteq H^*, \quad H_2 \subseteq H^*, \quad H_3 \subseteq H^*.$$

Further, let

$$\Phi_{123} = \{[((i,1),(j,2),(u,3)),((i+1,1),(j+1,2),(u+1,3))] : i,j,\ u \in Z_{n-1} \in H^*\},$$

$$\Phi_{231} = \{[((i,2),(j,3),(u,1)),((i+1,2),(j+1,3),(u+1,1))] : i,j,\ u \in Z_{n-1} \in H^*\},$$

$$\Phi_{312} = \{[((i,3),(j,1),(u,2)),((i+1,3),(j+1,1),(u+1,2))] : i,j,\ u \in Z_{n-1} \in H^*\}.$$

Just as in Theorem 8, it is possible to verify that the pair (X, H^*) is a $N(4,6)-\mathrm{HQS}_{(8,6,4)}(3n-2)$. The number of blocks in H^* is

$$\begin{aligned}|H^*| &= |H_1| + |H_2| + |H_3| + |\Phi_{123}| + |\Phi_{231}| + |\Phi_{312}| \\ &= \frac{3\binom{n}{2}}{8}8 + 3(n-1)^2 = \frac{3n(3n-1)}{2} + 3(n-1)^2 = \frac{9n^2-15n+6}{2}.\end{aligned}$$

This is exactly the number of blocks of a $N(4,6)-\mathrm{HQS}_{(8,6,4)}(3n-2)$. Further

$$\frac{\binom{3n-2}{2}}{8}8 = \frac{(3n-2)(3n-3)}{2} = \frac{9n^2-15n+6}{2}$$

which completes the proof. □

5. Non-existence of $N(4,6)-\mathrm{HQS}_{(8,6,4)}$ of order 6. Existence of $N(4,6)-\mathrm{HQS}_{(8,6,4)}$ of orders 9, 10, 15, 16

We prove that the minimum value for the existence of $N(4,6)-\mathrm{HQS}_{(8,6,4)}$ is $n=7$. Further we give examples of $N(4,6)-\mathrm{HQS}_{(8,6,4)}$ of orders 9, 10, 15, 16.

Theorem 10. *Does not there exist a $N(4,6)-\mathrm{HQS}_{(8,6,4)}$ of order $n=6$.*

Proof. Suppose that (Z_6, H) is a $N(4,6)-\mathrm{HQS}_{(8,6,4)}$ of order $n=6$ and let (X, C') be the 6-cycle system of index $\lambda = 6$ and (X, C'') the 4-cycle system of index $\mu = 4$, both nested in it. Indicate by $[(a,b,c),(\alpha,\beta,\gamma)]$ the blocks of H. Consider a 2-colouring

$$f : Z_6 \to \{A, B\}$$

defined as follows:

$$f(x) = \begin{cases} A & \text{if } x = 0,1,2, \\ B & \text{if } x = 3,4,5. \end{cases}$$

Observe that

(1) every $x \in Z_6$ belongs to all the blocks of H and, if T is the number of blocks of H containing x as element of degree two and M the number of blocks having x as element of degree three, then

$$\begin{cases} T + M = 15, \\ 2T + 3M = 40 \end{cases}$$

from which $T = 5$, $M = 10$;

(2) a fixed pair $\{x, y\} \in Z_6$ is contained:
   (i) 2 *times as an edge of* 4-*cycles of* C'', *but not of* 6-*cycles of* C';
   (ii) 2 *times as an edge of both* 6-*cycles of* C' *and* 4-*cycles of* C'';
   (iii) 4 *times as an edge of* 6-*cycles of* C', *but not of* 4-*cycles of* C''; in fact, if (i) or (ii) is not true, then $\lambda = 6$ and $\mu = 4$ imply $\rho \neq 8$;

(3) in every block of H there exists at least an edge whose vertices have the same colour; so, if $[(a, b, c), (\alpha, \beta, \gamma)] \in H$, the sequence of the colours of the blocks of H can be
   (1) $[(A, A, A), (B, B, B)]$;
   (2) $[(A, B, A), (B, B, A)]$;
   (3) $[(A, A, B), (B, B, A)]$;
   (4) $[(A, A, B), (A, B, B)]$;
   (5) $[(A, B, A), (B, A, B)]$.

Now, denote by

- v_1 the number of blocks of H of type (1), where 3 edges have both the vertices coloured by A,
- v_2 the number of blocks of type (2) and (3), where 2 edges have both the vertices coloured by A,
- v_3 the number of blocks of type (4) and (5), where only one edge has both the vertices coloured by A.

From (2i) and (2ii), since there are no pairs of type (2ii) in the blocks of type (1), (4) and (5), it follows necessarily that $y = 6$. So, after all, we have

$$\begin{cases} v_1 + v_2 + v_3 = 15, \\ 3v_1 + 2v_2 + v_3 = 24, \\ v_2 = 6 \end{cases}$$

and this does not give positive integer solutions. □

Theorem 11. *There exist* $N(4, 6) - \mathrm{HQS}_{(8,6,4)}(n)$, *for* $n = 9, 10, 15, 16$.

Proof. Case $n = 9$. Let H be the family of hexagon quadrangles defined on Z_9 as follows:

$$\begin{aligned} H = &\{b_{j,1} = [(j, j+3, j+1), (j+5, j+6, j+2)] : j \in Z_9\} \\ &\cup \{b_{j,2} = [(j, j+3, j+4), (j+1, j+6, j+8)] : j \in Z_9\} \\ &\cup \{b_{j,3} = [(j, j+7, j+6), (j+4, j+8, j+5)] : j \in Z_9\} \\ &\cup \{b_{j,4} = [(j, j+5, j+7), (j+8, j+2, j+3)] : j \in Z_9\}. \end{aligned}$$

Observe that the hexagon quadrangles of H can be obtained by difference methods, starting from the base blocks $b_{0,1}$, $b_{0,2}$, $b_{0,3}$, $b_{0,4}$. It is possible to verify that (Z_9, H) is a $N(4, 6) - \mathrm{HQS}_{(8,6,4)}$ of order 9.

Case $n = 10$. Let $Z_9 \cup \{\infty\}$ and let

$$\begin{aligned} b_{j,\infty,1} &= \{[(\infty, j+5, j+7), (j+8, j+2, j+3)] : j \in Z_9\}; \\ b_{j,\infty,2} &= \{[(j, j+5, j+7), (\infty, j+2, j+3)] : j \in Z_9\}; \\ b_{j,\infty,3} &= \{[(j, \infty, j+6), (j+4, j+8, j+5)] : j \in Z_9\}. \end{aligned}$$

If

$$H^* = (H - \{b_{j,3}, b_{j,4}\}) \cup \{b_{j,\infty,1}, b_{j,\infty,2}, b_{j,\infty,3}\},$$

then it is possible to verify that $(Z_9 \cup \{\infty\}, H^*)$ is a $N(4, 6) - \mathrm{HQS}_{(8,6,4)}$ of order 10.

Case $n = 15$: Let H be the family of hexagon quadrangles defined on Z_{15} as follows:

$$H = \{b_{j,1} = [(j, j+14, j+2), (j+8, j+10, j+1)] : j \in Z_{15}\}$$
$$\cup \{b_{j,2} = [(j, j+8, j+1), (j+6, j+4, j+2)] : j \in Z_{15}\}$$
$$\cup \{b_{j,3} = [(j, j+3, j+2), (j+9, j+5, j+1)] : j \in Z_{15}\}$$
$$\cup \{b_{j,4} = [(j, j+8, j+3), (j+9, j+13, j+4)] : j \in Z_{15}\}$$
$$\cup \{b_{j,5} = [(j, j+13, j+4), (j+9, j+6, j+3)] : j \in Z_{15}\}$$
$$\cup \{b_{j,6} = [(j, j+6, j+1), (j+8, j+3, j+2)] : j \in Z_{15}\}$$
$$\cup \{b_{j,7} = [(j, j+8, j+3), (j+7, j+11, j+12)] : j \in Z_{15}\}.$$

Observe that the hexagon quadrangles of H can be obtained by difference methods, starting from the base blocks $b_{0,1}, b_{0,2}, b_{0,3}, b_{0,4}, b_{0,5}, b_{0,6}, b_{0,7}$. It is possible to verify that (Z_{15}, H) is a $N(4,6) - \mathrm{HQS}_{(8,6,4)}$ of order 15.

Case $n = 16$: Let $Z_{15} \cup \{\infty\}$ and let

$$b_{j,\infty,1} = [(\infty, j+8, j+3), (j+7, j+11, j+12)] : j \in Z_{15}\};$$
$$b_{j,\infty,2} = [(j, j+8, j+3), (\infty, j+11, j+12)] : j \in Z_{15}\};$$
$$b_{j,\infty,3} = [(j, \infty, j+6), (j+14, j+4, j+13] : j \in Z_{15}\}.$$

If

$$H^* = (H - \{b_{j,6}, b_{j,7}\}) \cup \{b_{j,\infty,1}, b_{j,\infty,2}, b_{j,\infty,3}\},$$

then it is possible to verify that $(Z_{15} \cup \{\infty\}, H^*)$ is a $N(4,6) - \mathrm{HQS}_{(8,6,4)}$ of order 16. □

6. Use of PBDs to construct $\mathbf{HQS_{(8,6,4)}}$

Let K be a set of positive integers and let λ be a positive integer. *A pairwise balanced design* of order v with block sizes from K, briefly a $\mathrm{PBD}(v, K; \lambda)$ or (K, λ)-PBD, is a pair (X, B), where X is a finite set (*points*) of cardinality v and B is a family of subsets (*blocks*) of X which satisfy the properties:

(i) if $E \in B$, then $|E| \in K$;
(ii) every pair of distinct elements of X occurs in exactly λ blocks of B. The integer λ is the *index* of the PBD. A $\mathrm{PBD}(v, K)$ is a PBD of index $\lambda = 1$. The following theorem is a consequence of the important "Wilson Fundamental Construction" for PBDs.

Theorem 12. *Let* (X, B) *be a* $\mathrm{PBD}(v, K)$, *where* $K = \{n_1, n_2, \ldots, n_r\}$. *If, for every* $n_i \in K$, *there exists a* $N(4,6) - \mathrm{HQS}_{(\rho,\lambda,\mu)}$ *of order* n_i, *then there exists a* $N(4,6) - \mathrm{HQS}_{(\rho,\lambda,\mu)}$ *of order* v *defined on X.*

Proof. For every pair $x, y \in X$, $x \neq y$, there exists exactly one block $E \in B$, such that $x, y \in E$. If $|E| = n_i$, there exists a $N(4,6) - \mathrm{HQS}_{(\rho,\lambda,\mu)}$ of order n_i. The conclusion follows from the fact that no other block of B contains x, y. □

In the following theorems, we will consider the sets W_1, W_2 defined below.

$$W_1 = \{10, \ldots \to \ldots, 48\} \cup \{51, \ldots \to \ldots, 55\} \cup \{59, \ldots \to \ldots, 62\} \text{ and}$$
$$W_2 = \{93, \ldots \to \ldots, 111\} \cup \{116, 117, 118, 132\}\{138, \ldots \to \ldots, 168\}\}$$
$$\cup \{170, \ldots \to \ldots, 223\} \cup \{228, 229, 230\} \cup \{242, \ldots \to \ldots, 279\}$$
$$\cup \{283, 284, 285, 286, 298, 299, 300, 303, 304, 305, 306, 307\}$$
$$\cup \{311, \ldots \to \ldots, 335\}\} \cup \{339, 340, 341, 342\}.$$

Theorem 13 (*Colbourn and Dinitz [2]*). *If* $K = \{7, 8, 9\}$, *then for every* $v \in N$, $v \geqslant 7$, *there exist* $\mathrm{PBD}(v, K)$ *of order* v, *with the exceptions of* $v \in W_1$ *and the possible exceptions of* $v \in W_2$ (See [2, p. 209]).

Theorem 14. *There exists a* $N(4,6)-\mathrm{HQS}_{(\rho,\lambda,\mu)}$ *of order* v, *for every* $v\in N$, $v\geqslant 7$, *with the possible exceptions of* $v\in W_1\cup W_2$.

Proof. The statement follows from Theorems 4 (existence for $v=7$), 5 (existence for $v=8$), 11 (existence for $v=9$), 12 and 13. □

7. Conclusions

Collecting together the results of the previous sections, we have the following result:

Theorem 15. *There exists a* $N(4,6)-\mathrm{HQS}_{(\rho,\lambda,\mu)}$ *of order* v, *for every* $v\in N$, $v\geqslant 7$, *with the possible exceptions of the following values*: 26, 35, 55, 95, 105, 116, 146, 155, 165, 176, 185, 206, 215, 245, 266, 275, 285, 305, 315, 326, 335.

Proof. The statement follows directly from Theorems 4–9,11 and 14. □

If $Q_{\geqslant 7}=\{n\in N : n \text{ prime power}, n\geqslant 7\}$, $V=\{10, 12, 14, 15, 18, 20, 21, 22, 24, 26, 28, 30, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46, 48, 51, 52, 54, 55, 60, 62\}$, we can see in [2, p. 210] that:

Theorem 16. *For every* $v\in N$, $v\geqslant 7$, $v\notin V$, *there exists a* $\mathrm{PBD}(v)$ *having blocks of cardinality* $k\in Q_{\geqslant 7}$.

As a consequence of this result we have the following theorem.

Theorem 17. *There exists a* $N(4,6)-\mathrm{HQS}_{(\rho,\lambda,\mu)}$ *of order* v, *for every* $v\in N$, $v\geqslant 7$, $v\neq 26, 35$.

Proof. From Theorem 15, it is possible to verify that for every prime power k, $k\geqslant 7$, there exists a $N(4,6)-\mathrm{HQS}_{(8,6,4)}$ of order k. From Theorem 16 it follows that, if v is a positive integer $v\geqslant 7$, $v\notin V$, there exists a PBD of order v having blocks of cardinality k, for k a prime power, $k\geqslant 7$. So, from Theorem 13, the statement follows. □

Theorem 18. *There exist* $N(4,6)-\mathrm{HQS}_{(8,6,4)}(n)$, *for* $n=26, 35$.

Proof. Case $n=35$. Let H be the family of hexagon quadrangles defined on Z_{35} as follows:

$$
\begin{aligned}
H = &\{b_{j,1}=[(j, j+18, j+1), (j+2, j+3, j+4)] : j\in Z_{35}\}\\
&\cup\{b_{j,2}=[(j, j+1, j+2), (j+4, j+32, j+8)] : j\in Z_{35}\}\\
&\cup\{b_{j,3}=[(j, j+19, j+3), (j+6, j+10, j+12)] : j\in Z_{35}\}\\
&\cup\{b_{j,4}=[(j, j+2, j+4), (j+8, j+10, j+16)] : j\in Z_{35}\}\\
&\cup\{b_{j,5}=[(j, j+23, j+5), (j+10, j+25, j+20)] : j\in Z_{35}\}\\
&\cup\{b_{j,6}=[(j, j+3, j+6), (j+12, j+33, j+24)] : j\in Z_{35}\}\\
&\cup\{b_{j,7}=[(j, j+4, j+8), (j+16, j+9, j+32)] : j\in Z_{35}\}\\
&\cup\{b_{j,8}=[(j, j+22, j+9), (j+18, j+12, j+1)] : j\in Z_{35}\}\\
&\cup\{b_{j,9}=[(j, j+23, j+11), (j+22, j+14, j+9)] : j\in Z_{35}\}\\
&\cup\{b_{j,10}=[(j, j+6, j+12), (j+24, j+3, j+13)] : j\in Z_{35}\}\\
&\cup\{b_{j,11}=[(j, j+24, j+13), (j+26, j+4, j+17)] : j\in Z_{35}\}\\
&\cup\{b_{j,12}=[(j, j+8, j+16), (j+32, j+13, j+29)] : j\in Z_{35}\}\\
&\cup\{b_{j,13}=[(j, j+26, j+17), (j+34, j+25, j+33)] : j\in Z_{35}\}\\
&\cup\{b_{j,14}=[(j, j+7, j+21), (j+28, j+11, j+14)] : j\in Z_{35}\}\\
&\cup\{b_{j,15}=[(j, j+21, j+28), (j+14, j+10, j+7)] : j\in Z_{35}\}\\
&\cup\{b_{j,16}=[(j, j+5, j+15), (j+30, j+10, j+25)] : j\in Z_{35}\}\\
&\cup\{b_{j,17}=[(j, j+15, j+10), (j+20, j+30, j+5)] : j\in Z_{35}\}.
\end{aligned}
$$

Observe that the hexagon quadrangles of H can be obtained by difference methods, starting from the base blocks $b_{0,1}, b_{0,2}, \ldots, b_{0,17}$. It is straight forward to verify that (Z_{35}, H) is a $N(4,6) - \mathrm{HQS}_{(8,6,4)}$ of order 35.

Case $n = 26$: Let $Z_{25} \cup \{\infty\}$, $j \in Z_{25}$ and let

$$b_{j,\infty,1} = [(\infty, j+15, j+5), (j, j+6, j+12)],$$
$$b_{j,\infty,2} = [(\infty, j+20, j+5), (j, j+10, j+15)],$$
$$b_{j,\infty,3} = [(j+15, \infty, j+20), (j+10, j+5, j)],$$
$$b_{j,1} = [(j, j+13, j+1), (j+2, j+15, j+3)],$$
$$b_{j,2} = [(j, j+1, j+2), (j+4, j+5, j+6)],$$
$$b_{j,3} = [(j, j+14, j+3), (j+6, j+20, j+9)],$$
$$b_{j,4} = [(j, j+3, j+6), (j+12, j+15, j+18)],$$
$$b_{j,5} = [(j, j+16, j+7), (j+14, j+5, j+21)],$$
$$b_{j,6} = [(j, j+4, j+8), (j+16, j+20, j+24)],$$
$$b_{j,7} = [(j, j+17, j+9), (j+18, j+10, j+2)],$$
$$b_{j,8} = [(j, j+18, j+11), (j+22, j+15, j+8)],$$
$$b_{j,9} = [(j, j+2, j+4), (j+16, j+14, j+12)],$$
$$b_{j,10} = [(j, j+6, j+12), (j+16, j+21, j+11)].$$

If

$$H = \{b_{j,\infty,1}, b_{j,\infty,2}, b_{j,\infty,3}, b_{j,1}, \ldots, b_{j,10} : j \in Z_{25}\},$$

then it is possible to verify that $(Z_{25} \cup \{\infty\}, H)$ is a $N(4,6) - \mathrm{HQS}_{(8,6,4)}$ of order 26. □

Finally, we can enunciate the following conclusive theorem:

Theorem 19. *There exists a* $N(4,6) - \mathrm{HQS}_{(8,6,4)}$ *of order* v, *for every* $v \in N$, $v \geqslant 7$.

Proof. It is sufficient to see the statements of Theorems 17 and 18. □

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